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Non-commutative and polynomial multidimensional stronger central sets theorem

Sujan Pal[a]

[a]*Department of Mathematics, University of Kalyani, Kalyani, Nadia-741235, West Bengal, India*

Abstract. The Central Sets Theorem and Milliken-Taylor Theorem extends the famous Hindman's theorem in two different directions. In [1] Beiglboeck proved a joint extension of these two theorems. In this article, we prove the non-commutative version of Beiglboeck's result. Then we also prove the polynomial extension of stronger Central Sets Theorem for general commutative groups, which extents a recent result of [6].

1. Introduction

Van der Waerden's Theorem [13] states that for any partition of the positive integers $\mathbb{N}$ one of the cells of the partition contains arbitrarily length arithmetic progressions. By $P_f(X)$ we denote the set of all finite non-empty subsets of X. For a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ we set

$$\mathrm{FS}(x_n) = \left\{ \sum_{t\in\alpha} x_t : \alpha \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

Hindman's Theorem [8, Theorem 3.1] states that for any partition of the positive integers $\mathbb{N}$ one of the cells contains all possible finite sums of some sequence. The Central Sets Theorem provided a joint extension of this two theorems. To state the Central Sets Theorem let us formulate some notations. A set A is called an IP-set if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\mathrm{FS}(x_n) \subset A$. (This definitions makes perfect sense in any semigroup $(S, \cdot)$ and we use it in this context. FS is an abbreviation of finite sums and will be replaced by FP if we use multiplicative notation for the semigroup operation).

To state about the Central Sets Theorem, let us introduce some algebraic preliminaries on Stone-Čech compactification.

Let $(S, \cdot)$ be a discrete semigroup and βS be the Stone-Čech compactification of the discrete semigroup S and $\cdot$ on βS (which we represent by the same symbol on S) is the extension of $\cdot$ on S. The points of βS are ultrafilters and principal ultrafilters are identified by the points of S. The extension is unique extension for which $(\beta S, \cdot)$ is compact, right topological semigroup with S contained in its topological center. That is,

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The author acknowledge the Grant CSIR-UGC NET fellowship with file No. 09/106(0199)/2019-EMR-I.
Email address: sujan2016pal@gmail.com (Sujan Pal)
ORCID iD: https://orcid.org/0000-0001-7784-6600 (Sujan Pal)

for all $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and for all $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\lambda_x(q) = x \cdot q$. For $p, q \in \beta S$, $p \cdot q = \left\{A \subseteq S : \left\{x \in S : x^{-1}A \in q\right\} \in p\right\}$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

There is a famous theorem due to Ellis [10, Corollary 2.39] that if S is a compact right topological semigroup then the set of idempotents $E(S) \neq \emptyset$.

A non-empty subset I of a semigroup T is called a left ideal of S if $TI \subset I$, where $TI = \{ts : t \in T \text{ and } s \in I\}$, a right ideal if $IT \subset I$, where $IT = \{st : t \in T \text{ and } s \in I\}$ and a two sided ideal (or simply an ideal) if it is both a left and right ideal. A minimal left ideal is the left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideal.

Any compact Hausdorff right topological semigroup T has the smallest two sided ideal,

$$\begin{aligned} K(T) &= \bigcup \{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup \{R : R \text{ is a minimal right ideal of } T\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal $K(T)$ of T. See [10] for an elementary introduction to the algebra of βS and for any unfamiliar details.

Definition 1.1. *Let S be a discrete semigroup and let C be a subset of S. Then C is central if there is an idempotent p in $K(\beta S)$ such that $C \in p$.*

The notion of central sets originally introduced by Furstenberg [5] using topological dynamics and latter by Bergelson and Hindman proved this notation to be equivalent with algebraic definition using algebra of $\beta\mathbb{N}$.

Theorem 1.2. *(Original Central Sets Theorem) Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, ..., l\}$, let $(y_{i,n})_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Let C be a central subset of $\mathbb{N}$. Then there exist sequences $(a_n)_{n=1}^{\infty}$ in $\mathbb{N}$ and $(H_n)_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*
(1) *for all n, $\max H_n < \min H_{n+1}$ and*
(2) *for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, ..., l\}$,*

$$\sum_{n \in F} \left(a_n + \sum_{t \in H_n} y_{i,t} \right) \in C.$$

K. Milliken and A. Taylor ([11],[12]) found a quite natural common extension of the Theorems of Hindman and Ramsey: For a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ and $k \geq 1$ put

$$[\mathrm{FS}(x_n)]^k_< := \left\{ \left\{ \sum_{t \in \alpha_1} x_t, \ldots, \sum_{t \in \alpha_k} x_t \right\} : \alpha_1 < \ldots < \alpha_k \in \mathcal{P}_f(\mathbb{N}) \right\}$$

$\alpha < \beta$ for $\alpha, \beta \in \mathcal{P}_f(\mathbb{N})$ if and only if $\max \alpha < \min \beta$.

Let $[S]^k$ denote the k-element subsets of a set S. The Milliken-Taylor theorem says that for any finite partition $[\mathbb{N}]^k = \cup_{i=1}^{r} A_i$, there exist some $i \in \{1, 2, \ldots, r\}$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that $[\mathrm{FS}(x_n)]^k_< \subseteq A_i$.

Define $\Phi = \left\{ f \in \mathbb{N}^{\mathbb{N}} : \text{ for each } n \in \mathbb{N}, f(n) \leq n \right\}$.

As a combine extension of Milliken-Taylor Theorem and the Central Sets Theorem, Beiglboeck established the following result.

Theorem 1.3. *[1, Theorem 1.4] Let $(S, \cdot)$ be a commutative semigroup and assume that there exists a non-principal minimal idempotent in βS. For each $l \in \mathbb{N}$, let $\langle y_{l,n} \rangle_{n=0}^{\infty}$ be a sequence in S. Let $k, r \geq 1$ and let $[S]^k = \cup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \ldots, r\}$, a sequence $(x_n)_{n=1}^{\infty}$ in S and a sequence $\alpha_0 < \alpha_1 < \ldots$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $g \in \Phi$,*

$$\left[FP\left(\left\langle x_n \prod_{t \in \alpha_n} y_{g(n),t} \right\rangle_{n=0}^{\infty} \right) \right]^k_< \subseteq A_i.$$

There are several extensions of the Central Sets Theorem [2], [3], [4], [7], [9], [6] in the literature. In [9] authors established a version of the Central Sets Theorem for commutative semigroups considering countable infinitely many sequences at a time and in [2] authors established the non-commutative version of the Central Sets Theorem. We state the Theorem from [10].

For our purpose let us introduce the following notations. In a non-commutative semigroup, by $\prod_{t\in F} x_t$ we mean the product taken in increasing order of indices. In the following $\mathcal{T}$ is the collection of all sequences in S.

Definition 1.4. *Let $m \in \mathbb{N}$. Then we denote*

$$I_m = \left\{(H_1, H_2, \ldots, H_m) \in \mathcal{P}_f(\mathbb{N})^m : H_t < H_{t+1}, t \in \{1, 2, \ldots, m-1\}, m > 1\right\}, \text{ and}$$

$$J_m = \{(t(1), t(2), \ldots, t(m)) \in \mathbb{N}^m : t(1) < t(2) < \ldots < t(m)\}.$$

Let $(S, \cdot)$ be a semigroup.

(a) Given $n \in \mathbb{N}$, $a \in S^{m+1}$, $t \in J_m$ and $f \in \mathcal{T}$

$$x(m, a, t, f) = \left(\prod_{j=1}^{m} a(j) \cdot f(t(j))\right) \cdot a(m+1).$$

(b) $A \subset S$ is a J-set if and only if for each $F \in \mathcal{P}_f(\mathcal{T})$ there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in J_m$ such that for each $f \in F$, $x(m, a, t, f) \in A$.

(c) $J(S) = \{p \in \beta S :$ for all $A \in p$, then A is a J-set$\}$.

(d) $A \subset S$ is a C-set if and only if there exist

$$m : \mathcal{P}_f(\mathcal{T}) \to \mathbb{N}, \qquad \alpha \in \times_{F\in\mathcal{P}_f(\mathcal{T})} S^{m(F)+1}, \text{ and } \tau \in \times_{F\in\mathcal{P}_f(\mathcal{T})} I_{m(F)}$$

such that

(i) if $F \subsetneq G$ in $\mathcal{P}_f(\mathcal{T})$ then $\tau(F)(m(F)) < \tau(G)(1)$ and

(ii) for any $n \in \mathbb{N}$, if $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$ in $\mathcal{P}_f(\mathcal{T})$ and for each $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$ then

$$\prod_{i=1}^{n} x(m(G_i), \alpha(G_i), \tau(G_i), f_i) \in A.$$

Theorem 1.5. *[10, Theorem 14.15] Let S be a semigroup, let A be a central subset of S, and for each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=1}^{\infty}$ be a sequence in S. Given $l, m \in \mathbb{N}$, $a \in S^{m+1}$ and let $H \in I_m$, let*

$$w(a, H, l) = \left(\left(\prod_{i=1}^{m} a(i) \cdot \prod_{t\in H(t)} y_{l,t}\right)\right) \cdot a(m+1).$$

There exist sequences $\langle m(n)\rangle_{n=1}^{\infty}$, $\langle a_n\rangle_{n=1}^{\infty}$ and $\langle H_n\rangle_{n=1}^{\infty}$ such that

1. *for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$, $a_n \in S^{m(n)+1}$ and $H_n \in I_{m(n)}$ and $\max H_{n,m(n)} < \min H_{n+1,1}$ and*
2. *for each $f \in \Phi$, $FP\left(\langle w(a_n, H_n, f(n))\rangle_{n=1}^{\infty}\right) \subseteq A$.*

In [3] author established the polynomial version of the original Central Sets Theorem for commutative semigroups. In [4] authors gave a version of the Central Sets Theorem to stronger form using arbitrary many sequences at a time known as new and Stronger Central Sets Theorem.

Theorem 1.6. *[4, Theorem 2.2](Stronger Central Sets Theorem for commutative semigroup) Let $(S, +)$ be a commutative semigroup and let C be a central subset of S. Then there exist functions $\alpha : \mathcal{P}_f\left(S^{\mathbb{N}}\right) \to S$ and $H : \mathcal{P}_f\left(S^{\mathbb{N}}\right) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *if* $F, G \in \mathcal{P}_f\left(S^{\mathbb{N}}\right)$ *and* $F \subsetneq G$ *then* $\max H(F) < \min H(G)$ *and*
(2) *if* $m \in \mathbb{N}, G_1, G_2, \ldots, G_m \in \mathcal{P}_f\left(S^{\mathbb{N}}\right)$; $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$; *and for each* $i \in \{1, 2, \ldots, m\}, (y_{i,n})_{n=1}^{\infty} \in G_i$, *then*

$$\sum_{i=1}^{m}\left(\alpha(G_i) + \sum_{t \in H(G_i)} y_{i,t}\right) \in C.$$

In the same article the authors proved the non-commutative extension of the above theorem, as well as they found an example of a non-commutative semigroup and a set inside that semigroup which witness the non-commutative original Central Sets Theorem but it does not witness the stronger one.

Theorem 1.7. *[4, Corollary 3.10](Stronger Central Sets Theorem for non-commutative semigroup) Let* $(S, \cdot)$ *be a semigroup and let* C *be a central subset of* S. *Then there exist functions* $m : \mathcal{P}_f(\mathcal{T}) \to \mathbb{N}$, $\alpha \in \times_{F \in \mathcal{P}_f(\mathcal{T})} S^{m(F)+1}$ *and* $\tau \in \times_{F \in \mathcal{P}_f(\mathcal{T})} I_{m(F)}$ *such that*

1. *if* $F \subsetneq G$ *in* $\mathcal{P}_f(\mathcal{T})$ *then* $\tau(F)(m(F)) < \tau(G)(1)$ *and*
2. *for any* $n \in \mathbb{N}$, *if* $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$ *in* $\mathcal{P}_f(\mathcal{T})$ *and for each* $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$ *then*

$$\prod_{i=1}^{n} x\left(m(G_i), \alpha(G_i), \tau(G_i), \langle y_{i,t}\rangle_{t=1}^{\infty}\right) \in A.$$

In the second section of the article we shall establish multidimensional version of this Theorem. In the other direction in [6] authors established polynomial version of the above Theorem. The following theorem is the polynomial extension of the Theorem 1.6.

Theorem 1.8. *(Polynomial Stronger Central Sets Theorem) Let* $A \subseteq \mathbb{N}$ *be a central set and* $T \subseteq \mathcal{P}_f(\mathbb{P})$, $\mathbb{P}$ *is the set of polynomials from* $\mathbb{N}$ *to* $\mathbb{N}$ *vanishes as* 0. *Then there exist functions* $\alpha : \mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right) \to S$ *and* $H : \mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right) \to \mathcal{P}_f(\mathbb{N})$ *such that*
(1) *if* $F, G \in \mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)$ *and* $F \subsetneq G$ *then* $\max H(F) < \min H(G)$ *and*
(2) *if* $m \in \mathbb{N}, G_1, G_2, \ldots, G_m \in \mathcal{P}_f\left(\mathbb{N}^{\mathbb{N}}\right)$; $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$; *and for each* $i \in \{1, 2, \ldots, m\}, f_i \in G_i$, *for all* $P \in T$,

$$\sum_{i=1}^{m} \alpha(G_i) + P\left(\sum_{i=1}^{m} \sum_{t \in H(G_i)} f_i(t)\right) \in A.$$

In our work we will extend 1.3 in the context of non-commutative semigroup. In the third section of the article we shall establish multidimensional version of Theorem 1.8.

2. Non-commutative multidimensional Central Sets Theorem

Let us recall the following technical lemma from [1].

Lemma 2.1. *[1, Lemma 3.1] Let* S *be a set, let* $e \in \beta S \setminus S$, *let* $k, r \geq 1$ *and let* $[S]^k = \cup_{i=1}^{r} A_i$. *For each* $i \in \{1, 2, \ldots, r\}$, *each* $t \in \{1, 2, \ldots, k\}$ *and each* $E \in [S]^{t-1}$, *define* $B_t(E, i)$ *by downward induction on* t:

1. *for* $E \in [S]^{k-1}$, $B_k(E, i) := \{y \in S \setminus E : E \cup \{y\} \in A_i\}$.
2. *for* $1 \leq t < k$ *and* $E \in [S]^{t-1}$,

$$B_t(E, i) := \{y \in S \setminus E : B_{t+1}(E \cup \{y\}, i) \in e\}.$$

Then there exists some $i \in \{1, 2, \ldots, r\}$ *such that* $B_1(\emptyset, i) \in e$.

We need to introduce a few notations before introducing a key lemma. Let S be a set. A set of ordered tuples T of S is said to be a tree if all of its initial segments belong to T. More formally let by $S^{<\omega}$, we mean collection of all functions from $\{0,\ldots,n-1\}$ to S, where $n\in\mathbb{N}$. Now a non empty set $T\subseteq S^{<\omega}$ is a tree in S if $f\in T$ if and only if for any $A=\{0,1,\ldots,m\}\subseteq dom f$, $f\mid_A\in T$.

A function $f\in S^{\{0,\ldots,n-1\}}$ can be identified by $\langle f(0),f(1),\ldots,f(n-1)\rangle$.

If $s\in S$, then $f^\frown s:=\langle f(0),f(1),\ldots,f(n-1),s\rangle$. For $f\in S^{<\omega}$, we put $T(f)=\{s\in S: f^\frown s\in T\}$.

The following is an important lemma for our purpose.

Lemma 2.2. *Let $(S,\cdot)$ be a semigroup such that there exists an idempotent $e\in\beta S\backslash S$, let $k,r\geq 1$ and let $[S]^k=\bigcup_{i=1}^r A_i$. Then there exist $i\in\{1,2,\ldots,r\}$ and a tree $T\subseteq S^{<\omega}$ such that for all $f\in T$, and $\alpha_1<\alpha_2<\ldots<\alpha_k\subseteq dom f$, $\alpha_j\in\mathcal{P}_f(\omega)$, $j\in\{1,2,\ldots,k\}$ one has:*

1. $T(f)\in e$.
2. $\left\{\prod_{t\in\alpha_1}f(t),\prod_{t\in\alpha_2}f(t),\ldots,\prod_{t\in\alpha_k}f(t)\right\}\in A_i$.

Proof. [1, Lemma 3.2]. □

The following theorem is non-commutative extension of Beiglboeck theorem. In fact, we will be using the same techniques for induction.

Theorem 2.3. *Let $(S,\cdot)$ be a commutative semigroup and assume that there exists a non-principal minimal idempotent in βS. Let $k,r\geq 1$ be integers and let $[S]^k=\cup_{l=1}^r A_l$. There exist $l\in\{1,2,\ldots,r\}$ and functions*

$$m:\mathcal{P}_f(\mathcal{T})\to\mathbb{N},\qquad \alpha\in\times_{F\in\mathcal{P}_f(\mathcal{T})}S^{m(F)+1},\text{ and }\tau\in\times_{F\in\mathcal{P}_f(\mathcal{T})}I_{m(F)}$$

such that

1. *if $F\subsetneq G$ in $\mathcal{T}$, then $\tau(F)(m(F))<\tau(G)(1)$ and*
2. *for any sequence $G_1\subsetneq G_2\subsetneq\ldots\subsetneq G_n$ in $\mathcal{P}_f(\mathcal{T})$, $f_i\in G_i$ and $\alpha_1<\alpha_2<\cdots<\alpha_k\leq\{n\}$ in $\mathcal{P}_f(\mathbb{N})$, we have*

$$\left(\prod_{i\in\alpha_1}x(m(G_i),\alpha(G_i),\tau(G_i),f_i),\ldots,\prod_{i\in\alpha_k}x(m(G_i),\alpha(G_i),\tau(G_i),f_i)\right)\in A_l.$$

Proof. Fix a minimal idempotent $e\in\beta S\setminus S$. Let $i\in\{1,2,\ldots,r\}$ and $T\subseteq S^{<\omega}$ be a tree as provided by lemma 2.2. We shall inductively construct sequences $G_1\subsetneq G_2\subsetneq\ldots\subsetneq G_n$ in $\mathcal{P}_f(\mathcal{T})$ and $\alpha_1<\alpha_2<\cdots<\alpha_k\leq\{n\}$. Assume we have constructed $G_1\subsetneq G_2\subsetneq\ldots\subsetneq G_{n-1}$ and $\alpha_1<\alpha_2<\cdots<\alpha_{k-1}<\{n\}$ such that

$$\left\langle\prod_{i\in\alpha_1}x(m(G_i),\alpha(G_i),\tau(G_i),f_i),\ldots,\prod_{i\in\alpha_{k-1}}x(m(G_i),\alpha(G_i),\tau(G_i),f_i)\right\rangle\in T$$

By lemma 2.2 we have

$$G=T\left(\left\langle\prod_{i\in\alpha_1}x(m(G_i),\alpha(G_i),\tau(G_i),f_i),\ldots,\prod_{i\in\alpha_{k-1}}x(m(G_i),\alpha(G_i),\tau(G_i),f_i)\right\rangle\right)\in e.$$

Then G is a central set in S. Then we have $f_i\in G_i$, $\prod_{i\in\alpha_k}x(m(G_i),\alpha(G_i),\tau(G_i),f_i)\in G$, where $\alpha_{k-1}<\alpha_k\leq\{n\}$. Then

$$\left\langle\prod_{i\in\alpha_1}x(m(G_i),\alpha(G_i),\tau(G_i),f_i),\ldots,\prod_{i\in\alpha_k}x(m(G_i),\alpha(G_i),\tau(G_i),f_i)\right\rangle\in T.$$

using lemma 2.2 we get our desire result. □

3. Multidimensional polynomial Central Sets Theorem

In [6] authors proved a polynomial generalization of Stronger Central Sets Theorem followed by polynomial generalization of the Central Sets Theorem in [3]. In [3] authors proved the polynomial version of the Central sets Theorem for countable commutative group taking finitely many sequences at a time. But in [6] authors deal with semigroup $(\mathbb{N},+)$ taking all sequences at a time. Almost verbatim we get the polynomial generalization of Stronger Central Sets Theorem for countable commutative group. First let us recall the definition of polynomial for countable commutative group.

Definition 3.1. *A map $f : G \to H$ between countable commutative groups we say that f is a polynomial map of degree 0 if it is constant. We say that f is a polynomial map of degree d, $d \in \mathbb{N}$, if it is not a polynomial map of degree $d-1$ and for every $h \in H$, the map $x \to f(x+h) - f(x)$ is a polynomial of degree $\leq d-1$. Finally we denote by $\mathbb{P}(G,H)$ the set of all polynomial maps $f : G \to H$ with $f(0) = 0$. Note that homomorphisms are elements of $\mathbb{P}(G,H)$ having degree 1.*

Theorem 3.2. *(Polynomial CST) Let $(S,+)$ be a countable commutative group and $F \in \mathcal{P}_f\left(\left(S^j\right)^{\mathbb{N}}\right)$, let $T \in \mathcal{P}_f(\mathbb{P})$, where $\mathbb{P}$ is set of all polynomials from S^j to S vanishes at zero and let A be a central subset of S. Then there exist sequences $\langle b_n\rangle_{n=1}^{\infty}$ in S and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

1. *for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and*
2. *for each $f \in F$, each $P \in T$ and each $K \in \mathcal{P}_f(\mathbb{N})$*

$$\sum_{n\in K} b_n + P\left(\sum_{n\in K}\sum_{t\in H_n} f(t)\right) \in A.$$

In [6] authors established a stronger version of the above theorem in $(\mathbb{N},+)$ which we extend for countable commutative group. First we need the following lemma.

Lemma 3.3. *Let $j,m \in \mathbb{N}$, let S be a countable commutative group and let $\mathbb{P}$ be a set of all polynomials from S^j to S vanishes at $\mathbf{0}$. Let $A \subset S$ be a central set in S and $F \in \mathcal{P}_f\left(\left(S^j\right)^{\mathbb{N}}\right)$. Then for every $T \in \mathcal{P}_f(\mathbb{P})$ there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $a + P\left(\sum_{t\in H} f(t)\right) \in A$, for all $f \in F$ and $P \in T$.*

Proof. [3, Corollary 2.12]. □

The lemma has a following stronger version.

Lemma 3.4. *Let $j,m \in \mathbb{N}$, let S be a countable commutative group and let $\mathbb{P}$ be a set of all polynomials from S^j to S vanishes at $\mathbf{0}$. Let $A \subset S$ be a central set in S and $F \in \mathcal{P}_f\left(\left(S^j\right)^{\mathbb{N}}\right)$. Then for every $T \in \mathcal{P}_f(\mathbb{P})$ there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H > m$ such that $a + P\left(\sum_{t\in H} f(t)\right) \in A$, for all $f \in F$ and $P \in T$.*

Proof. Let $j,m \in \mathbb{N}$, $T \in \mathcal{P}_f(\mathbb{P})$ and $F \in \mathcal{P}_f\left(\left(S^j\right)^{\mathbb{N}}\right)$, for each $f \in F$ define $g_f \in \left(S^j\right)^{\mathbb{N}}$ by $g_f(t) = f(t+m)$, $t \in \mathbb{N}$. For this $K = \left\{g_f : f \in F\right\} \in \mathcal{P}_f\left(\left(S^j\right)^{\mathbb{N}}\right)$, there exist $a \in S$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $a + P\left(\sum_{t\in L} g_f(t)\right) \in A$, for all $f \in F$ and $P \in T$. Therefore $a + P\left(\sum_{t\in L} f(t+m)\right) \in A$, for all $f \in F$ and $P \in T$, i.e. $a + P\left(\sum_{t\in H} f(t)\right) \in A$, for all $f \in F$ and $P \in T$, where $H = L + m > m$. □

In the following $\mathcal{T}_j$ is the collection of all sequences in S^j.

Theorem 3.5. *Let $(S,\cdot)$ be a countable commutative semigroup, let $T \in \mathcal{P}_f(\mathbb{P})$, where $\mathbb{P}$ be a set of all polynomials from S^j to S vanishes at $\mathbf{0}$. Let $A \subset S$ be a central set. Then there exist functions $\alpha : \mathcal{P}_f\left(\mathcal{T}_j\right) \to S$, $H : \mathcal{P}_f\left(\mathcal{T}_j\right) \to \mathcal{P}_f(\mathbb{N})$, such that*

1. *if $\emptyset \neq F \subsetneq G$ then $H(F) < H(G)$ and*

2. *for any* $n \in \mathbb{N}$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$ *in* $\mathcal{P}_f\left(\mathcal{T}_j\right)$, *we have for each* $i \in \{1, 2, ..., n\}$, $f_i \in G_i$, *and for all* $P \in T$,

$$\sum_{i=1}^{n} \alpha(G_i) + P\left(\sum_{i=1}^{n} \sum_{t \in H(G_i)} f_i(t)\right) \in A.$$

Proof. Choose a minimal idempotent $p \in \beta S$ with $A \in p$. For $F \in \mathcal{P}_f\left(\mathcal{T}_j\right)$, we shall use induction on cardinality of F , define $\alpha(F) \in S$ and $H(F) \in \mathcal{P}_f(\mathbb{N})$ for witnessing $(1, 2)$.

If $F = \{f\}$, as p is minimal idempotent, the set $A^* = \{x \in A : -x + A \in p\}$ belongs to p [10, Corollary 4.14]. Hence A^* is a central set. So by [3, Corollary 2.12], there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that

$$\forall P \in T,\ a + P\left(\sum_{t \in H} f(t)\right) \in A^*.$$

By setting $\alpha(\{f\}) = a$ and $H(\{f\}) = H$, conditions (1) and (2) are satisfied.

Now assume that $| F |> 1$, $\alpha(G)$ and $H(G)$ have been defined for all proper subsets G of F. Let $K = \bigcup\{H(G) : \emptyset \neq G \subsetneq F\} \in \mathcal{P}_f(\mathbb{N})$, $m = \max K$ and

Let

$$R = \left\{ \begin{array}{c} \sum_{i=1}^{n} \sum_{t \in H(G_i)} f_i(t) \mid n \in \mathbb{N} \\ \emptyset \neq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq F, \\ f_i \in G_i, \forall i = 1, 2, .., n. \end{array} \right\}$$

$$M = \left\{ \begin{array}{c} \sum_{i=1}^{n} \alpha(G_i) + P\left(\sum_{i=1}^{n} \sum_{t \in H(G_i)} f_i(t)\right) \mid n \in \mathbb{N} \\ \emptyset \neq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n \subsetneq F, \\ f_i \in G_i, \forall i = 1, 2, .., n,\ P \in T. \end{array} \right\}$$

Then R is a finite subset of S^j, M is a finite subset of S and by induction hypothesis , $M \subseteq A^*$.

Let

$$B = A^* \cap \left(\bigcap_{x \in M} (-x + A^*)\right) \in p.$$

For $P \in T$ and $d \in R$, let us define the polynomial $Q_{p,d} \in \mathbb{P}\left(S^j, S\right)$ by

$$Q_{p,d}(y) = P(y + d) - P(d).$$

Degree of $Q_{p,d}$ is one degree lesser than P.

Let $D = T \cup \left\{Q_{p,d} \mid P \in T \text{ and } d \in R\right\}$.

From lemma 3.4, there exist $\gamma \in \mathcal{P}_f(\mathbb{N})$ with $\min(\gamma) > m$ and $a \in S$ such that

$$\forall Q \in D, f \in F\ a + Q\left(\sum_{t \in \gamma} f(t)\right) \in B.$$

We set $\alpha(F) = a$ and $H(F) = \gamma$. Now we verify conditions (1) and (2).

Since $\min(\gamma) > m$, (1) is satisfied.

To verify (2), let $n \in \mathbb{N}$ and $G_1, G_2, ..., G_n \in \mathcal{P}_f\left(\mathcal{T}_j\right)$, $G_1 \subsetneq G_2 \subsetneq ..... \subsetneq G_n = F$ and $f_i \in G_i, i = 1, 2, ..., n$.

For $n = 1$, $G_1 = G_n = F$, $\alpha(G_n) + P\left(\sum_{t \in H(G_n)} f_n(t)\right) = a + P\left(\sum_{t \in \gamma} f(t)\right) \in B \subseteq A^*$ for all $P \in T \subseteq D$, and $f \in F$.

If $n > 1$, then

$$C_A = \sum_{i=1}^{n} \alpha(G_i) + P\left(\sum_{i=1}^{n} \sum_{t \in H(G_i)} f_i(t)\right)$$

$$= \alpha(G_n) + \sum_{i=1}^{n-1} \alpha(G_i) + P\left(\sum_{t\in H(G_n)} f_i(t) + \sum_{i=1}^{n-1}\sum_{t\in H(G_i)} f_i(t)\right)$$

$$= a + \sum_{i=1}^{n-1} \alpha(G_i) + P\left(\sum_{t\in\gamma} f_n(t) + \sum_{i=1}^{n-1}\sum_{t\in H(G_i)} f_i(t)\right)$$

Since $G_n = F$ and $\alpha(F) = a, H(F) = \gamma$.

$$C_A = a + \sum_{i=1}^{n-1} \alpha(G_i) + P\left(\sum_{i=1}^{n-1}\sum_{t\in H(G_i)} f_i(t)\right) + P\left(\sum_{t\in\gamma} f_n(t) + \sum_{i=1}^{n-1}\sum_{t\in H(G_i)} f_i(t)\right) - P\left(\sum_{i=1}^{n-1}\sum_{t\in H(G_i)} f_i(t)\right)$$

$$= a + y + Q_{p,d}\left(\sum_{t\in\gamma} f_n(t)\right),$$

where $y = \sum_{i=1}^{n-1} \alpha(G_i) + P\left(\sum_{i=1}^{n-1}\sum_{t\in H(G_i)} f_i(t)\right) \in M$, $d = \sum_{i=1}^{n-1}\sum_{t\in H(G_i)} f_i(t) \in R$ and $P \in T$ so $Q_{p,d} \in D$. So we have

$$a + Q_{p,d}\left(\sum_{t\in\gamma} f_n(t)\right) \in B \subseteq -y + A^*.$$

Therefore

$$\sum_{i=1}^{n} \alpha(G_i) + P\left(\sum_{i=1}^{n}\sum_{t\in H(G_i)} f_i(t)\right) \in A^*.$$

This completes the induction argument, hence the proof. □

Now we will extend Beiglboeck Multidimensional Central Sets Theorem for countable commutative group and polynomial version of Stronger Central Sets Theorem.

Theorem 3.6. *Let $(S, \cdot)$ be a countable commutative group and assume that there exists a non principal minimal idempotent in βS. Let $k, r \in \mathbb{N}$ and let $[S]^k = \cup_{l=1}^{r} A_l$. There exist $l \in \{1, 2, \ldots, r\}$ and functions $\alpha : \mathcal{P}_f(\mathcal{T}_j) \to S$, $H : \mathcal{P}_f(\mathcal{T}_j) \to \mathcal{P}_f(\mathbb{N})$ such that*

1. *if $\emptyset \neq F \subsetneq G$ then $H(F) < H(G)$*
2. *for $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_{n_k}$ in $\mathcal{P}_f(\mathcal{T}_j)$, for every $\beta_1 < \beta_2 < \ldots < \beta_k \leq \{n_k\}$ in $\mathcal{P}_f(\mathbb{N})$ and for every $F \in \mathcal{P}_f(\mathbb{P}(S^j, S))$, $f_i \in G_i$ we have*

$$\left\langle \begin{array}{c} \sum_{i\in\beta_1} \alpha(G_i) + P\left(\sum_{i\in\beta_1}\sum_{t\in H(G_i)} f_i(t)\right), \ldots, \\ \sum_{i\in\beta_k} \alpha(G_i) + P\left(\sum_{i\in\beta_k}\sum_{t\in H(G_i)} f_i(t)\right) \end{array} \right\rangle \in A_l,$$

for all $P \in F$.

Proof. As the theorem 2.2 we fix a minimal idempotent $e \in \beta S \setminus S$. Let $i \in \{1, 2, \ldots, r\}$ and $T \subseteq S^{<\omega}$ be a tree as provided by lemma 2.2. We fixed $F \in \mathcal{P}_f(\mathbb{P}(S^j, S))$ and will inductively construct the sequence $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_{n_k}$ in $\mathcal{P}_f(\mathcal{T}_j)$ and $\beta_1 < \beta_2 < \ldots < \beta_k \leq \{n_k\}$. Assume we have inductively constructed the sequence $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_{n_{k-1}}$ and $\beta_1 < \beta_2 < \ldots < \beta_{k-1} < \{n_k\}$ such that

$$\left\langle \begin{array}{c} \sum_{i\in\beta_1}\alpha(G_i)+P\left(\sum_{i\in\beta_1}\sum_{t\in H(G_i)}f_i(t)\right),\ldots, \\ \sum_{i\in\beta_{k-1}}\alpha(G_i)+P\left(\sum_{i\in\beta_{k-1}}\sum_{t\in H(G_i)}f_i(t)\right) \end{array} \right\rangle \in T.$$

$f_i \in G_i$, for all $P \in F$.

By lemma 2.2, we have

$$C = T\left(\left\langle \begin{array}{c} \sum_{i\in\beta_1}\alpha(G_i)+P\left(\sum_{i\in\beta_1}\sum_{t\in H(G_i)}f_i(t)\right),\ldots, \\ \sum_{i\in\beta_{k-1}}\alpha(G_i)+P\left(\sum_{i\in\beta_{k-1}}\sum_{t\in H(G_i)}f_i(t)\right) \end{array} \right\rangle\right) \in e.$$

Then C is a central set in S. Then for $F \in \mathcal{P}_f\left(\mathbb{P}\left(S^j, S\right)\right)$ and $G_{n_{k-1}+1} \subsetneq G_{n_{k-1}+2} \subsetneq \ldots \subsetneq G_{n_k}$, $f_i \in G_i$, $\sum_{i\in\beta_k}\left(\alpha(G_i)+P\left(\sum_{i\in\beta_k}\sum_{t\in H(G_i)}f_i(t)\right)\right) \in C$ for all $P \in F$, where $\beta_{k-1} < \beta_k \leq \{n_k\}$.

So we have,

$$\left\langle \begin{array}{c} \sum_{i\in\beta_1}\alpha(G_i)+P\left(\sum_{i\in\beta_1}\sum_{t\in H(G_i)}f_i(t)\right),\ldots, \\ \sum_{i\in\beta_k}\alpha(G_i)+P\left(\sum_{i\in\beta_k}\sum_{t\in H(G_i)}f_i(t)\right) \end{array} \right\rangle \in T,$$

as we wanted to show. □

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