Filomat 39:6 (2025), 2029–2038
https://doi.org/10.2298/FIL2506029K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The characterization of the weighted modified Morrey spaces

Abdulhamit Kucukaslan[a]

[a]*Ankara Yildirim Beyazit University, Ankara, Turkiye*

Abstract. In this study, firstly we define the weighted modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)$ for the weight function μ in the class $A_p(\mathbb{R}^n)$ and we prove the boundedness of some classical operators as the generalized fractional maximal operator M_ρ and the generalized fractional integral operator I_ρ from the weighted modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)$ with $\mu^q \in A_{1+\frac{q}{p'}}(\mathbb{R}^n)$ and from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)$ to weighted weak modified Morrey spaces $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)$, with $\mu \in A_{1,q}(\mathbb{R}^n)$ by proving the appropriate weighted norm inequalities.

1. Introduction

Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ were given by Morrey in [20] as the following: For $1 \le p < \infty, 0 \le \lambda < n, f \in L_{p,\lambda}(\mathbb{R}^n)$ if $f \in L_p^{loc}(\mathbb{R}^n)$ and

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty$$

holds. The most important properties of these spaces is to be useful in the study of local behavior properties of the solutions of second order elliptic PDEs. Morrey spaces have important applications to potential theory [1] and [2], elliptic equations [4], regularity problems [23, 24] and [25], Navier-Stokes equations [29] and Shrödinger equations [26]. For more information about the Morrey spaces can be seen in the book [28].

Weighted Morrey spaces $L_{p,\kappa}(\mathbb{R}^n,\mu)$ were introduced in [19] as follows: For $1 \le p < \infty, 0 < \kappa < 1$ and μ be a weight, $f \in L_{p,\kappa}(\mathbb{R}^n,\mu)$ if $f \in L_p^{loc}(\mathbb{R}^n,\mu)$ and

$$\|f\|_{L_{p,\kappa}(\mathbb{R}^n,\mu)} = \sup_{x\in\mathbb{R}^n, r>0} \mu(B(x,r))^{-\frac{\kappa}{p}} \|f\|_{L_p(B(x,r),\mu)} < \infty.$$

In recent years, it has been working on the behaviors of classical operators of harmonic analysis in the modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$. Some examples of these works can be seen in [3, 7, 8] and [27]. The boundednesses of fractional maximal operator M_α and Riesz potential operator in the modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ were investigated in [6]. The generalization of these two-operators as the generalized

2020 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B25, 42B35.
Keywords. Weighted modified Morrey spaces, generalized fractional integral operator, generalized fractional maximal operator, Muckenhoupt-Wheeden classes.
Received: 14 October 2024; Accepted: 15 October 2024
Communicated by Maria Alessandra Ragusa
Email address: a.kucukaslan@aybu.edu.tr (Abdulhamit Kucukaslan)
ORCID iD: https://orcid.org/0000-0002-9207-8977 (Abdulhamit Kucukaslan)

fractional maximal operator M_ρ and the generalized fractional integral operator I_ρ, respectively, in the modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ proved in [13].

Let $f \in L^1_{loc}(\mathbb{R}^n)$ which is the set of locally integrable functions. For a measurable function $\rho : (0,\infty) \to (0,\infty)$ the generalized fractional maximal operator M_ρ is defined by

$$M_\rho f(x) := \sup_{r>0} \frac{\rho(r)}{r^n} \int_{B(x,r)} |f(y)| dy$$

and the generalized fractional integral operator (so called, generalized Riesz potential operator) I_ρ by

$$I_\rho f(x) := \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y) dy$$

for any measurable function f on $\mathbb{R}^n$, respectively. If $\rho(r) \equiv r^\alpha$, then $I_\rho \equiv I_\alpha$ is Riesz potential and $M_\rho \equiv M_\alpha$ is the fractional maximal operator, respectively. If $\rho(r) \equiv 1$, then $M_\rho \equiv M$ is the Hardy-Littlewood maximal operator.

There is a stong relation between these two operators M_ρ and I_ρ (see [10], pp. 78), such that

$$M_\rho f(x) \leq C\, I_\rho(\big|f\big|)(x). \tag{1.1}$$

The generalized fractional integral operator I_ρ was initilally investigated in [15]. Nowadays many authors have been working on these two operators I_ρ and M_ρ especially in connection with Morrey-type spaces. For some of these studies can be looked at [9–14, 16, 17].

In the present work, first we define the weighted modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu)$ when μ in the Muckenhoupt's class $A_p(\mathbb{R}^n)$ and then we prove the boundedness of these two operator I_ρ and M_ρ from the weighted modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu^p)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ with $\mu^q \in A_{1+\frac{q}{p'}}$ for $1 < p < q < \infty$ and from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n, \mu)$ to the weighted weak modified Morrey spaces $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$, with $\mu \in A_{1,q}$ for $p = 1, 1 < q < \infty$ (see, Theorem 3.4 and Theorem 3.6, respectively) by proving the appropriate weighted norm inequalities (see, Lemma 3.3). Our all weight functions are belong to Muckenhoupt-Wheeden classes $A_{p,q}(\mathbb{R}^n)$.

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence.

2. Preliminaries

Let $x \in \mathbb{R}^n$ and $r > 0$, then $B(x,r) \subset \mathbb{R}^n$ is denoted as the open ball centered at x of radius r. We show the Lebesgue measure of ball $B(x,r)$ by $|B(x,r)|$, the Euclidean space by $\mathbb{R}^n$ and the characteristic function of E by χ_E. A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight μ and a measurable set E, the weighted measure of set is

$$\mu(E) = \int_E \mu(x) dx,$$

the special case of $\mu \equiv 1$ is $\mu(E) = |E|$.

Let $f \in L_1^{loc}(\mathbb{R}^n)$ and $1 \leq p < \infty$. Then we denote by $L_p^{loc}(\mathbb{R}^n, \mu)$ the weighted Lebesgue space defined by the norm

$$\|f\chi_{B(x,r)}\|_{L_p(\mathbb{R}^n,\mu)} = \left(\int_{B(x,r)} |f(x)|^p \mu(x) dx\right)^{\frac{1}{p}} < \infty.$$

A weight function μ belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$ (see [21]) for $1 < p < \infty$, if

$$\sup_B \left(\frac{1}{|B|}\int_B \mu(x)dx\right)\left(\frac{1}{|B|}\int_B \mu(x)^{1-p'}dx\right)^{p-1} \leq C,$$

and a weight function μ belongs to the Muckenhoupt-Wheeden class $A_{p,q}(\mathbb{R}^n)$ (see [22]) for $1 < p < q < \infty$, if

$$\sup_B \left(\frac{1}{|B|}\int_B \mu(x)^q dx\right)^{\frac{1}{q}} \left(\frac{1}{|B|}\int_B \mu(x)^{-p'}dx\right)^{\frac{1}{p'}} \leq C, \tag{2.1}$$

where the supremum is taken with respect to all balls B and $C > 0$. Note that, for all balls B by Hölder's inequality we get

$$|B|^{\frac{1}{p}-\frac{1}{q}-1}\|\mu\|_{L_q(B)}\|\mu^{-1}\|_{L_{p'}(B)} \geq 1. \tag{2.2}$$

If $p = 1$, w is in the $A_{1,q}(\mathbb{R}^n)$ with $1 < q < \infty$ if

$$\sup_B \left(\frac{1}{|B|}\int_B \mu(x)^q dx\right)^{\frac{1}{q}} \left(\text{ess sup}_{x\in B}\frac{1}{\mu(x)}\right) \leq C,$$

where the supremum is taken with respect to all balls B and $C > 0$.

The following lemma show that the equivalence conditions for relation parameters p, q, r, s.

Lemma 2.1. *[18] If $\mu \in A_{p,q}(\mathbb{R}^n)$ with $1 < p < q < \infty$, then the following statements are true.*
(i) $\mu^q \in A_r(\mathbb{R}^n)$ with $r = 1 + \frac{q}{p'}$.
(ii) $\mu^{-p'} \in A_{r'}(\mathbb{R}^n)$ with $r' = 1 + \frac{p}{q'}$.
(iii) $\mu^p \in A_s(\mathbb{R}^n)$ with $s = 1 + \frac{p}{q'}$.
(iv) $\mu^{-q'} \in A_{s'}(\mathbb{R}^n)$ with $s' = 1 + \frac{q'}{p}$.

The weight function μ satisfies the reverse doubling condition if there exist constants $\alpha_1 > 1$ and $\alpha_2 < 1$ such that

$$\mu(B(x,r)) \leq \alpha_2\mu(B(x,\alpha_1 r)) \tag{2.3}$$

for arbitrary $x \in \mathbb{R}^n$ and $r > 0$.

We give the definitions of weighted modified Morrey spaces and weighted weak modified Morrey spaces as follows.

Definition 2.2. *Let $1 \leq p < \infty$, $0 \leq \lambda \leq n$, $\mu \in A_p(\mathbb{R}^n)$. We denote by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)$ the weighted modified Morrey space, as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norm*

$$\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)} := \sup_{x\in\mathbb{R}^n, t>0} [\min\{1, (\mu(B(x,t)))^{\frac{1}{n}}\}]^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,t),\mu)}.$$

Note that, by inequality (2.1) we get that

$$\widetilde{L}_{p,0}(\mathbb{R}^n,\mu) = L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n,\mu),$$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu) \hookrightarrow L_{p,\lambda}(\mathbb{R}^n,\mu)$$

and

$$\max\{\|f\|_{L_{p,\lambda}(\mathbb{R}^n,\mu)}, \|f\|_{L_p(\mathbb{R}^n,\mu)}\} \leq \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)}$$

and if $\lambda < 0$ or $\lambda > n$, then the space $L_{p,\lambda}(\mathbb{R}^n,\mu) \equiv \widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)$ is trivial.

Definition 2.3. *Let $1 \le p < \infty$, $0 \le \lambda \le n$, $\mu \in A_p(\mathbb{R}^n)$. We denote by $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)$ the weighted weak modified Morrey space as the set of all local integrable functions $f(x)$, $x \in \mathbb{R}^n$ with finite norm*

$$\|f\|_{W\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)} := \sup_{x\in\mathbb{R}^n,t>0} [\min\{1,(\mu(B(x,t)))^{\frac{1}{n}}\}]^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t),\mu)}.$$

Notice that the following are valid. By the inequality (2.2) we have that

$$WL_p(\mathbb{R}^n,\mu) = WL_{p,0}(\mathbb{R}^n,\mu) = W\widetilde{L}_{p,0}(\mathbb{R}^n,\mu),$$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu) \subset W\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu) \text{ and } \|f\|_{W\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)} \le \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)}.$$

Remark 2.4. *In the Definition 2.2 and Definition 2.3, in the special case, if we choose Lebesgue measure of $B(x,t)$, $|B(x,t)|$ instead of weighted measure $\mu(B(x,t))$ then we get definition of modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ and weak modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, respectively.*

3. The operators I_ρ and M_ρ in weighted modified Morrey spaces

When we consider our two-operators I_ρ and M_ρ, we will always assume that ρ satisfies *the Dini condition*:

$$\int_0^1 \rho(t)\frac{dt}{t} < \infty, \quad \int_1^\infty \frac{\rho(t)}{t^n}\frac{dt}{t} < \infty, \tag{3.1}$$

respectively, to ensure the presence of both operators I_ρ and M_ρ in the defined domain, we choose at least for characteristic functions $1/|x|^{2n}$ of complementary balls,

$$f(x) = \frac{\chi_{\mathbb{R}^n\setminus B(0,1)}(x)}{|x|^{2n}}.$$

Also ρ satisfies *the growth condition*: there exist constants $C > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{\frac{r}{2}<s\le\frac{3r}{2}} \frac{\rho(s)}{s^n} \le C\int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t^{n+1}}dt, r>0; \quad \sup_{r<s\le 2r} \frac{\rho(s)}{s^n} \le C \sup_{k_1 r<t<k_2 r} \frac{\rho(t)}{t^n}, \ \ r>0 \tag{3.2}$$

for the operators I_ρ and M_ρ, respectively. Also we will put the following conditions on ρ (see, [15]):

$$\int_0^r \frac{\rho(t)}{t}dt \le C\rho(r), \ \ r,t>0 \tag{3.3}$$

and

$$\frac{\rho(r)}{r^n} \le C\frac{\rho(s)}{s^n}, \ \ s \le r \tag{3.4}$$

so that the sufficient conditions for the boundedness of generalized fractional integral operator I_ρ and generalized fractional maximal operator M_ρ are satisfied on the weighted modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu)$.

Lemma 3.1. *[17] (i) Let $1 < p < q < \infty$. Then the operator I_ρ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that for all $r > 0$*

$$\rho(r) \le Cr^{\frac{n}{p}-\frac{n}{q}}. \tag{3.5}$$

(ii) Let $1 < q < \infty$. Then the operator I_ρ is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that for all $r > 0$

$$\rho(r) \le Cr^{n-\frac{n}{q}}. \tag{3.6}$$

The following lemma can be derived from Theorem 1.6 in [5] which is valid for the operators I_ρ and M_ρ.

Lemma 3.2. *[5] Let $1 \le p < q < \infty$, $\mu^q \in A_{1+\frac{q}{p'}}(\mathbb{R}^n)$ satisfies (2.3), the function ρ satisfies the conditions(3.2)-(3.4), and $f \in L_1^{loc}(\mathbb{R}^n, \mu)$. Then there exist $C > 0$ for all $B(x,r) \subset \mathbb{R}^n$ such that the inequality*

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{\rho(r)}{r^n} \left(\int_{B(x,r)} \mu^q(x)dx \right)^{\frac{1}{q}} \left(\int_{B(x,r)} \mu(x)^{-p'}dx \right)^{\frac{1}{p'}} \le C \tag{3.7}$$

is necessary and sufficient condition for the boundedness of generalized Riesz potential operator I_ρ and generalized fractional maximal operator M_ρ from $L_p(\mathbb{R}^n, \mu^p)$ to $WL_q(\mathbb{R}^n, \mu^q)$ for $1 \le p < q < \infty$, and from $L_p(\mathbb{R}^n, \mu^p)$ to $L_q(\mathbb{R}^n, \mu^q)$ for $1 < p < q < \infty$, $\mu^q \in A_{1+\frac{q}{p'}}(\mathbb{R}^n)$, where the constant C does not depend on f.

The following weighted norm inequalities are weighted local $L_p(\mathbb{R}^n, \mu)$-estimate for the operator I_ρ.

Lemma 3.3. *Let $1 \le p < q < \infty$, $\mu^q \in A_{1+\frac{q}{p'}}(\mathbb{R}^n)$ and $\rho(t)$ satisfy the conditions (3.2)-(3.4).*

(i) If the condition (3.7) is fulfill, then weighted norm inequality

$$\begin{aligned} \|I_\rho f\chi_{B(x,r)}\|_{L_q(\mathbb{R}^n,\mu^q)} \lesssim \|f\chi_{B(x,2r)}\|_{L_p(\mathbb{R}^n,\mu^p)} \\ + (\mu^q(B(x,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\chi_{B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} (\mu^q(B(x,t)))^{-\frac{1}{q}} \frac{\rho(t)}{t^n}\frac{dt}{t} \end{aligned} \tag{3.8}$$

holds for the ball $B(x,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n, \mu)$.

(ii) If the condition (3.7) is fulfill, then for $p = 1$ the weighted norm inequality

$$\begin{aligned} \|I_\rho f\chi_{B(x,r)}\|_{WL_q(\mu^q)} \lesssim \|f\chi_{B(x,2r)}\|_{L_1(\mu)} \\ + (\mu^q(B(x,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\chi_{B(x,t)}\|_{L_1(\mu)} (\mu^q(B(x,t)))^{-\frac{1}{q}} \frac{\rho(t)}{t^n}\frac{dt}{t} \end{aligned} \tag{3.9}$$

holds for the ball $B(x,r)$ and for all $f \in L_1^{loc}(\mathbb{R}^n, \mu)$.

Proof. Let $1 \le p < q < \infty$ and $\mu^q \in A_{1+\frac{q}{p'}}$. For the set $B \equiv B(x,r)$ for the ball centered at x and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence, by the Minkowski inequality we have

$$\|I_\rho f\chi_B\|_{L_q(\mathbb{R}^n,\mu^q)} \le \|I_\rho f_1\chi_B\|_{L_q(\mathbb{R}^n,\mu^q)} + \|I_\rho f_2\chi_B\|_{L_q(\mathbb{R}^n,\mu^q)}.$$

Since $f_1 \in L_p(\mathbb{R}^n, \mu^p)$, $I_\rho f_1 \in L_q(\mathbb{R}^n, \mu^q)$ and from condition (3.7) we get the boundedness of I_ρ from $L_p(\mathbb{R}^n, \mu^p)$ to $L_q(\mathbb{R}^n, \mu^q)$ (see Lemma 3.2) and it follows that:

$$\|I_\rho f_1\chi_B\|_{L_q(\mathbb{R}^n,\mu^q)} \le \|I_\rho f_1\|_{L_q(\mathbb{R}^n,\mu^q)} \le C\|f_1\|_{L_p(\mu^p,\mathbb{R}^n)} = C\|f\chi_{2B}\|_{L_p(\mathbb{R}^n,\mu^p)},$$

where constant $C > 0$ is independent of f.

It's clear that $z \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x-y| \le |z-y| \le \frac{3}{2}|x-y|$. Then from conditions (3.1), (3.2) and by Fubini's theorem we have

$$|I_\rho f_2(z)| \lesssim \int_{\complement_{(2B)}} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|dy \lesssim \int_{2r}^{\infty} \int_{B(x,t)} |f(y)|dy \frac{\rho(t)}{t^n}\frac{dt}{t}.$$

Applying Hölder's inequality and from (2.2), we get

$$\begin{aligned} \int_{\complement_{(2B)}} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|dy &\lesssim \int_{2r}^{\infty} \|f\chi_{B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} \|\mu^{-1}\chi_{B(x,t)}\|_{L_{p'}(\mathbb{R}^n)} \frac{\rho(t)}{t^n}\frac{dt}{t} \\ &\lesssim \int_{2r}^{\infty} \|f\chi_{B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} (\mu^q(B(x,t)))^{-\frac{1}{q}} \frac{\rho(t)}{t^n}\frac{dt}{t}. \end{aligned}$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\|I_\rho f_2 \chi_B\|_{L_q(\mathbb{R}^n,\mu^q)} \lesssim (\mu^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \|f \chi_{B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} \left(\mu^q(B(x,t))\right)^{-\frac{1}{q}} \frac{\rho(t)}{t^n} \frac{dt}{t}$$

is valid. Thus

$$\|I_\rho f \chi_B\|_{L_q(\mathbb{R}^n,\mu^q)} \lesssim \|f \chi_{2B}\|_{L_p(\mathbb{R}^n,\mu^p)} + (\mu^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \|f \chi_{B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} \|\mu^{-1} \chi_{B(x,t)}\|_{L_{p'}(\mathbb{R}^n)} \frac{\rho(t)}{t^n} \frac{dt}{t}.$$

On the other hand,

$$\begin{aligned}
\|f \chi_{2B}\|_{L_p(\mathbb{R}^n,\mu^p)} &\approx \frac{r^{\frac{n}{p}}}{\rho(r)} \|f \chi_{2B}\|_{L_p(\mathbb{R}^n,\mu^p)} \int_r^{\infty} \frac{\rho(t)}{t^n} \frac{dt}{t} \\
&\leq \frac{r^{\frac{n}{p}}}{\rho(r)} \int_{2r}^{\infty} \|f \chi_{2B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} \frac{\rho(t)}{t^n} \frac{dt}{t} \\
&\lesssim (\mu^q(B))^{\frac{1}{q}} \|\mu^{-1} \chi_B\|_{L_{p'}(\mathbb{R}^n)} \int_{2r}^{\infty} \|f \chi_{2B(x,t)}\|_{L_p(\mu^p)} \frac{\rho(t)}{t^n} \frac{dt}{t} \\
&\lesssim (\mu^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \|f \chi_{2B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} \|\mu^{-1} \chi_{B(x,t)}\|_{L_{p'}(\mathbb{R}^n)} \frac{\rho(t)}{t^n} \frac{dt}{t} \\
&\lesssim (\mu^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \|f \chi_{2B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} \left(\mu^q(B(x,t))\right)^{-\frac{1}{q}} \frac{\rho(t)}{t^n} \frac{dt}{t}.
\end{aligned} \tag{3.10}$$

Hence by

$$\|I_\rho f \chi_B\|_{L_q(\mathbb{R}^n,\mu^q)} \lesssim (\mu^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \|f \chi_{B(x,t)}\|_{L_p(\mathbb{R}^n,\mu^p)} \left(\mu^q(B(x,t))\right)^{-\frac{1}{q}} \frac{\rho(t)}{t^n} \frac{dt}{t}$$

we get the inequality (3.8).

Now let $p = 1$ and $\mu \in A_{1,q}$. In this case by (3.7) we obtain

$$\begin{aligned}
\|I_\rho f_1 \chi_B\|_{WL_q(\mathbb{R}^n,\mu^q)} &\leq \|I_\rho f_1\|_{WL_q(\mathbb{R}^n,\mu^q)} \\
&\lesssim \|f_1\|_{L_1(\mathbb{R}^n,\mu)} \\
&= \|f \chi_B\|_{L_1(\mathbb{R}^n,\mu)} \\
&\approx \frac{r^n}{\rho(r)} \|f \chi_{2B}\|_{L_1(\mathbb{R}^n,\mu)} \int_r^{\infty} \frac{\rho(t)}{t^n} \frac{dt}{t} \\
&\leq \frac{r^n}{\rho(r)} \int_{2r}^{\infty} \|f \chi_{2B(x,t)}\|_{L_1(\mathbb{R}^n,\mu)} \frac{\rho(t)}{t^n} \frac{dt}{t} \\
&\lesssim (\mu^q(B))^{\frac{1}{q}} \|\mu^{-1} \chi_B\|_{L_\infty} \int_{2r}^{\infty} \|f \chi_{2B(x,t)}\|_{L_1(\mathbb{R}^n,\mu)} \frac{\rho(t)}{t^n} \frac{dt}{t} \\
&\lesssim (\mu^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \|f \chi_{2B(x,t)}\|_{L_1(\mathbb{R}^n,\mu)} \|\mu^{-1} \chi_{B(x,t)}\|_{L_\infty} \frac{\rho(t)}{t^n} \frac{dt}{t} \\
&\lesssim (\mu^q(B))^{\frac{1}{q}} \int_{2r}^{\infty} \|f \chi_{2B(x,t)}\|_{L_1(\mathbb{R}^n,\mu)} \left(\mu^q(B(x,t))\right)^{-\frac{1}{q}} \frac{\rho(t)}{t^n} \frac{dt}{t}.
\end{aligned} \tag{3.11}$$

Then from (3.10) and (3.11) we get the inequality (3.9). □

In the following theorem, we give the sufficient conditions for the boundedness of the generalized Riesz potential operator in the weighted modified Morrey spaces. We prove Theorem 3.4 using the weighted norm inequalities given in Lemma 3.3.

Theorem 3.4. *Let $0 \le \lambda < n$, $1 \le p < q < \infty$, $\mu^q \in A_{1+\frac{q}{p'}}(\mathbb{R}^n)$ and the function ρ be a positive, measurable function and $\rho(t)$ satisfy the conditions* (3.2)-(3.4) *and $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu)$.*

(*i*) *If $1 < p < q < \infty$ and ρ satisfies the condition (3.5) and $\mu \in A_{1,q}(\mathbb{R}^n)$, then the generalized Riesz potential operator I_ρ is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu^p)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ and the following weighted norm inequality satisfies, i.e.,*

$$\|I_\rho f\|_{\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)} \le C\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)}.$$

(*ii*) *If $p = 1, 1 < q < \infty$ and ρ satisfies the condition (3.6), then the generalized Riesz potential operator I_ρ is bounded from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ and the following weighted norm inequality satisfies, i.e.,*

$$\|I_\rho f\|_{W\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)} \le C\|f\|_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)}.$$

Proof. (*i*) Let $0 \le \lambda < n$, $1 \le p < q < \infty$, $\mu^q \in A_{1+\frac{q}{p'}}(\mathbb{R}^n)$, $\rho(t)$ satisfy the conditions (3.2)-(3.4) and $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu)$. If we write for first part of the inequality of the Lemma 3.3 in the form of maximal function $Mf(x)$ then we get

$$\begin{aligned}
\|I_\rho f\|_{\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)} &= \sup_{x\in\mathbb{R}^n,t>0} [\min\{1, \mu(B(x,t))^{\frac{1}{n}}\}]^{-\frac{\lambda}{q}} \|I_\rho f\|_{L_q(B(x,t),\mu^q)} \\
&= \sup_{x\in\mathbb{R}^n,t>0} \left([\min\{1, \mu(B(x,t))^{\frac{1}{n}}\}]^{-\lambda} \int_{B(x,t)} |I_\rho f(y)|^q dy \right)^{\frac{1}{q}} \\
&\le C \sup_{x\in\mathbb{R}^n,t>0} [\min\{1, \mu(B(x,t))^{\frac{1}{n}}\}]^{-\frac{\lambda}{q}} \\
&\times \left(\int_{B(x,t)} \left(\rho(r)Mf(y) + \int_r^\infty \|f\|_{L_p(B(x,\tau),\mu^p)} \frac{\rho(\tau)}{\tau^{\frac{n}{p}+1}} d\tau \right)^q dy \right)^{1/q} \\
&\le C \sup_{x\in\mathbb{R}^n,t>0} [\min\{1, \mu(B(x,t))^{\frac{1}{n}}\}]^{-\frac{\lambda}{q}} \\
&\times \left(\int_{B(x,t)} \left(\rho(r)Mf(y) + \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)} \min\left\{ \int_r^\infty \frac{\rho(\tau)}{\tau^{\frac{n}{p}+1}} d\tau, \int_r^\infty \frac{\rho(\tau)}{\tau^{\frac{n-\lambda}{p}+1}} d\tau \right\} \right)^q dy \right)^{1/q} \\
&= \sup_{x\in\mathbb{R}^n,t>0} [\min\{1, t\}]^{-\frac{\lambda}{q}} \\
&\times \left(\int_{B(x,t)} \left(\rho(r)Mf(y) + \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)} \min\left\{ \frac{\rho(r)}{r^{\frac{n}{p}}}, \frac{\rho(r)}{r^{\frac{n-\lambda}{p}}} \right\} \right)^q dy \right)^{1/q}.
\end{aligned}$$

Thus choosing $\rho(r) = \left(\frac{\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)}}{Mf(y)} \right)^{\frac{q-p}{q}}$ for all $y \in B(x,t)$ we obtain

$$\|I_\rho f\|_{\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)} \le C \sup_{x\in\mathbb{R}^n,t>0} [\min\{1, \mu(B(x,t))^{\frac{1}{n}}\}]^{-\frac{\lambda}{q}} \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)}^{1-\frac{p}{q}} \|Mf\|_{L_p(B(x,t),\mu^p)}^{\frac{p}{q}}.$$

Hence from the boundedness of Hardy-Littlewood maximal operator M in the spaces $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ (see [6], pp. 493) we get that

$$\|I_\rho f\|_{\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)} \leq C\|f\|^{1-\frac{p}{q}}_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)}\|f\|^{\frac{p}{q}}_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)} = \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)},$$

which completes the boundedness of the generalized Riesz potential operator I_ρ from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)$.

(*ii*) Let $p = 1$ and $1 < q < \infty$. From the Theorem 3.3 we have

$$\begin{aligned}|I_\rho f(x)| &\leq C\left(\rho(t)Mf(x) + \int_t^\infty \|f\|_{L_1(B(x,r),\mu)}\frac{\rho(r)}{r^{n+1}}dr\right)\\ &\leq C\left(\rho(t)Mf(x) + \|f\|_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)}\min\left\{\frac{\rho(t)}{t^n},\frac{\rho(t)}{t^{n-\lambda}}\right\}\right).\end{aligned}$$

Thus choosing $\rho(r) = \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)}}{Mf(y)}\right)^{\frac{q-1}{q}}$ for all $y \in B(x,t)$ we obtain

$$|I_\rho f(x)| \leq C(Mf(x))^{1/q}\|f\|^{1-1/q}_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)}. \tag{3.12}$$

From the inequality (3.12) and Theorem 1 (*ii*) in ([6] pp. 494) we get

$$\begin{aligned}\|I_\rho f\|^q_{W\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)} &= \sup_{x\in\mathbb{R}^n,t>0}[\min\{1,\mu(B(x,t))^{\frac{1}{n}}\}]^{-\lambda}\|I_\rho f\|^q_{WL_q(B(x,t),\mu^q)}\\ &= \sup_{r>0} r^q \sup_{x\in\mathbb{R}^n,t>0}[\min\{1,\mu(B(x,t))^{\frac{1}{n}}\}]^{-\lambda}\left|\{y\in B(x,t): |I_\rho f(y)| > r\}\right|\\ &\leq C\sup_{r>0} r^q \sup_{x\in\mathbb{R}^n,t>0}[\min\{1,\mu(B(x,t))^{\frac{1}{n}}\}]^{-\lambda}\\ &\times\left|\{y\in B(x,t): (Mf(y))^{1/q}\|f\|^{1-1/q}_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)} > r\}\right|\\ &= \sup_{r>0} r^q \sup_{x\in\mathbb{R}^n,t>0}[\min\{1,\mu(B(x,t))^{\frac{1}{n}}\}]^{-\lambda}\\ &\times\left|\left\{y\in B(x,t): Mf(y) > \left(\frac{r}{\|f\|^{1-1/q}_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)}}\right)^q\right\}\right|\\ &\leq C\sup_{r>0} r^q\left(\frac{\|f\|^{1-\frac{1}{q}}_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)}}{r}\right)^q\|f\|_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)}\\ &= \|f\|^q_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)},\end{aligned}$$

which completes the boundedness of generalized Riesz potential operator I_ρ from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)$ to the weak space $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)$. □

In the special case of our operator I_ρ for $\rho(r) = r^\alpha$ we get a new result in the weighted modified Morrey spaces for Riesz potential operator I_α.

Corollary 3.5. *In the Theorem 3.4, in the special case if we choose* $\rho(t) = t^{\alpha}$ *then we get sufficient condition for the boundedness of Riesz potential operator* I_{α} *from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu^p)$ *to* $\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ *for* $1 < p < q < \infty$, *and from* $\widetilde{L}_{1,\lambda}(\mathbb{R}^n, \mu)$ *to* $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ *for* $p = 1, 1 < q < \infty$.

In the following theorem, we give the sufficient conditions for the boundedness of the generalized fractional maximal operator in the weighted modified Morrey spaces. We prove Theorem 3.6 using the weighted norm inequalities given in Lemma 3.3.

Theorem 3.6. *Let* $0 \leq \lambda < n$, $1 \leq p < q < \infty$, $\mu^q \in A_{1+\frac{q}{p'}}$ *and the function* ρ *be a positive, measurable function and* $\rho(t)$ *satisfy the conditions* (3.2)-(3.4) *and* $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu)$.

(i) *If* $1 < p < q < \infty$ *and* ρ *satisfies the condition (3.5), then the generalized fractional maximal operator* M_{ρ} *is bounded from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu^p)$ *to* $\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ *and the following weighted norm inequality satisfies, i.e.,*

$$\|M_{\rho} f\|_{\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)} \leq C\|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n,\mu^p)}. \tag{3.13}$$

(ii) *If* $p = 1, 1 < q < \infty$ *and* ρ *satisfies the condition (3.6), then the generalized fractional maximal operator* M_{ρ} *is bounded from* $\widetilde{L}_{1,\lambda}(\mathbb{R}^n, \mu)$ *to the weak space* $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ *and the following weighted norm inequality satisfies, i.e.,*

$$\|M_{\rho} f\|_{W\widetilde{L}_{q,\lambda}(\mathbb{R}^n,\mu^q)} \leq C\|f\|_{\widetilde{L}_{1,\lambda}(\mathbb{R}^n,\mu)}.$$

Proof. The inequality (1.1) implies that the generalized fractional maximal operator M_{ρ} is dominated by the generalized Riesz potential operator I_{ρ}. Hence the proof of Theorem 3.6 can be seen by following step by step in the proof of Theorem 3.4 for strong-type boundedness at (i) and for weak-type boundedness at (ii), respectively. □

In the special case of our operator M_{ρ} for $\rho(r) = r^{\alpha}$ we get a new result in the weighted modified Morrey spaces for fractional maximal operator M_{α}.

Corollary 3.7. *In the Theorem 3.6, in the special case if we choose* $\rho(t) = t^{\alpha}$ *then we get sufficient condition for the boundedness of fractional maximal operator* M_{α} *from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu^p)$ *to* $\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ *for* $1 < p < q < \infty$, *and from* $\widetilde{L}_{1,\lambda}(\mathbb{R}^n, \mu)$ *to* $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ *for* $p = 1, 1 < q < \infty$.

In the special case of our operator M_{ρ} for $\rho(r) \equiv 1$ we get a new result in the weighted modified Morrey spaces.

Corollary 3.8. *In the Theorem 3.6, in the special case if we choose* $\rho(t) \equiv 1$ *then we get sufficient conditions for the boundedness of Hardy-Littlewood maximal operator* M *from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu^p)$ *to* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n, \mu^p)$ *for* $1 < p < \infty$, *and from* $\widetilde{L}_{1,\lambda}(\mathbb{R}^n, \mu)$ *to* $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n, \mu^q)$ *for* $p = 1, 1 < q < \infty$.

Conflict of Interests: The author declare that there is no conflict of interests regarding the publication of this paper.

References

[1] D.R. Adams, *A note on Riesz potentials*, Duke Math. **42**(4)(1975), 765-778.
[2] D.R. Adams, Choquet integrals in potential theory, Publ. Mat. **42**(1998), 3-66.
[3] E.V. Guliyev, Y.Y. Mammadov, *Some embeddings into the Morrey and modified Morrey spaces sssociated with the Dunkl operator*, Abstr. Appl. Anal., Volume 2010, Article ID 291345, 10 pages.
[4] G. Di Fazio, D.K. Palagachev, and M.A. Ragusa, *Global Morrey ragularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients*, J. Funct. Anal. , **166**(2)(1999), 179-196.
[5] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, *Solution of two-weight problems for integral transforms with positive kernels*, Georgian Math. J., **3**(4)(1996), 31-342.
[6] V.S. Guliyev, J. Hasanov and Y. Zeren, *Necessary and sufficient conditions for the boundedness of Riesz potential in modified Morrey spaces*, Journal of Mathematical Inequalities **5**(4)(2011), 491-506.

[7] V.S. Guliyev and Y.Y. Mammadov, *Riesz potential on the Heisenberg group and modified Morrey spaces*, An. St. Univ. Ovidius Constanta, **20**(1)(2012), 189-212.

[8] V.S. Guliyev, M.N. Omarova, M.A. Ragusa, A. Scapellato, *Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces*, Analysis and Mathematical Physics, (2021) 11:13. https://doi.org/10.1007/s13324-020-00433-9

[9] V.S. Guliyev, A.F. Ismayilova, A. Kucukaslan, A. Serbetci, *Generalized fractional integral operators on generalized local Morrey spaces*, J. Funct. Spaces, Article ID 594323, (2015), 8 pages.

[10] A. Kucukaslan, V.S. Guliyev, A. Serbetci, *Generalized fractional maximal operators on generalized local Morrey spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **69**(1)(2020), pp. 73-87.

[11] A. Kucukaslan, *Equivalence of norms of the generalized fractional integral operator and the generalized fractional maximal operator on generalized weighted Morrey spaces*, Ann. Funct. Anal. **11**(2020) pp. 1007-1026.

[12] A. Kucukaslan, *Two-type estimates for the boundedness of generalized Riesz potential operators on generalized weighted local Morrey spaces*, Tbilisi Math. Journal, **14**(4)(2021), pp. 111-134.

[13] A. Kucukaslan, *Generalized Riesz Potential Operator in the Modified Morrey Spaces*, Mathematical Methods for Engineering Applications, Springer Proceedings in Mathematics Statistics, vol 384. Springer, Cham. https://doi.org/10.1007/978-3-030-96401-6-12

[14] A. Kucukaslan, *Generalized fractional integrals in the vanishing generalized weighted local and global Morrey spaces*, Filomat, **37**(6)(2023), pp. 1893–1905.

[15] E. Nakai, *On generalized fractional integrals*, Taiwanese J. Math., **5**(2001), 587-602.

[16] E. Nakai, *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr., **287**(2-3) (2014), 339–351.

[17] A. Eridani, H. Gunawan, E. Nakai, Y. Sawano, *Characterizations for the generalized fractional integral operators on Morrey spaces*, Math. Inequal. Appl., **17**(2)(2014), 761-777.

[18] J. Garcia-Cuerva, J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math., 16, Amsterdam, 1985.

[19] T.Y. Komori, S. Shirai, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr., **282**(2)(2009), 219-231.

[20] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**(1938), 126-166.

[21] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165**(1972), 261-274.

[22] B. Muckenhoupt, R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192**(1974), 207-226.

[23] M.A. Ragusa, *Regularity of solutions of divergence form elliptic equations*, Proc. Amer. Math. Soc. 128, (1999) 533-540.

[24] M.A. Ragusa, Necessary and sufficient condition for a VMO function. Appl.Math. Comput. **218**(24), (2012) 11952?11958.

[25] V.S. Guliyev, M.N. Omarova, M.A. Ragusa, A. Scapellato, *Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces*, Anal.Math.Phys. 11, 13 (2021). https://doi.org/10.1007/s13324-020-00433-9

[26] A. Ruiz, L. Vega, *Unique continuation for Schrödinger operators with potentials in the Morrey class*, Publ. Math., **35**(2)(1991), 291-298, Conference of Mathematical Analysis (El Escorial, 1989).

[27] Y. Sawano, T. Shimomurahi, H. Tanaka, *A remark on modified Morrey spaces on metric measure spaces,* Hokkaido Math. J., **47**(1) (2018), 1-15. https://doi.org/10.14492/hokmj/1520928055

[28] Y. Sawano, G. Di Fazio, and D.I. Hakim, *Morrey Spaces Introduction and Applications to Integral Operators and PDEs*, Volume I, Chapman and Hall/CRC, 1st Edition (2020).

[29] M.E. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations **17**(1992), 1407-1456.