Filomat 39:6 (2025), 2061–2076
https://doi.org/10.2298/FIL2506061Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Self-adaptive algorithms for solving split pseudomonotone equilibrium problems and pseudocontractive operators

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Abstract. In this paper, we apply self-adaptive methods to solve split pseudomonotone equilibrium problems subject to fixed point problems of pseudocontractive operators in Hilbert spaces. First, we use linear search rules to avoid the requirement of Lipschitz type conditions of bifunctions. Secondly, we just need to assume that two pseudocontractive operators are Lipschitz continuous, without knowing the sizes of the Lipschitz constants. We present a self-adaptive algorithm for solving the investigated split problem. Under some standard conditions, we show that the sequence generated by the algorithm converges weakly to a solution of the split problem.

1. Introduction

In this paper, we focus on the following equilibrium problem of finding a point $x^* \in E$ such that

$$g(x^*, x) \geq 0, \ \forall x \in E, \tag{1}$$

where E is a nonempty, closed, and convex subset of a real Hilbert space H and $g\colon E \times E \to \mathbb{R}$ is a bifunction. Throughout, we use $\mathrm{Ep}(E, g)$ to denote the solution set of (1).

Equilibrium problem becomes a topic of general interest in the fields of science and engineering. Especially, it includes several important fields, such as fixed point problems, optimization problems, inverse problems, the Kirszbraun problem, Browder variational inclusions and variational inequality problems as special cases ([4, 9, 12, 14, 16, 19, 24, 25, 28, 30, 31, 33, 35, 38, 39]). Several methods have been proposed to solve equilibrium problem ([4, 13, 15, 17, 20, 22, 23, 27, 34]).

We introduce now the problem to be studied. Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty, closed, and convex subsets of H_1 and H_2, respectively. Let $\phi_1 : C \times C \to \mathbb{R}$ and $\phi_2 : Q \times Q \to \mathbb{R}$

2020 *Mathematics Subject Classification*. Primary 47J25; Secondary 65K10, 90C99.
Keywords. split problem, equilibrium problem, fixed point, pseudomonotone operator, pseudocontractive operator.
Received: 01 October 2024; Revised: 30 November 2024; Accepted: 14 December 2024
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be two bifunctions, and $A\colon H_1 \to H_2$ be a bounded linear operator. In this article, our purpose is to solve the following split problem which is to find a point $q^* \in C$ such that

$$q^* \in \mathrm{Ep}(C, \phi_1) \cap \mathrm{Fix}(\varphi) \text{ and } Aq^* \in \mathrm{Ep}(Q, \phi_2) \cap \mathrm{Fix}(\psi), \tag{2}$$

where $\mathrm{Fix}(\varphi)$ and $\mathrm{Fix}(\psi)$ are the fixed point sets of two nonlinear operators $\varphi\colon C \to C$ and $\psi\colon Q \to Q$, respectively.

A way for solving (2) is the gap function method presented by Zhu and Marcotte [37] which converts the original problem into an optimization problem. A technique to solve (2) is based on auxiliary problem principle by involving a strongly monotone bifunction and satisfying a certain Lipschitz-type condition, see Rockafellar [20]. An approach for solving (2) is the proximal point method ([23]) in which the subsequent problems needed to be solved are related to strongly monotone equilibrium. Another remarkable way to solve (2) is the extragradient method of Korpelevich [15], based on auxiliary problem principle of Flam and Antipin [11], which generates a sequence $\{x_n\}$ defined by

$$\begin{cases} y_n = \arg\min_{x\in E}\{\lambda g(x_n, x) + \Phi(x_n, x)\}, \\ x_{n+1} = \arg\min_{x\in E}\{\lambda g(y_n, x) + \Phi(x_n, x)\},\ n \geq 0 \end{cases}$$

where $\Phi(x, y)$ is the Bregman distance function ([10]), and x_0 is given.

Note that the prototype of (2) is the split feasibility problem arising from signal processing and image restoration [5]. Meanwhile, the split problem (2) includes the split equilibrium problem studied by Yao *et al.* [32] and the split fixed point problem of Censor and Segal [7] as special cases. There has been growing interest in the split problems due to their powerful applications. Iterative algorithms for finding the solution of the split problems have been investigated extensively, see, e.g., [1, 2, 6, 8, 16–18, 26, 29, 33, 38, 39]). Especially, Yao, Li and Postolache [32] provided a unified framework for solving (2) where the involved equilibrium bifunctions ϕ_1 and ϕ_2 are pseudomonotone and monotone, respectively, and the operators φ and ψ are Lipschitz pseudocontractive.

In this paper, we continue to investigate the split problem (2). We have two objectives: the first one is to extend the bifunction ϕ_2 from monotone to pseudomonotone and the second one is to use self-adaptive techniques to relax the restrictions of Lipschitz constants of pseudocontractive operators φ and ψ. We just need to assume that the operators φ and ψ are Lipschitz continuous, without knowing the sizes of the Lipschitz constants. At the same time, we use linear search rules to avoid the requirement of Lipschitz type conditions of bifunctions ϕ_1 and ϕ_2. We present a self-adaptive algorithm for solving (2). Under some adequate conditions, we show that the sequence $\{x_n\}$ generated by the algorithm weakly converges to a solution of (2).

2. Preliminaries

In this section, we enumerate the related nations and lemmas which will be used in the third section.
In the following, H is a real Hilbert space and C is a nonempty, closed, and convex subset of H.
The normal cone of C at $u \in C$ is defined by

$$N_C(u) = \{x \in H : \langle x, y - u\rangle \leq 0, \forall y \in C\}. \tag{3}$$

Let $f\colon C \to (-\infty, +\infty]$ be a function. f is said to be

- proper if $\{x \in C : f(x) < +\infty\} \neq \emptyset$.

- lower semicontinuous if $\{x \in C : f(x) \leq r\}$ is closed for each $r \in \mathbb{R}$.

- convex if $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$, for any $t \in [0, 1]$, and $x, y \in C$.

If $f: C \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function, then the subdifferential ∂f of f at the point $p \in C$, is defined by

$$\partial f(p) := \{x \in H : f(y) - f(p) \geq \langle x, y - p\rangle, \forall y \in C\}. \tag{4}$$

It is well known that

$$u^\dagger = \arg\min_{u \in C} f(u) \Leftrightarrow 0 \in \partial f(u^\dagger) + N_C(u^\dagger). \tag{5}$$

Let $T: C \to C$ be an operator. Recall that T is said to be

- L-Lipschitz if there is $L > 0$ such that

$$\|T(x) - T(y)\| \leq L\|x - y\|, \ \forall x, y \in C.$$

- pseudocontractive if

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C. \tag{6}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that, for every $y \in C$,

$$\|x - P_C(x)\| \leq \|x - y\|.$$

It is known that P_C is 1-Lipschitz.

Let $h: C \times C \to \mathbb{R}$ be a bifunction. Recall that h is said to be

- monotone if

$$h(x, y) + h(y, x) \leq 0, \ \forall x, y \in C.$$

- pseudomonotone if

$$h(x, y) \geq 0 \text{ implies } h(y, x) \leq 0, \forall x, y \in C.$$

Let $\{x_n\}$ be a sequence in C. Let $\omega_w(x_n)$ be the set of weak cluster points of $\{x_n\}$, i.e.,

$$\omega_w(x_n) = \{u : \exists\{x_{n_i}\} \subset \{x_n\} \text{ such that } x_{n_i} \rightharpoonup u \text{ as } i \to \infty\},$$

where "$\rightharpoonup$" and "$\to$" denote weak convergence and strong convergence, respectively.

A bifunction $h: C \times C \to \mathbb{R}$ is said to be jointly sequently weakly continuous, if

$$x_n \rightharpoonup x^\dagger \text{ and } y_n \rightharpoonup y^\dagger \Rightarrow h(x_n, y_n) \to h(x^\dagger, y^\dagger).$$

Let $h: C \times C \to \mathbb{R}$ be a bifunction. It is said that h satisfies Cond(C, h) if

(i) $h(x, x) = 0, \ \forall x \in C$;

(ii) h is pseudomonotone;

(iii) h is jointly sequently weakly continuous;

(iv) $\forall x \in C$, $h(x, \cdot)$ is convex and subdifferentiable (that is $\partial h(x, p)$ is not void, for any $p \in C$).

Lemma 2.1 ([22]). *Let $h\colon C\times C\to\mathbb{R}$ be a bifunction satisfying* Cond(C,h). *For $b_n\in C$, set*

$$a_n=\arg\min_{x\in C}\Big\{h(b_n,x)+\frac{1}{2\kappa_n}\|b_n-x\|^2\Big\},\ \kappa_n\in[t_1,t_2]\subset(0,1].$$

If $\{b_n\}$ is bounded, then $\{a_n\}$ is bounded.

Lemma 2.2 ([27]). *Let $h\colon C\times C\to\mathbb{R}$ be a bifunction satisfying* Cond(C,h). *Suppose there are two sequences $\{c_n\}\subset C$ and $\{d_n\}\subset C$, so that $c_n\rightharpoonup x^*\in C$ and $d_n\rightharpoonup y^*\in C$, respectively. Then, for any $\epsilon>0$, there exist $\epsilon_1>0$ and $N_\epsilon\in\mathbb{N}$ such that*

$$\partial_2h(c_n,d_n)\subset\partial_2h(x^*,y^*)+\frac{\epsilon}{\epsilon_1}D,$$

for every $n\geq N_\epsilon$, where ∂_2 designates the subdifferential with respect to the second variable, and $D:=\{x\in H:\|x\|\leq 1\}$.

Lemma 2.3 ([36]). *Let $T\colon C\to C$ be a continuous pseudocontractive operator. Then, T is demiclosed, that is if $\{x_n\}\subseteq C$ is a sequence which converges weakly to x, and $\lim_{n\to\infty}\|Tx_n-v\|=0$, then $x\in C$, and $Tx=v$.*

Lemma 2.4 ([3]). *Let Ω be a nonempty, closed, and convex subset of a real Hilbert space H, and $\{q_n\}\subset H$ be a sequence. Suppose that the following conditions are satisfied*

(i) For each $q\in\Omega$, $\lim_{n\to\infty}\|q_n-q\|$ exists;

(ii) $w_\omega(q_n)\subset\Omega$.

Then $\{q_n\}$ converges weakly to a point in Ω.

3. Main results

In this section, we will propose our algorithm for solving the split problem (2), and prove its convergence. Let H_1 and H_2 be two real Hilbert spaces, and C, Q be two nonempty, closed, and convex subsets of H_1 and H_2, respectively. Let $A\colon H_1\to H_2$ be a bounded linear operator with its adjoint A^*, and $\phi_1\colon C\times C\to\mathbb{R}$, $\phi_2\colon Q\times Q\to\mathbb{R}$ be two bifunctions satisfying Cond(C,ϕ_1) and Cond(Q,ϕ_2), respectively. Let $\varphi\colon C\to C$ and $\psi\colon Q\to Q$ be two pseudocontractive operators with Lipschitz constants L_1 and L_2, respectively.

Suppose the following restrictions are satisfied:

(r1): $\sigma_1\in(0,1)$, $\sigma_2\in(0,1)$, $\eta\in\Big(0,\frac{1-\sigma_1^2}{2}\Big)$, $\zeta\in\Big(0,\frac{1-\sigma_2^2}{2}\Big)$, $\beta_1\in(0,1)$, $\beta_2\in(0,1)$, $\varpi\in(0,1)$, $\delta\in(0,1)$, $\vartheta_1\in(0,1)$, $\vartheta_2\in(0,1)$ and $\alpha\in(0,1/\|A\|^2)$,

(r2): For all $n\geq0$, $\lambda_n\in[c_1,d_1]\subset(0,1]$, $\tau_n\in[c_2,d_2]\subset(0,1]$, $\gamma_n\in[c_3,d_3]\subset(0,2)$ and $\epsilon_n\in[c_4,d_4]\subset(0,2)$,

(r3): $\Gamma:=\{p|p\in\mathrm{Ep}(C,\phi_1)\cap\mathrm{Fix}(\varphi),Ap\in\mathrm{Ep}(Q,\phi_2)\cap\mathrm{Fix}(\psi)\}\neq\emptyset$.

Next, we present an iterative algorithm for solving (2).

Algorithm 3.1. *Let $x_0\in C$ be an initial guess.*

Step 1. Let x_n be given. Calculate

$$v_n=\Big(1-\frac{\rho_n}{2}\Big)x_n+\frac{\rho_n}{2}\varphi(p_n),\tag{7}$$

where

$$p_n=(1-\rho_n)x_n+\rho_n\varphi(x_n),\tag{8}$$

in which $\rho_n=\eta\beta_1^i$ and $i=\min\{0,1,2,\cdots\}$ such that

$$\rho_n\|\varphi(p_n)-\varphi(x_n)\|\leq\sigma_1\|p_n-x_n\|.\tag{9}$$

Step 2. Compute

$$y_n = \arg\min_{u\in C}\Big\{\phi_1(v_n, u) + \frac{1}{2\lambda_n}\|v_n - u\|^2\Big\}. \tag{10}$$

If $y_n = v_n$, then set $u_n = v_n$ and go to Step 5. Otherwise, continue to the next step.
Step 3. Set $z_{n,k} = (1-\varpi^k)v_n + \varpi^k y_n$, where $k = \min\{1, 2, \cdots\}$ such that

$$2\lambda_n(\phi_1(z_n, v_n) - \phi_1(z_n, y_n)) \geq \vartheta_1\|v_n - y_n\|^2. \tag{11}$$

Write $\varpi_n = \varpi^k$ and $z_n = z_{n,k}$, i.e., $z_n = (1-\varpi_n)v_n + \varpi_n y_n$.
Step 4. Compute

$$u_n = P_C(v_n - \gamma_n \iota_n \nu_n), \tag{12}$$

where $\nu_n \in \partial_2\phi_1(z_n, v_n)$ and $\iota_n = \dfrac{\phi_1(z_n, v_n)}{\|\nu_n\|^2}$.
Step 5. Compute

$$w_n = \arg\min_{v\in Q}\Big\{\phi_2(P_Q(Au_n), v) + \frac{1}{2\tau_n}\|P_Q(Au_n) - v\|^2\Big\}. \tag{13}$$

If $w_n = P_Q(Au_n)$, then set $t_n = w_n$ and go to Step 8. Otherwise, continue to the next step.
Step 6. Set $d_{n,l} = (1-\delta^l)P_Q(Au_n) + \delta^l w_n$, where $l = \min\{1, 2, \cdots\}$ such that

$$2\tau_n(\phi_2(d_n, P_Q(Au_n)) - \phi_2(d_n, w_n)) \geq \vartheta_2\|P_Q(Au_n) - w_n\|^2. \tag{14}$$

Write $\delta_n = \delta^l$ and $d_n = d_{n,l}$, i.e., $d_n = (1-\delta_n)P_Q(Au_n) + \delta_n w_n$.
Step 7. Compute

$$t_n = P_Q(P_Q(Au_n) - \epsilon_n \mu_n \varsigma_n), \tag{15}$$

where $\varsigma_n \in \partial_2\phi_2(d_n, P_Q(Au_n))$ and $\mu_n = \dfrac{\phi_2(d_n, P_Q(Au_n))}{\|\varsigma_n\|^2}$.
Step 8. Compute

$$q_n = (1 - \frac{\varrho_n}{2})t_n + \frac{\varrho_n}{2}\psi(r_n), \tag{16}$$

where

$$r_n = (1-\varrho_n)t_n + \varrho_n\psi(t_n), \tag{17}$$

in which $\varrho_n = \zeta\beta_2^j$ and $j = \min\{0, 1, 2, \cdots\}$ such that

$$\varrho_n\|\psi(r_n) - \psi(t_n)\| \leq \sigma_2\|r_n - t_n\|. \tag{18}$$

Step 9. Compute

$$x_{n+1} = P_C(u_n + \alpha A^*(q_n - Au_n)), \tag{19}$$

set $n := n+1$, and return to Step 1.

The next properties are to be mentioned in the context of Algorithm 3.1.

Proposition 3.2. *We have the following statements:*

(p1): *$y_n = v_n \Rightarrow y_n \in \mathrm{Ep}(C, \phi_1)$ and $y_n \neq v_n \Rightarrow 0 \notin \partial_2\phi_1(z_n, v_n)$, in this case $\nu_n \neq 0$.*

(p2): *$w_n = P_Q(Au_n) \Rightarrow w_n \in \mathrm{Ep}(Q, \phi_2)$ and $w_n \neq P_Q(Au_n) \Rightarrow 0 \notin \partial_2\phi_2(d_n, P_Q(Au_n))$, in this case $\varsigma_n \neq 0$.*

(p3): *There are i and j satisfying* (9) *and* (18), *respectively and*

$$\min\left\{\eta, \frac{\beta_1\sigma_1}{L_1}\right\} \leq \rho_n \leq \eta,\ \min\left\{\zeta, \frac{\beta_2\sigma_2}{L_2}\right\} \leq \varrho_n \leq \zeta,\ n \geq 0.$$

(p4): *There exist k and l such that* (11) *and* (14) *hold, respectively.*

Proof. For a proof of items (p1), (p2), and (p4), please see [21].

(p3) In fact, if $p_n = x_n$, we can choose $i = 0$. Next, we consider the case of $p_n \neq x_n$. In this situation, suppose that (9) does not hold for any $i \in \min\{0, 1, 2, \cdots\}$, namely,

$$\eta\beta_1^i\|\varphi(p_n) - \varphi(x_n)\| > \sigma_1\|p_n - x_n\|, \text{ for all } i \geq 0. \tag{20}$$

By (8), we have

$$\|p_n - x_n\| = \rho_n\|\varphi(x_n) - x_n\| = \eta\beta_1^i\|\varphi(x_n) - x_n\|, \tag{21}$$

which, together with $p_n \neq x_n$, implies that

$$\|\varphi(x_n) - x_n\| > 0. \tag{22}$$

Combining (20) and (21), we obtain

$$\eta\beta_1^i\|\varphi(p_n) - \varphi(x_n)\| > \sigma_1\|p_n - x_n\| = \sigma_1\eta\beta_1^i\|\varphi(x_n) - x_n\|, \text{ for all } i \geq 0,$$

which yields that

$$\|\varphi(p_n) - \varphi(x_n)\| > \sigma_1\|\varphi(x_n) - x_n\|, \text{ for all } i \geq 0. \tag{23}$$

Noting that $\beta_1 \in (0, 1)$ and φ is L_1-Lipschitz, we have

$$\begin{aligned}\lim_{i\to+\infty}\|\varphi(x_n) - \varphi(p_n)\| &= \lim_{i\to+\infty}\|\varphi(x_n) - \varphi(x_n + \eta\beta_1^i(\varphi(x_n) - x_n))\| \\ &= \lim_{i\to+\infty}\|\varphi(x_n) - \varphi(x_n)\| = 0,\end{aligned}$$

which together with (23) implies that $\|\varphi(x_n) - x_n\| \leq 0$. This is a contradiction with (22). Hence, there is i such that inequality (9) holds.

Since φ is L_1-Lipschitz, we have

$$\rho_n\|\varphi(p_n) - \varphi(x_n)\| \leq \rho_n L_1\|p_n - x_n\|. \tag{24}$$

At the same time,

$$\frac{\rho_n}{\beta_1}\|\varphi(p_n) - \varphi(x_n)\| > \sigma_1\|p_n - x_n\|. \tag{25}$$

From (24) and (25), we have

$$\beta_1\sigma_1\|p_n - x_n\| < \rho_n L_1\|p_n - x_n\|.$$

If $p_n = x_n$, then $i = 0$ and $\rho_n = \eta$. If $p_n \neq x_n$, then $\rho_n > \frac{\beta_1\sigma_1}{L_1}$. So, there is i satisfying (9) and $\min\left\{\eta, \frac{\beta_1\sigma_1}{L_1}\right\} \leq \rho_n \leq \eta, n \geq 0$.

Similarly, we can show that there is j satisfying (18) and $\min\left\{\zeta, \frac{\beta_2\sigma_2}{L_2}\right\} \leq \varrho_n \leq \zeta,\ n \geq 0$. □

The next properties will be useful in the development of our study.

Proposition 3.3 ([13, 21, 22]). *Suppose that* $\mathrm{Ep}(C, \phi_1) \neq \emptyset$ *and* $\mathrm{Ep}(Q, \phi_2) \neq \emptyset$. *Then, we have*

(i) $\phi_1(z_n, v_n) > 0$ *and* $\phi_2(d_n, P_Q(Au_n)) > 0$,

(ii) $\|u_n - x^*\|^2 \leq \|v_n - x^*\|^2 - \gamma_n(2 - \gamma_n)(\iota_n\|\nu_n\|)^2,\ \forall x^* \in \mathrm{Ep}(C, \phi_1)$,

(iii) $\|t_n - y^*\|^2 \le \|P_Q(Au_n) - y^*\|^2 - \epsilon_n(2-\epsilon_n)(\mu_n\|\varsigma_n\|)^2,\ \forall y^* \in \mathrm{Ep}(Q,\phi_2)$.

For a proof, we address the reader to [21].

To prove our main theorem, we first prove several lemmas. In the sequel, let x^* be a point in Γ.

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, we have the following estimates:*

(a1): $\|v_n - x^*\|^2 \le \|x_n - x^*\|^2 - \frac{\rho_n^2}{4}\|\varphi(p_n) - x_n\|^2$,

(a2): $\|q_n - Ax^*\|^2 \le \|t_n - Ax^*\|^2 - \frac{\varrho_n^2}{4}\|t_n - \psi(r_n)\|^2$.

Proof. **Proof of (a1).** Since $x^* \in \mathrm{Fix}(\varphi)\cap \mathrm{Ep}(C,\phi_1)$ and $Ax^* \in \mathrm{Fix}(\psi)\cap\mathrm{Ep}(Q,\phi_2)$, from (7) and (8), we have

$$\begin{aligned}\|v_n - x^*\|^2 &= (1-\frac{\rho_n}{2})\|x_n - x^*\|^2 + \frac{\rho_n}{2}\|\varphi(p_n) - x^*\|^2\\ &\quad - (1-\frac{\rho_n}{2})\frac{\rho_n}{2}\|\varphi(p_n) - x_n\|^2,\end{aligned} \tag{26}$$

$$\begin{aligned}\|p_n - x^*\|^2 &= (1-\rho_n)\|x_n - x^*\|^2 + \rho_n\|\varphi(x_n) - x^*\|^2\\ &\quad - \rho_n(1-\rho_n)\|\varphi(x_n) - x_n\|^2,\end{aligned} \tag{27}$$

and

$$\begin{aligned}\|p_n - \varphi(p_n)\|^2 &= (1-\rho_n)\|x_n - \varphi(p_n)\|^2 + \rho_n\|\varphi(x_n) - \varphi(p_n)\|^2\\ &\quad - \rho_n(1-\rho_n)\|\varphi(x_n) - x_n\|^2.\end{aligned} \tag{28}$$

Since φ is pseudocontractive, we get

$$\|\varphi(x_n) - x^*\|^2 \le \|x_n - x^*\|^2 + \|x_n - \varphi(x_n)\|^2, \tag{29}$$

and

$$\|\varphi(p_n) - x^*\|^2 \le \|p_n - x^*\|^2 + \|p_n - \varphi(p_n)\|^2. \tag{30}$$

Combining with (27) and (29), we have

$$\|p_n - x^*\|^2 \le \|x_n - x^*\|^2 + \rho_n^2\|x_n - \varphi(x_n)\|^2.$$

By (8), $p_n - x_n = \rho_n(\varphi(x_n) - x_n)$ which together with (9) implies that $\rho_n\|\varphi(p_n) - \varphi(x_n)\|^2 \le \sigma_1^2\rho_n\|\varphi(x_n) - x_n\|^2$. Submitting it into (28) to deduce

$$\begin{aligned}\|p_n - \varphi(p_n)\|^2 &\le (1-\rho_n)\|x_n - \varphi(p_n)\|^2\\ &\quad - \rho_n(1-\sigma_1^2-\rho_n)\|\varphi(x_n) - x_n\|^2,\end{aligned}$$

which together with (30) yields

$$\begin{aligned}\|\varphi(p_n) - x^*\|^2 &\le \|x_n - x^*\|^2 + (1-\rho_n)\|x_n - \varphi(p_n)\|^2\\ &\quad - \rho_n(1-\sigma_1^2-2\rho_n)\|\varphi(x_n) - x_n\|^2.\end{aligned} \tag{31}$$

Since $\rho_n \le \eta < \frac{1-\sigma_1^2}{2}$, $1-\sigma_1^2-2\rho_n > 0$. It follows from (31) that

$$\|\varphi(p_n) - x^*\|^2 \le \|x_n - x^*\|^2 + (1-\rho_n)\|x_n - \varphi(p_n)\|^2,$$

which combines with (26) to get (a1).

Proof of (a2). From (16) and (17), we have

$$\begin{aligned}\|q_n - Ax^*\|^2 &= (1-\frac{\varrho_n}{2})\|t_n - Ax^*\|^2 + \frac{\varrho_n}{2}\|\psi(r_n) - Ax^*\|^2\\ &\quad - \frac{\varrho_n}{2}(1-\frac{\varrho_n}{2})\|t_n - \psi(r_n)\|^2,\end{aligned} \tag{32}$$

$$\|r_n - Ax^*\|^2 = (1-\varrho_n)\|t_n - Ax^*\|^2 + \varrho_n\|\psi(t_n) - Ax^*\|^2 - \varrho_n(1-\varrho_n)\|t_n - \psi(t_n)\|^2, \tag{33}$$

and

$$\|r_n - \psi(r_n)\|^2 = (1-\varrho_n)\|t_n - \psi(r_n)\|^2 + \varrho_n\|\psi(t_n) - \psi(r_n)\|^2 - \varrho_n(1-\varrho_n)\|t_n - \psi(t_n)\|^2. \tag{34}$$

Owing to the pseudocontractiveness of ψ, we obtain

$$\|\psi(t_n) - Ax^*\|^2 \le \|t_n - Ax^*\|^2 + \|t_n - \psi(t_n)\|^2, \tag{35}$$

and

$$\|\psi(r_n) - Ax^*\|^2 \le \|r_n - Ax^*\|^2 + \|r_n - \psi(r_n)\|^2. \tag{36}$$

Substituting (35) into (33) to deduce

$$\|r_n - Ax^*\|^2 \le \|t_n - Ax^*\|^2 + \varrho_n^2\|t_n - \psi(t_n)\|^2.$$

We can rewrite (17) as $r_n - t_n = \varrho_n(\psi(t_n) - t_n)$. Then,

$$\varrho_n\|\psi(r_n) - \psi(t_n)\|^2 \le \sigma_2^2\varrho_n\|\psi(t_n) - t_n\|^2.$$

Hence, by (34), we have

$$\|r_n - \psi(r_n)\|^2 \le (1-\varrho_n)\|t_n - \psi(r_n)\|^2 - \varrho_n(1-\sigma_2^2-\varrho_n)\|t_n - \psi(t_n)\|^2,$$

which combines with (36) to obtain

$$\|\psi(r_n) - Ax^*\|^2 \le \|t_n - Ax^*\|^2 + (1-\varrho_n)\|t_n - \psi(r_n)\|^2 - \varrho_n(1-\sigma_2^2-2\varrho_n)\|t_n - \psi(t_n)\|^2. \tag{37}$$

Since $\varrho_n \le \zeta < \frac{1-\sigma_2^2}{2}$, $1-\sigma_2^2-2\varrho_n > 0$, it follows from (32) and (37) that

$$\begin{aligned}\|q_n - Ax^*\|^2 &\le (1-\frac{\varrho_n}{2})\|t_n - Ax^*\|^2 + \frac{\varrho_n}{2}(\|t_n - Ax^*\|^2 + (1-\varrho_n)\|t_n - \psi(r_n)\|^2) \\ &\quad - \frac{\varrho_n}{2}(1-\frac{\varrho_n}{2})\|t_n - \psi(r_n)\|^2 \\ &\le \|t_n - Ax^*\|^2 - \frac{\varrho_n^2}{4}\|t_n - \psi(r_n)\|^2,\end{aligned}$$

which is exactly (a2). □

Lemma 3.5. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. Then, $\lim_{n\to\infty}\|x_n - x^*\|$ exists and $\{x_n\}$ is bounded.*

Proof. Note that $Ax^* \in \mathrm{Ep}(Q,\phi_2) \cap \mathrm{Fix}(\psi) \subset Q$. Since P_Q is firmly nonexpansive, we have

$$\begin{aligned}\|P_Q(Au_n) - Ax^*\|^2 &= \|P_Q(Au_n) - P_Q(Ax^*)\|^2 \\ &\le \langle P_Q(Au_n) - P_Q(Ax^*), Au_n - Ax^*\rangle \\ &= \frac{1}{2}[\|P_Q(Au_n) - Ax^*\|^2 + \|Au_n - Ax^*\|^2 - \|P_Q(Au_n) - Au_n\|^2],\end{aligned}$$

and hence

$$\|P_Q(Au_n) - Ax^*\|^2 \le \|Au_n - Ax^*\|^2 - \|P_Q(Au_n) - Au_n\|^2.$$

This together with Proposition 3.3 and Lemma 3.4 (a2) imply that

$$\begin{aligned}\|q_n - Ax^*\|^2 &\le \|P_Q(Au_n) - Ax^*\|^2 - \epsilon_n(2-\epsilon_n)(\mu_n\|\varsigma_n\|)^2 \\ &\quad - \frac{\varrho_n^2}{4}\|t_n - \psi(r_n)\|^2 \\ &\le \|Au_n - Ax^*\|^2 - \|P_Q(Au_n) - Au_n\|^2 - \epsilon_n(2-\epsilon_n)(\mu_n\|\varsigma_n\|)^2 \\ &\quad - \frac{\varrho_n^2}{4}\|t_n - \psi(r_n)\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}2\langle u_n - x^*, A^*(q_n - Au_n)\rangle &= 2\langle q_n - Au_n, Au_n - Ax^*\rangle \\ &= \|q_n - Ax^*\|^2 - \|Au_n - Ax^*\|^2 - \|q_n - Au_n\|^2 \\ &\le -\epsilon_n(2-\epsilon_n)(\mu_n\|\varsigma_n\|)^2 - \frac{\varrho_n^2}{4}\|t_n - \psi(r_n)\|^2 \\ &\quad - \|q_n - Au_n\|^2 - \|P_Q(Au_n) - Au_n\|^2.\end{aligned} \tag{38}$$

By Proposition 3.3 and Lemma 3.4 (a1), we have

$$\begin{aligned}\|u_n - x^*\|^2 &\le \|v_n - x^*\|^2 - \gamma_n(2-\gamma_n)(\iota_n\|\nu_n\|)^2 \\ &\le \|x_n - x^*\|^2 - \frac{\rho_n^2}{4}\|\varphi(p_n) - x_n\|^2 \\ &\quad - \gamma_n(2-\gamma_n)(\iota_n\|\nu_n\|)^2.\end{aligned} \tag{39}$$

Taking into advantage relation (19), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|P_C(u_n + \alpha A^*(q_n - Au_n)) - P_C(x^*)\|^2 \\ &\le \|u_n - x^* + \alpha A^*(q_n - Au_n)\|^2 \\ &= \|u_n - x^*\|^2 + 2\alpha\langle A^*(q_n - Au_n), u_n - x^*\rangle \\ &\quad + \|\alpha A^*(q_n - Au_n)\|^2.\end{aligned} \tag{40}$$

Based on (38)-(40), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 - \alpha(1 - \alpha\|A\|^2)\|q_n - Au_n\|^2 \\ &\quad - \alpha\epsilon_n(2-\epsilon_n)(\mu_n\|\varsigma_n\|)^2 \\ &\le \|x_n - x^*\|^2 - \alpha(1 - \alpha\|A\|^2)\|q_n - Au_n\|^2 \\ &\quad - \alpha\epsilon_n(2-\epsilon_n)(\mu_n\|\varsigma_n\|)^2 - \alpha\|P_Q(Au_n) - Au_n\|^2 \\ &\le \|x_n - x^*\|^2,\end{aligned} \tag{41}$$

which implies that $\lim_{n\to\infty}\|x_n - x^*\|$ exists and so $\{x_n\}$ is bounded. □

Lemma 3.6. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. Then,*

(a3): $\lim_{n\to\infty}\|x_n - \varphi(x_n)\| = 0$,

(a4): $\lim_{n\to\infty}\|Au_n - \psi(Au_n)\| = 0$,

(a5): $\lim_{n\to\infty}\|v_n - x_n\| = \lim_{n\to\infty}\|u_n - x_n\| = 0$.

Proof. **Proof of (a3).** Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists, $\|x_{n+1} - x^*\| - \|x_n - x^*\| \to 0$. By virtue of (41), we derive

$$\alpha(1-\alpha\|A\|^2)\|q_n - Au_n\|^2 + \alpha\epsilon_n(2-\epsilon_n)(\mu_n\|\varsigma_n\|)^2 + \alpha\|P_Q(Au_n) - Au_n\|^2 \\ \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0,$$

which results in the equality

$$\lim_{n\to\infty}\|q_n - Au_n\| = 0, \tag{42}$$

$$\lim_{n\to\infty}\mu_n\|\varsigma_n\| = 0. \tag{43}$$

and

$$\lim_{n\to\infty}\|P_Q(Au_n) - Au_n\| = 0. \tag{44}$$

By Lemmas 3.4 and 3.5, we have

$$\|x_{n+1} - x^*\| \leq \|u_n - x^*\| \leq \|v_n - x^*\| \leq \|x_n - x^*\|.$$

Therefore,

$$\lim_{n\to\infty}\|u_n - x^*\| = \lim_{n\to\infty}\|v_n - x^*\| = \lim_{n\to\infty}\|x_n - x^*\|,$$

which also indicates that the sequences $\{u_n\}$ and $\{v_n\}$ are all bounded.

By (15), we have $\|t_n - P_Q(Au_n)\| \leq \epsilon_n\mu_n\|\varsigma_n\|$. It follows from (43) that $\lim_{n\to\infty}\|t_n - P_Q(Au_n)\| = 0$, which together with (44) implies that

$$\lim_{n\to\infty}\|t_n - Au_n\| = 0, \tag{45}$$

Thanks to (39), we obtain

$$\frac{\rho_n^2}{4}\|\varphi(p_n) - x_n\|^2 + \gamma_n(2-\gamma_n)(\iota_n\|\nu_n\|)^2 \leq \|v_n - x^*\|^2 - \|u_n - x^*\|^2 \to 0.$$

Thus,

$$\lim_{n\to\infty}\|\varphi(p_n) - x_n\| = 0, \tag{46}$$

and

$$\lim_{n\to\infty}\iota_n\|\nu_n\| = 0. \tag{47}$$

Due to (8) and (9), we have

$$\|\varphi(x_n) - x_n\| \leq \|\varphi(x_n) - \varphi(p_n)\| + \|\varphi(p_n) - x_n\| \\ \leq \sigma_1\|\varphi(x_n) - x_n\| + \|\varphi(p_n) - x_n\|.$$

It follows that $\|\varphi(x_n) - x_n\| \leq \frac{1}{1-\sigma_1}\|\varphi(p_n) - x_n\|$. It combines with (46) to conclude (a3).

Proof of (a4). Taking into account (42) and (45), we have $\|q_n - t_n\| \to 0$. This together with (16) implies that

$$\lim_{n\to\infty}\|\psi(r_n) - t_n\| = 0. \tag{48}$$

According to (17) and (18), we obtain

$$\|t_n - \psi(t_n)\| \leq \|t_n - \psi(r_n)\| + \|\psi(r_n) - \psi(t_n)\| \\ \leq \|t_n - \psi(r_n)\| + \sigma_2\|t_n - \psi(t_n)\|,$$

which leads to $\|t_n - \psi(t_n)\| \le \frac{1}{1-\sigma_2}\|t_n - \psi(r_n)\|$. It follows from (48) that

$$\lim_{n\to\infty} \|t_n - \psi(t_n)\| = 0,$$

which combines with (45) to reach (a4).

Proof of (a5). Since $\|v_n - x_n\| \le \frac{\rho_n}{2}\|\varphi(p_n) - x_n\|$, it follows from (46) that

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{49}$$

Taking into account equality (12), we have

$$\|u_n - x_n\| \le \|u_n - v_n\| + \|v_n - x_n\| \le \gamma_n \iota_n \|\nu_n\| + \|v_n - x_n\|.$$

Hence, by (47) and (49), we get $\lim_{n\to\infty} \|u_n - x_n\| = 0$. □

Lemma 3.7. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, $\omega_w(x_n) \subset \Gamma$.*

Proof. Pick up any $z^* \in \omega_w(x_n)$, i.e., there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup z^* \in C$. Utilizing Lemma 3.6, we deduce that $u_{n_i} \rightharpoonup z^*$, $v_{n_i} \rightharpoonup z^*$, $Au_{n_i} \rightharpoonup Az^*$, $Av_{n_i} \rightharpoonup Az^*$ and $t_{n_i} \rightharpoonup Az^*$.

Step 1. $z^* \in \mathrm{Fix}(\varphi)$.

From (a3), we have $\lim_{i\to\infty} \|x_{n_i} - \varphi(x_{n_i})\| = 0$. According to Lemma 2.3, we deduce $z^* \in \mathrm{Fix}(\varphi)$, as φ is Lipschitz pseudocontractive.

Step 2. $z^* \in \mathrm{Ep}(C, \phi_1)$.

Applying Lemma 2.1 and Lemma 2.2, we can deduce that the sequences $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{\nu_n\}$, $\{Au_n\}$, $\{w_n\}$, $\{q_n\}$, $\{d_n\}$ and $\{\varsigma_n\}$ are all bounded. According to (47), we have

$$\lim_{i\to\infty} \phi_1(z_{n_i}, v_{n_i}) = \lim_{i\to\infty}(\iota_{n_i}\|\nu_{n_i}\|)\|\nu_{n_i}\| = 0. \tag{50}$$

By using the convexity of $\phi_1(z_{n_i}, \cdot)$, we have

$$\begin{aligned} 0 = \phi_1(z_{n_i}, z_{n_i}) &= \phi_1(z_{n_i}, (1 - \varpi_{n_i})v_{n_i} + \varpi_{n_i} y_{n_i}) \\ &\le (1 - \varpi_{n_i})\phi_1(z_{n_i}, v_{n_i}) + \varpi_{n_i}\phi_1(z_{n_i}, y_{n_i}). \end{aligned}$$

So, from (11), we obtain that

$$\phi_1(z_{n_i}, v_{n_i}) \ge \varpi_{n_i}(\phi_1(z_{n_i}, v_{n_i}) - \phi_1(z_{n_i}, y_{n_i})) \ge \frac{\vartheta_1 \varpi_{n_i}}{2\lambda_{n_i}}\|v_{n_i} - y_{n_i}\|^2,$$

which combines with (50) to get

$$\lim_{i\to\infty} \varpi_{n_i}\|v_{n_i} - y_{n_i}\|^2 = 0. \tag{51}$$

Applying the equivalent relation (5) to (10) we obtain

$$0 \in \partial_2\phi_1(v_{n_i}, y_{n_i}) + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) + N_C(y_{n_i}). \tag{52}$$

From relation (3) related to the normal cone N_C and inclusion (52), there exists $\hat{v}_{n_i} \in \partial_2\phi_1(v_{n_i}, y_{n_i})$ such that

$$\langle \hat{v}_{n_i}, y - y_{n_i}\rangle + \frac{1}{\lambda_{n_i}}\langle y_{n_i} - v_{n_i}, y - y_{n_i}\rangle \ge 0, \ \forall y \in C. \tag{53}$$

Owing to the fact that $\hat{v}_{n_i} \in \partial_2\phi_1(v_{n_i}, y_{n_i})$, by the subdifferential inequality (4), we get

$$\phi_1(v_{n_i}, y) - \phi_1(v_{n_i}, y_{n_i}) \ge \langle \hat{v}_{n_i}, y - y_{n_i}\rangle, \ \forall y \in C. \tag{54}$$

Combining (53) and (54) to conclude

$$\phi_1(v_{n_i}, y) - \phi_1(v_{n_i}, y_{n_i}) + \frac{1}{\lambda_{n_i}}\langle y_{n_i} - v_{n_i}, y - y_{n_i}\rangle \ge 0, \ \forall y \in C. \tag{55}$$

It follows that

$$\phi_1(v_{n_i}, y) - \phi_1(v_{n_i}, y_{n_i}) + \frac{1}{\lambda_{n_i}}\|y_{n_i} - v_{n_i}\|\|y - y_{n_i}\| \geq 0, \ \forall y \in C. \tag{56}$$

Next, we consider two cases:

Case 1: $\limsup_{i\to\infty} \varpi_{n_i} > 0$ and **Case 2:** $\lim_{i\to\infty} \varpi_{n_i} = 0$.

Case 1: In this situation, there is a subsequence of $\{\varpi_{n_i}\}$, still denoted by $\{\varpi_{n_i}\}$ such that $\varpi_{n_i} > a > 0$ when $i \geq N_1$ for some positive integer N_1. So, by (51), we deduce

$$\lim_{i\to\infty} \|v_{n_i} - y_{n_i}\| = 0, \tag{57}$$

which implies that $y_{n_i} \rightharpoonup z^*$. Letting $i \to \infty$ in (56), we have

$$\phi_1(z^*, y) \geq \phi_1(z^*, z^*) = 0, \ \forall y \in C,$$

which yields that $z^* \in \text{Ep}(C, \phi_1)$.

Case 2: Here, we may assume without loss of generality that $y_{n_i} \rightharpoonup z^\dagger$ as $i \to \infty$ because of the boundedness of $\{y_{n_i}\}$. Since $v_{n_i} \in C$, replacing y by v_{n_i} in (55), we get

$$\phi_1(v_{n_i}, y_{n_i}) \leq -\frac{1}{\lambda_{n_i}}\|y_{n_i} - v_{n_i}\|^2. \tag{58}$$

From (11), for $k_{n_i} - 1$, we have

$$\phi_1(z_{n_i,k_{n_i}-1}, v_{n_i}) - \phi_1(z_{n_i,k_{n_i}-1}, y_{n_i}) < \frac{\vartheta_1}{2\lambda_{n_i}}\|y_{n_i} - v_{n_i}\|^2. \tag{59}$$

Combining (58) and (59) to attain

$$\phi_1(v_{n_i}, y_{n_i}) \leq \frac{2}{\vartheta_1}(\phi_1(z_{n_i,k_{n_i}-1}, y_{n_i}) - \phi_1(z_{n_i,k_{n_i}-1}, v_{n_i})). \tag{60}$$

Since $v_{n_i} \rightharpoonup z^*$, $z_{n_i,k_{n_i}-1} \rightharpoonup z^*$ and $y_{n_i} \rightharpoonup z^\dagger$, letting $i \to \infty$ in (60), we have $\phi_1(z^*, z^\dagger) \leq \frac{2}{\vartheta_1}\phi_1(z^*, z^\dagger)$, which yields that $\phi_1(z^*, z^\dagger) \geq 0$. This together with relations (56) and (58) implies that $\lim_{i\to\infty} \|y_{n_i} - v_{n_i}\| = 0$. Therefore, $z^* \in \text{Ep}(C, \phi_1)$.

Step 3. $Az^* \in \text{Fix}(\psi)$. In fact, by (a4), we have $\lim_{i\to\infty} \|Au_{n_i} - \psi(Au_{n_i})\| = 0$. Since $Au_{n_i} \rightharpoonup Az^*$, utilizing Lemma 2.3, we conclude the desired result.

Step 4. $Az^* \in \text{Ep}(Q, \phi_2)$.

Since $\phi_2(d_{n_i}, P_Q(Au_{n_i})) = \mu_{n_i}\|\varsigma_{n_i}\|^2$, from (43), we have

$$\lim_{i\to\infty} \phi_2(d_{n_i}, P_Q(Au_{n_i})) = 0. \tag{61}$$

Thus, from the convexity of $\phi_2(d_{n_i}, \cdot)$, we receive

$$\begin{aligned} 0 = \phi_2(d_{n_i}, d_{n_i}) &= \phi_2(d_{n_i}, (1 - \delta_{n_i})P_Q(Au_{n_i}) + \delta_{n_i}w_{n_i}) \\ &\leq (1 - \delta_{n_i})\phi_2(d_{n_i}, P_Q(Au_{n_i}) + \delta_{n_i}\phi_2(d_{n_i}, w_{n_i}). \end{aligned}$$

This together with (14) leads to

$$\phi_2(d_{n_i}, P_Q(Au_{n_i}) \geq \delta_{n_i}(\phi_2(d_{n_i}, P_Q(Au_{n_i}) - \phi_2(d_{n_i}, w_{n_i})) \geq \frac{\vartheta_2}{2\tau_{n_i}}\delta_{n_i}\|P_Q(Au_{n_i}) - w_{n_i}\|^2.$$

Due to (61), we deduce

$$\lim_{i\to\infty} \delta_{n_i}\|P_Q(Au_{n_i}) - w_{n_i}\|^2 = 0. \tag{62}$$

Applying the equivalent relation (5) to (13) to acquire

$$0 \in \partial_2\phi_2(P_Q(Au_{n_i}), w_{n_i}) + \frac{1}{\tau_{n_i}}(w_{n_i} - P_Q(Au_{n_i}) + N_Q(w_{n_i}).$$

Consequently, in view of the definition of the normal cone $N_Q(w_{n_i})$, we have

$$\phi_2(P_Q(Au_{n_i}), x) - \phi_2(P_Q(Au_{n_i}), w_{n_i}) + \frac{1}{\tau_{n_i}}\langle w_{n_i} - P_Q(Au_{n_i}), x - w_{n_i}\rangle \geq 0,\ \forall x \in Q. \tag{63}$$

It follows that

$$\phi_2(P_Q(Au_{n_i}), x) - \phi_2(P_Q(Au_{n_i}), w_{n_i}) + \frac{1}{\tau_{n_i}}\|w_{n_i} - P_Q(Au_{n_i})\|\|x - w_{n_i}\| \geq 0,\ \forall x \in Q. \tag{64}$$

Next, we have to take into consideration two cases:

Case (i): $\limsup_{i\to\infty} \delta_{n_i} > 0$ and **Case (ii):** $\lim_{i\to\infty} \delta_{n_i} = 0$.

For Case (i), there exists a subsequence of $\{\delta_{n_i}\}$, still denoted by $\{\delta_{n_i}\}$ such that $\delta_{n_i} > b > 0$ when $i \geq N_2$ for some positive integer N_2. By virtue of (62), we obtain

$$\lim_{i\to\infty} \|P_Q(Au_{n_i}) - w_{n_i}\| = 0, \tag{65}$$

which results in $w_{n_i} \rightharpoonup Az^*$ because $Au_{n_i} \rightharpoonup Az^* \in Q$.

Taking the limit in (64) as $i \to \infty$, we have

$$\phi_2(Az^*, x) \geq \phi_2(Az^*, Az^*) = 0,\ \forall x \in Q,$$

which yields that $Az^* \in \mathrm{Ep}(Q, \phi_2)$.

In Case (ii), without loss of generality, we may assume that $w_{n_i} \rightharpoonup \hat{z}$ as $i \to \infty$. Replacing x by $P_Q(Au_{n_i})$ in (63), we get

$$\phi_2(P_Q(Au_{n_i}), w_{n_i}) \leq -\frac{1}{\tau_{n_i}}\|w_{n_i} - P_Q(Au_{n_i})\|^2. \tag{66}$$

By (14), for $l_{n_i} - 1$, we have

$$\phi_2(d_{n_i, l_{n_i}-1}, P_Q(Au_{n_i})) - \phi_2(d_{n_i, l_{n_i}-1}, w_{n_i}) < \frac{\vartheta_2}{2\tau_{n_i}}\|w_{n_i} - P_Q(Au_{n_i})\|^2. \tag{67}$$

Combining (66) and (67) to obtain

$$\phi_2(P_Q(Au_{n_i}), w_{n_i}) \leq \frac{2}{\vartheta_2}(\phi_2(d_{n_i, l_{n_i}-1}, w_{n_i}) - \phi_2(d_{n_i, l_{n_i}-1}, P_Q(Au_{n_i}))). \tag{68}$$

Taking the limit in (68) as $i \to \infty$, we obtain $\phi_2(Az^*, \hat{z}) \leq \frac{2}{\vartheta_2}\phi_2(Az^*, \hat{z})$. Therefore, $\phi_2(Az^*, \hat{z}) \geq 0$ and $\lim_{i\to\infty} \|w_{n_i} - P_Q(Au_{n_i})\| = 0$ by (66). Similarly, we can deduce that $Az^* \in \mathrm{Ep}(Q, \phi_2)$.

To this end, we have $z^* \in \mathrm{Fix}(\varphi) \cap \mathrm{Ep}(C, \phi_1)$ and $Az^* \in \mathrm{Fix}(\psi) \cap \mathrm{Ep}(Q, \phi_2)$, i.e., $z^* \in \Gamma$ and so $\omega_w(x_n) \subset \Gamma$. □

Theorem 3.8. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to some point in Γ.*

Proof. We have proved that $\lim_{n\to\infty} \|x_n - x^*\| (\forall x^* \in \Gamma)$ exists and $\omega_w(x_n) \subset \Gamma$. Finally, using Lemma 2.4, we can conclude that $\{x_n\}$ converges weakly to some point in Γ. This completes the proof. □

According to Algorithm 3.1 and Theorem 3.8, we obtain the following algorithms and corollaries, by taking precise values for ϕ_1, ϕ_2, φ, and ψ.

Algorithm 3.9. *Let $x_0 \in C$ be an initial guess.*

Step 1. Let x_n be given. Calculate

$$v_n = (1 - \frac{\rho_n}{2})x_n + \frac{\rho_n}{2}\varphi(p_n),$$

where

$$p_n = (1 - \rho_n)x_n + \rho_n\varphi(x_n),$$

in which $\rho_n = \eta\beta_1^i$ *and* $i = \min\{0, 1, 2, \cdots\}$ *such that*

$$\rho_n\|\varphi(p_n) - \varphi(x_n)\| \leq \sigma_1\|p_n - x_n\|.$$

Step 2. Compute

$$q_n = (1 - \frac{\varrho_n}{2})v_n + \frac{\varrho_n}{2}\psi(r_n),$$

where

$$r_n = (1 - \varrho_n)v_n + \varrho_n\psi(v_n),$$

in which $\varrho_n = \zeta\beta_2^j$ *and* $j = \min\{0, 1, 2, \cdots\}$ *such that*

$$\varrho_n\|\psi(r_n) - \psi(v_n)\| \leq \sigma_2\|r_n - v_n\|.$$

Step 3. Compute

$$x_{n+1} = P_C(x_n + \alpha A^*(q_n - Ax_n)),$$

and set $n := n + 1$ *and return to Step 1.*

Corollary 3.10. *Suppose that* $\Gamma_1 := \{x^* \in \mathrm{Fix}(\varphi), Ax^* \in \mathrm{Fix}(\psi)\} \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 3.9 converges weakly to some point in* Γ_1.

Algorithm 3.11. *Let* $x_0 \in C$ *be an initial guess.*

Step 1. Let x_n *be given. Compute*

$$y_n = \arg\min_{u\in C}\Big\{\phi_1(x_n, u) + \frac{1}{2\lambda_n}\|x_n - u\|^2\Big\}.$$

If $y_n = x_n$, *then set* $u_n = x_n$ *and go to Step 4. Otherwise, continue to the next step.*

Step 2. Set $z_{n,k} = (1 - \varpi^k)x_n + \varpi^k y_n$, *where* $k = \min\{1, 2, \cdots\}$ *such that*

$$2\lambda_n(\phi_1(z_n, x_n) - \phi_1(z_n, y_n)) \geq \vartheta_1\|x_n - y_n\|^2$$

Write $\varpi_n = \varpi^k$ *and* $z_n = z_{n,k}$, *i.e.,* $z_n = (1 - \varpi_n)x_n + \varpi_n y_n$.

Step 3. Compute

$$u_n = P_C(x_n - \gamma_n\iota_n\nu_n),$$

where $\nu_n \in \partial_2\phi_1(z_n, x_n)$ *and* $\iota_n = \dfrac{\phi_1(z_n, x_n)}{\|\nu_n\|^2}$.

Step 4. Compute

$$w_n = \arg\min_{v\in Q}\Big\{\phi_2(P_Q(Au_n), v) + \frac{1}{2\tau_n}\|P_Q(Au_n) - v\|^2\Big\}.$$

If $w_n = P_Q(Au_n)$, *then set* $t_n = w_n$ *and go to Step 7. Otherwise, continue to the next step.*

Step 5. Set $d_{n,l} = (1 - \delta^l)P_Q(Au_n) + \delta^l w_n$, *where* $l = \min\{1, 2, \cdots\}$ *such that*

$$2\tau_n(\phi_2(d_n, P_Q(Au_n)) - \phi_2(d_n, w_n)) \geq \vartheta_2\|P_Q(Au_n) - w_n\|^2.$$

Write $\delta_n = \delta^l$ *and* $d_n = d_{n,l}$, *i.e.,* $d_n = (1 - \delta_n)P_Q(Au_n) + \delta_n w_n$.

Step 6. Compute

$$t_n = P_Q(P_Q(Au_n) - \epsilon_n\mu_n\varsigma_n),$$

where $\varsigma_n \in \partial_2\phi_2(d_n, P_Q(Au_n))$ *and* $\mu_n = \dfrac{\phi_2(d_n, P_Q(Au_n))}{\|\varsigma_n\|^2}$.

Step 7. Compute

$$x_{n+1} = P_C(u_n + \alpha A^*(t_n - Au_n)),$$

and set $n := n + 1$ *and return to Step 1.*

Corollary 3.12. *Suppose that* $\Gamma_2 := \{x^* \in \mathrm{Ep}(C, \phi_1), Ax^* \in \mathrm{Ep}(Q, \phi_2)\} \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 3.11 converges weakly to some point in* Γ_2.

4. Conclusions

The paper presents a study on split pseudomonotone equilibrium problems along sets of fixed points of two nonlinear operators with pseudocontractive properties. The results are obtained without the use of the size of the Lipschits constants of the two operators. A self-adaptive algorithm is designed to solve this problem, and its convergence is proved. Some known results in literature are also obtained as consequences of these outcomes.

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