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On the weighted contraharmonic means

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Abstract. Let $\mathscr{A}$ be a unital C^*-algebra with unit e and let $\nu \in [0,1]$. We introduce the concept of the ν-weighted contraharmonic mean of two positive definite elements a and b of $\mathscr{A}$ by

$$C_\nu(a,b) := 2\Big((1-\nu)a + \nu b\Big) - \Big((1-\nu)a^{-1} + \nu b^{-1}\Big)^{-1}.$$

When $\nu \in (0,1)$, we show that

$$C_\nu(a,b) = \max_{x+y=e}\Big\{2(1-\nu)a - (1-\nu)^{-1}x^*ax + 2\nu b - \nu^{-1}y^*by\Big\},$$

and then apply it to present some properties of this weighted mean.

1. Introduction and Preliminaries

The theory of (weighted) means for numbers is a classical and very well developed area in mathematical analysis (see, e.g., [6, Chapters II-III]). A mean of positive scalars α and β may be introduced in many different ways. One of the most important is a concept of the Gini–Beckenbach–Lehmer mean ([2, 4, 9]):

$$M_s(\alpha,\beta) = \frac{\alpha^s + \beta^s}{\alpha^{s-1} + \beta^{s-1}}.$$

Notice that the harmonic mean (H), geometric mean (G), arithmetic mean (A) and contraharmonic mean (C), which is frequently used in this paper, can be associated with this mean, respectively, by letting $s = 0$, $s = \frac{1}{2}$, $s = 1$ and $s = 2$. That is,

$$\begin{aligned} H(\alpha,\beta) &= M_0(\alpha,\beta) = \frac{2\alpha\beta}{\alpha+\beta}, \\ G(\alpha,\beta) &= M_{\frac{1}{2}}(\alpha,\beta) = \sqrt{\alpha\beta}, \\ A(\alpha,\beta) &= M_1(\alpha,\beta) = \frac{\alpha+\beta}{2}, \\ C(\alpha,\beta) &= M_2(\alpha,\beta) = \frac{\alpha^2+\beta^2}{\alpha+\beta}. \end{aligned}$$

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Let $\mathscr{A}$ be a C^*-algebra. An element a of $\mathscr{A}$ is positive, in short $0 \leq a$, if $a = b^*b$ for some $b \in \mathscr{A}$. If $0 \leq a$, then we denote by $a^{1/2}$ the unique positive square root of a. If a and b are self-adjoint elements of $\mathscr{A}$ such that $0 \leq a - b$, we write $b \leq a$. An element a of $\mathscr{A}$ is also said to be positive definite if a is positive and invertible. Averaging operations are of interest in the context of von Neumann algebras and C^*-algebras as well, and various notions of (weighted) means of positive definite elements have been studied (see [3, 7, 8, 10] and the references therein).

Let $\nu \in [0, 1]$. For two positive definite elements a and b of $\mathscr{A}$ the (ν-weighted) harmonic mean H_ν, (ν-weighted) geometric mean G_ν and (ν-weighted) arithmetic mean A_ν are defined by

$$H_\nu(a, b) = \left((1-\nu)a^{-1} + \nu b^{-1}\right)^{-1},$$
$$G_\nu(a, b) = a^{\frac{1}{2}}\left(a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{\nu} a^{\frac{1}{2}},$$
$$A_\nu(a, b) = (1-\nu)a + \nu b.$$

In this note, inspired by the definition for the contraharmonic mean of matrices [1, 5], we introduce the concept of the ν-weighted contraharmonic mean in the setting of C^*-algebras. We investigate some properties of this weighted mean and prove inequalities involving it.

2. Results

As we have already mentioned, the contraharmonic mean of two positive scalars is defined by the formula

$$C(\alpha, \beta) = \frac{\alpha^2 + \beta^2}{\alpha + \beta}.$$

This may be rewritten as

$$C(\alpha, \beta) = 2A(\alpha, \beta) - H(\alpha, \beta).$$

This motivates the following definition.

Definition 2.1. *Let $\mathscr{A}$ be a unital C^*-algebra with unit e and let $\nu \in [0, 1]$. The ν-weighted contraharmonic mean of two positive definite elements a and b of $\mathscr{A}$ is defined by*

$$C_\nu(a, b) = 2A_\nu(a, b) - H_\nu(a, b).$$

Remark 2.2. *In the sequel, a, b, c and d denote positive definite elements of a unital C^*-algebra $\mathscr{A}$ with unit e. Also, from now on, $\nu \in [0, 1]$, unless stated otherwise.*

Remark 2.3. *The following properties of the weighted contraharmonic mean are obvious:*

(i) $C_0(a, b) = a$, $C_1(a, b) = b$ *and* $C_\nu(a, a) = a$.

(ii) $C_\nu(a, b) = C_{1-\nu}(b, a)$.

(iii) $C_\nu(\alpha e, \beta e) = C_\nu(\alpha, \beta)e$ *for any* $\alpha, \beta > 0$.

(iv) $C_\nu(ra, rb) = rC_\nu(a, b)$ *for any* $r > 0$.

Remark 2.4. *Obviously, $C_\nu(a, b) \leq 2A_\nu(a, b)$. It is also easy to see that*

$$H_\nu\left(\left\|a^{-1}\right\|^{-1}, \left\|b^{-1}\right\|^{-1}\right)e \leq H_\nu(a, b).$$

Thus, by Definition 2.1, we have

$$C_\nu(a, b) \leq 2A_\nu(a, b) - H_\nu\left(\left\|a^{-1}\right\|^{-1}, \left\|b^{-1}\right\|^{-1}\right)e.$$

If α and β are two positive scalars, then the contraharmonic mean $C(\alpha, \beta)$ can be stated by the solution of the following variational problem:

$$C(\alpha, \beta) = \max_{s+t=1} \left\{ \alpha - 2\alpha s^2 + \beta - 2\beta t^2 \right\}.$$

Motivated by this expression for the contraharmonic mean of scalars, we establish the following theorem.

Theorem 2.5. *Let $\nu \in (0, 1)$. Then*

$$C_\nu(a, b) = \max_{x+y=e} \left\{ 2(1-\nu)a - (1-\nu)^{-1} x^* a x + 2\nu b - \nu^{-1} y^* b y \right\}.$$

Proof. Note first that, by direct computations we have

$$a\Big(\nu a + (1-\nu)b\Big)^{-1} b = H_\nu(a, b) = b\Big(\nu a + (1-\nu)b\Big)^{-1} a \tag{1}$$

and

$$\begin{aligned} &\nu^{-1}(1-\nu)a^{-\frac{1}{2}} b a^{-\frac{1}{2}} - \left(e + \nu(1-\nu)^{-1} a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}\right)^{-1} \\ &\quad = \left(e + \nu^{-1}(1-\nu)a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \left(\nu^{-1}(1-\nu)a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^2 \left(e + \nu^{-1}(1-\nu)a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{-\frac{1}{2}}. \end{aligned} \tag{2}$$

Set $z = (1-\nu)\Big(\nu a + (1-\nu)b\Big)^{-1} b$ and $w = \nu\Big(\nu a + (1-\nu)b\Big)^{-1} a$. Then $z + w = e$. By (1), we have

$$\begin{aligned} &\max_{x+y=e} \left\{ 2(1-\nu)a - (1-\nu)^{-1} x^* a x + 2\nu b - \nu^{-1} y^* b y \right\} \\ &\quad \geq 2(1-\nu)a - (1-\nu)^{-1} z^* a z + 2\nu b - \nu^{-1} w^* b w \\ &\quad = 2(1-\nu)a - (1-\nu)b\Big(\nu a + (1-\nu)b\Big)^{-1} a\Big(\nu a + (1-\nu)b\Big)^{-1} b \\ &\qquad\quad + 2\nu b - \nu a\Big(\nu a + (1-\nu)b\Big)^{-1} b\Big(\nu a + (1-\nu)b\Big)^{-1} a \\ &\quad = 2\Big((1-\nu)a + \nu b\Big) - (1-\nu)b\Big(\nu a + (1-\nu)b\Big)^{-1} H_\nu(a, b) \\ &\qquad\quad - \nu a\Big(\nu a + (1-\nu)b\Big)^{-1} H_\nu(a, b) \\ &\quad = 2A_\nu(a, b) - \Big((1-\nu)b + \nu a\Big)\Big(\nu a + (1-\nu)b\Big)^{-1} H_\nu(a, b) \\ &\quad = 2A_\nu(a, b) - H_\nu(a, b) = C_\nu(a, b), \end{aligned}$$

and hence

$$C_\nu(a, b) \leq \max_{x+y=e} \left\{ 2(1-\nu)a - (1-\nu)^{-1} x^* a x + 2\nu b - \nu^{-1} y^* b y \right\}. \tag{3}$$

Now, suppose $x, y \in \mathscr{A}$ with $x + y = e$. Let us put

$$h := \left(e + \nu^{-1}(1-\nu)a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{\frac{1}{2}} a^{\frac{1}{2}} x a^{-\frac{1}{2}} - \nu^{-1}(1-\nu)a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \left(e + \nu^{-1}(1-\nu)a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

By exploiting (2) we have

$$\begin{aligned}
&a^{\frac{1}{2}}\left(e+\nu(1-\nu)^{-1}a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)^{-1}a^{\frac{1}{2}}+a^{\frac{1}{2}}h^*ha^{\frac{1}{2}}\\
&\quad=a^{\frac{1}{2}}\left(e+\nu(1-\nu)^{-1}a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)^{-1}a^{\frac{1}{2}}+x^*a^{\frac{1}{2}}\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)a^{\frac{1}{2}}x\\
&\qquad-x^*a^{\frac{1}{2}}\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{\frac{1}{2}}\left(\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{-\frac{1}{2}}a^{\frac{1}{2}}\\
&\qquad-a^{\frac{1}{2}}\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\left(\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{\frac{1}{2}}a^{\frac{1}{2}}x\\
&\qquad+a^{\frac{1}{2}}\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\left(\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{2}\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{-\frac{1}{2}}a^{\frac{1}{2}}\\
&\quad=a^{\frac{1}{2}}\left(e+\nu(1-\nu)^{-1}a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)^{-1}a^{\frac{1}{2}}+x^*ax\\
&\qquad+a^{\frac{1}{2}}\left(e-a^{-\frac{1}{2}}x^*a^{\frac{1}{2}}\right)\left(\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)\left(e-a^{\frac{1}{2}}xa^{-\frac{1}{2}}\right)a^{\frac{1}{2}}-\nu^{-1}(1-\nu)b\\
&\qquad+a^{\frac{1}{2}}\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\left(\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{2}\left(e+\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{-\frac{1}{2}}a^{\frac{1}{2}}\\
&\quad=a^{\frac{1}{2}}\left(e+\nu(1-\nu)^{-1}a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)^{-1}a^{\frac{1}{2}}+x^*ax\\
&\qquad+a^{\frac{1}{2}}\left(a^{-\frac{1}{2}}y^*a^{\frac{1}{2}}\right)\left(\nu^{-1}(1-\nu)a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)\left(a^{\frac{1}{2}}ya^{-\frac{1}{2}}\right)a^{\frac{1}{2}}-\nu^{-1}(1-\nu)b\\
&\qquad+\nu^{-1}(1-\nu)b-a^{\frac{1}{2}}\left(e+\nu(1-\nu)^{-1}a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)^{-1}a^{\frac{1}{2}}\\
&\quad=x^*ax+\nu^{-1}(1-\nu)y^*by,
\end{aligned}$$

and wherefrom

$$a^{\frac{1}{2}}\left(e+\nu(1-\nu)^{-1}a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)^{-1}a^{\frac{1}{2}}+a^{\frac{1}{2}}h^*ha^{\frac{1}{2}}=x^*ax+\nu^{-1}(1-\nu)y^*by.$$

This implies

$$(1-\nu)^{-1}a^{\frac{1}{2}}h^*ha^{\frac{1}{2}}=(1-\nu)^{-1}x^*ax+\nu^{-1}y^*by-(1-\nu)^{-1}a^{\frac{1}{2}}\left(e+\nu(1-\nu)^{-1}a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)^{-1}a^{\frac{1}{2}}. \tag{4}$$

Since $0\le(1-\nu)^{-1}a^{\frac{1}{2}}h^*ha^{\frac{1}{2}}$, by (4) we obtain

$$\begin{aligned}
&2(1-\nu)a-(1-\nu)^{-1}x^*ax+2\nu b-\nu^{-1}y^*by\\
&\quad\le 2(1-\nu)a-(1-\nu)^{-1}x^*ax+2\nu b-\nu^{-1}y^*by+(1-\nu)^{-1}a^{\frac{1}{2}}h^*ha^{\frac{1}{2}}\\
&\quad=2A_\nu(a,b)-(1-\nu)^{-1}a^{\frac{1}{2}}\left(e+\nu(1-\nu)^{-1}a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}\right)^{-1}a^{\frac{1}{2}}\\
&\quad=2A_\nu(a,b)-\left((1-\nu)a^{-1}+\nu b^{-1}\right)^{-1}\\
&\quad=2A_\nu(a,b)-H_\nu(a,b)=C_\nu(a,b),
\end{aligned}$$

and so

$$2(1-\nu)a-(1-\nu)^{-1}x^*ax+2\nu b-\nu^{-1}y^*by\le C_\nu(a,b)\qquad(x+y=e). \tag{5}$$

It follows from (5) that

$$\max_{x+y=e}\left\{2(1-\nu)a-(1-\nu)^{-1}x^*ax+2\nu b-\nu^{-1}y^*by\right\}\le C_\nu(a,b). \tag{6}$$

Now, by (3) and (6), we deduce the desired result. □

As a consequence of Theorem 2.5, we have the following result.

Corollary 2.6. *Let p be a projection in $\mathscr{A}$. Then*

$$C_\nu(a,b) \le 2A_\nu(a,b) - (1-\nu)^{-1}pap - \nu^{-1}(e-p)b(e-p).$$

Proof. This follows immediately form Theorem 2.5 by setting $x = p$ and $y = e - p$. □

Remark 2.7. *Setting $b = a$ the inequality in Corollary 2.6 reduces to the familiar inequality*

$$(1-\nu)^{-1}pap + \nu^{-1}(e-p)a(e-p) \le a.$$

Here, we use Theorem 2.5 to obtain operator inequalities.

Corollary 2.8. *Let $\mu \in [0,1]$. Then*

$$C_\nu\Big(A_\mu(a,b), A_\mu(c,d)\Big) \le A_\mu\Big(C_\nu(a,c), C_\nu(b,d)\Big).$$

Proof. We may assume that $\nu \in (0,1)$ otherwise, by Remark 2.3(i), the desired inequality trivially holds. Let $x + y = e$. By Theorem 2.5 we have

$$\begin{aligned}
A_\mu\Big(C_\nu(a,c), C_\nu(b,d)\Big) &= (1-\mu)C_\nu(a,c) + \mu C_\nu(b,d) \\
&\ge (1-\mu)\Big(2(1-\nu)a - (1-\nu)^{-1}x^*ax + 2\nu c - \nu^{-1}y^*cy\Big) \\
&\quad + \mu\Big(2(1-\nu)b - (1-\nu)^{-1}x^*bx + 2\nu d - \nu^{-1}y^*dy\Big) \\
&= 2(1-\nu)\Big((1-\mu)a + \mu b\Big) - (1-\nu)^{-1}x^*\Big((1-\mu)a + \mu b\Big)x \\
&\quad + 2\nu\Big((1-\mu)c + \mu d\Big) - \nu^{-1}y^*\Big((1-\mu)c + \mu d\Big)y
\end{aligned}$$

and hence

$$2(1-\nu)A_\mu(a,b) - (1-\nu)^{-1}x^*A_\mu(a,b)x + 2\nu A_\mu(c,d) - \nu^{-1}y^*A_\mu(c,d)y \le A_\mu\left(C_\nu(a,c), C_\nu(b,d)\right).$$

Thus,

$$\max_{x+y=e}\Big\{2(1-\nu)A_\mu(a,b) - (1-\nu)^{-1}x^*A_\mu(a,b)x + 2\nu A_\mu(c,d) - \nu^{-1}y^*A_\mu(c,d)y\Big\} \le A_\mu\Big(C_\nu(a,c), C_\nu(b,d)\Big).$$

Now, from Theorem 2.5 we obtain $C_\nu\Big(A_\mu(a,b), A_\mu(c,d)\Big) \le A_\mu\Big(C_\nu(a,c), C_\nu(b,d)\Big)$. □

Remark 2.9. *Let $\mu \in [0,1]$. By Corollary 2.8 we have*

$$C_\nu\Big(A_\mu(a,a), A_\mu(a,b)\Big) \le A_\mu\Big(C_\nu(a,a), C_\nu(a,b)\Big). \tag{7}$$

Since $A_\mu(a,a) = C_\nu(a,a) = a$, from (7) it follows that

$$C_\nu\Big(a, A_\mu(a,b)\Big) \le A_\mu\Big(a, C_\nu(a,b)\Big).$$

Another consequence of Theorem 2.5 can be stated as follows.

Corollary 2.10. *Let z be an invertible element of $\mathscr{A}$. Then*

$$C_\nu\Big(z^*az, z^*bz\Big) = z^*C_\nu(a,b)z.$$

Proof. Since by Remark 2.3(i) the desired identity trivially holds when $\nu = 0$ and $\nu = 1$, we may assume that $\nu \in (0,1)$. Let $x + y = e$. Put $x_0 = zxz^{-1}$ and $y_0 = zyz^{-1}$. Then $x_0 + y_0 = e$. So, by Theorem 2.5, we have

$$\begin{aligned} z^*C_\nu(a,b)z &= z^*\left(\max_{x+y=e}\left\{2(1-\nu)a - (1-\nu)^{-1}x^*ax + 2\nu b - \nu^{-1}y^*by\right\}\right)z \\ &\geq z^*\left(2(1-\nu)a - (1-\nu)^{-1}x_0^*ax_0 + 2\nu b - \nu^{-1}y_0^*by_0\right)z \\ &= 2(1-\nu)(z^*az) - (1-\nu)^{-1}x^*(z^*az)x + 2\nu(z^*bz) - \nu^{-1}y^*(z^*bz)y, \end{aligned}$$

and so

$$2(1-\nu)(z^*az) - (1-\nu)^{-1}x^*(z^*az)x + 2\nu(z^*bz) - \nu^{-1}y^*(z^*bz)y \leq z^*C_\nu(a,b)z.$$

Therefore,

$$\max_{x+y=e}\left\{2(1-\nu)(z^*az) - (1-\nu)^{-1}x^*(z^*az)x + 2\nu(z^*bz) - \nu^{-1}y^*(z^*bz)y\right\} \leq z^*C_\nu(a,b)z.$$

Now, by Theorem 2.5, we conclude that $C_\nu\left(z^*az, z^*bz\right) \leq z^*C_\nu(a,b)z$. By a similar argument, we get $z^*C_\nu(a,b)z \leq C_\nu\left(z^*az, z^*bz\right)$ and the proof is completed. □

Our next assertion is interesting on its own right.

Corollary 2.11. *There exists a contraction z in $\mathscr{A}$ such that*

$$A_\nu(a,b) = z^*C_\nu(a,b)z.$$

Proof. Let $\nu \in (0,1)$. Since $(1-\nu)e + \nu e = e$, by Theorem 2.5, we have

$$\begin{aligned} C_\nu(a,b) &\geq 2(1-\nu)a - (1-\nu)^{-1}\left((1-\nu)e\right)^*a\left((1-\nu)e\right) + 2\nu b - \nu^{-1}\left(\nu e\right)^*b\left(\nu e\right) \\ &= 2(1-\nu)a - (1-\nu)a + 2\nu b - \nu b = A_\nu(a,b), \end{aligned}$$

and so

$$A_\nu(a,b) \leq C_\nu(a,b). \tag{8}$$

By Remark 2.3(i), the inequality (8) holds trivially for $\nu = 0$ and $\nu = 1$. Put

$$z := \left(C_\nu(a,b)\right)^{-\frac{1}{2}}\left(A_\nu(a,b)\right)^{\frac{1}{2}}.$$

From (8) it follows that

$$z^*z = \left(A_\nu(a,b)\right)^{\frac{1}{2}}\left(C_\nu(a,b)\right)^{-1}\left(A_\nu(a,b)\right)^{\frac{1}{2}} \leq e,$$

and hence z is a contraction. It is also easy to check that $z^*C_\nu(a,b)z = A_\nu(a,b)$. □

Finally, we state an inequality for non-zero positive linear functionals.

Corollary 2.12. *Let φ be a non-zero positive linear functional on $\mathscr{A}$. Then*

$$C_\nu\left(\varphi(a), \varphi(b)\right) \leq \varphi\left(C_\nu(a,b)\right).$$

Proof. We may assume that $\nu \in (0,1)$ otherwise, by Remark 2.3(i), the desired inequality trivially holds. Set $x_0 = \frac{(1-\nu)\varphi(b)}{\varphi(\nu a+(1-\nu)b)}e$ and $y_0 = \frac{\nu\varphi(a)}{\varphi(\nu a+(1-\nu)b)}e$. Then $x_0 + y_0 = e$. Hence, by Theorem 2.5, we have

$$\begin{aligned}
\varphi\Big(C_\nu(a,b)\Big) &= \varphi\left(\max_{x+y=e}\Big\{2(1-\nu)a-(1-\nu)^{-1}x^*ax+2\nu b-\nu^{-1}y^*by\Big\}\right)\\
&\geq \varphi\Big(2(1-\nu)a-(1-\nu)^{-1}x_0^*ax_0+2\nu b-\nu^{-1}y_0^*by_0\Big)\\
&= 2\Big((1-\nu)\varphi(a)+\nu\varphi(b)\Big)-\left(\frac{(1-\nu)\varphi^2(b)\varphi(a)}{\varphi^2\left(\nu a+(1-\nu)b\right)}+\frac{\nu\varphi^2(a)\varphi(b)}{\varphi^2\left(\nu a+(1-\nu)b\right)}\right)\\
&= 2A_\nu\Big(\varphi(a),\varphi(b)\Big)-\frac{\varphi(a)\varphi(b)}{\varphi\left(\nu a+(1-\nu)b\right)}\\
&= 2A_\nu\Big(\varphi(a),\varphi(b)\Big)-H_\nu\Big(\varphi(a),\varphi(b)\Big)=C_\nu\Big(\varphi(a),\varphi(b)\Big).
\end{aligned}$$

□

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