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Another version discrete Orlicz–Morrey spaces

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Abstract. The aim of this paper is to define Discrete Orlicz–Morrey spaces. Furthermore, we also present the sufficient and necessary conditions for the inclusion properties of among these spaces. Computing the norms of the characteristic sequences is one of the keys results. Similar results for Discrete weak Orlicz–Morrey spaces are also obtained.

1. Introduction and Preliminaries

The Orlicz–Morrey spaces are generalizations of Orlicz spaces and Morrey spaces. Many researchers have made important observations about Orlicz spaces and Morrey spaces, for example ([2, 4–6, 8–10, 17, 22], etc.). Besides the 'continuous' Orlicz spaces and 'continuous' Morrey spaces, several authors have made study about discrete Orlicz spaces (see [13, 19]) and discrete Morrey spaces (see [1, 7]).

Recently, Fatimah *et al.* [3] have proved inclusion relations between two generalized discrete Orlicz–Morrey spaces and between two generalized weak Discrete Morrey spaces for Nakai's version. Related results on discrete Orlicz spaces can be found in [13]. In this paper, we are interested to study another discrete Orlicz–Morrey spaces and weak discrete Orlicz–Morrey spaces. In particular, we present some sufficient and necessary conditions for inclusion properties on these spaces. For related works, interested researcher can see [3, 7, 10, 16].

First, we recall definition of Young function. A function $\Psi : [0, \infty) \to [0, \infty)$ is called a Young function if Ψ is convex, left-continuous, $\Psi(0) = 0 = \lim_{t\to 0} \Psi(t)$, and $\lim_{t\to\infty} \Psi(t) = \infty$. Given two Young functions Ψ_1, Ψ_2, we write $\Psi_1 \prec \Psi_2$ if there exists a constant $C > 0$ such that $\Psi_1(t) \le \Psi_2(Ct)$ for all $t > 0$.

Let $m \in \mathbb{Z}$, $N \in \omega := \mathbb{N} \cup \{0\}$, we write $S_{m,N} := \{m-N, ..., m, ..., m+N\}$ and $|S_{m,N}| = 2N+1$ for the cardinality of $S_{m,N}$. Let G_ψ be the set of all functions $\psi : 2\omega + 1 \to (0, \infty)$ such that $\psi(2N + 1)$ is nondecreasing but for any $2M + 1 \in 2\omega + 1$, $\frac{\psi(2(N+M)+1)}{\Psi^{-1}(\frac{2(N+M)+1}{2N+1})}$ is nonincreasing.

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For $\psi_1, \psi_2 : 2\omega + 1 \to (0, \infty)$, we write $\psi_1 \preceq \psi_2$ if there exists a constant $C > 0$ such that $\psi_1(2N+1) \leq C\psi_2(2N+1)$ for all $N \in \omega$. If $\psi_1 \preceq \psi_2$ and $\psi_2 \preceq \psi_1$, then we write $\psi_1 \approx \psi_2$.

Now, let Ψ be a Young function and $\psi \in G_\psi$. The Discrete Orlicz–Morrey spaces $\ell_{\psi,\Psi}(\mathbb{R})$ is the set of all sequences $x = (x_k)_{k=1}^{\infty}$ taking values in $\mathbb{R}$ such that for every $m \in \mathbb{Z}$ and $N \in \omega$, we have

$$\|x\|_{\ell_{\psi,\Psi}} := \sup_{m\in\mathbb{Z}, N\in\omega} \psi(2N+1)\|x\|_{(\Psi,m,N)} < \infty,$$

where $\|x\|_{(\Psi,m,N)} := \inf\Big\{b > 0 : \frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi\Big(\frac{|x_k|}{b}\Big) \leq 1\Big\}$.

In next section, we will show the Discrete Orlicz–Morrey spaces $\ell_{\psi,\Psi}(\mathbb{R})$ is a Banach space with respect to $\|x\|_{\ell_{\psi,\Psi}}$.

For $\psi(2N+1) := 2N+1$, the space $\ell_{\psi,\Psi}(\mathbb{R})$ is the Discrete Orlicz space $\ell_\Psi(\mathbb{R})$. Meanwhile, for $\Psi(t) = t^p$, the space $\ell_{\psi,\Psi}(\mathbb{R})$ reduces to the generalized Discrete Morrey space $\ell^p_\psi(\mathbb{R})$.

Throughout this paper, the letter C denotes a constant that may vary in values from line to line. To keep track of some constants, we use subscripts, such as C_1 and C_2.

2. Inclusion properties of (strong) Discrete Orlicz–Morrey spaces

First, we will show $\|\cdot\|_{\ell_{\psi,\Psi}}$ defines a norm on $\ell_{\psi,\Psi}(\mathbb{R})$. For getting a result, we present some lemmas in the following.

Lemma 2.1. *If $x = (x_k)_{k=1}^{\infty} \in \ell_{\psi,\Psi}(\mathbb{R})$, then*

$$\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi\Big(\frac{|x_k|}{\|x\|_{(\Psi,m,N)}}\Big) \leq 1$$

for any $m \in \mathbb{Z}$ and $N \in \omega$. Furthermore, $\|x\|_{(\Psi,m,N)} \leq 1$ if and only if $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi(|x_k|) \leq 1$ for any $m \in \mathbb{Z}$ and $N \in \omega$.

Proof. Let x be an element of $\ell_{\psi,\Psi}(\mathbb{R})$ and take an arbitrary $\epsilon > 0$, then there exists $b_\epsilon > 0$ such that $b_\epsilon \leq \|x\|_{(\Psi,m,N)} + \epsilon$ and $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi\Big(\frac{|x_k|}{b_\epsilon}\Big) \leq 1$ for any $m \in \mathbb{Z}$ and $N \in \omega$. Because Ψ is increasing, we have

$$\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi\Big(\frac{|x_k|}{\|x\|_{(\Psi,m,N)} + \epsilon}\Big) \leq \frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi\Big(\frac{|x_k|}{b_\epsilon}\Big) \leq 1.$$

Since $\epsilon > 0$ is arbitrary, we can conclude

$$\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi\Big(\frac{|x_k|}{\|x\|_{(\Psi,m,N)}}\Big) \leq 1$$

for any $m \in \mathbb{Z}$ and $N \in \omega$.

Next, if $\|x\|_{(\Psi,m,N)} \leq 1$ for any $m \in \mathbb{Z}$ and $N \in \omega$, then

$$\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi(|x_k|) \leq \frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi\Big(\frac{|x_k|}{\|x\|_{(\Psi,m,N)}}\Big) \leq 1.$$

Now, assume that $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi(|x_k|) \leq 1$ holds for any $m \in \mathbb{Z}$ and $N \in \omega$. Clearly, we have $\|x\|_{(\Psi,m,N)} \leq 1$. □

Lemma 2.2. *Let Ψ be a Young function, $\psi \in G_\psi$, $m \in \mathbb{Z}$ and $N \in \omega$. Then the following statements are equivalent:*
(1) $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\frac{|x_k|}{\epsilon}\Big) \le 1$ *for every $\epsilon > 0$.*
(2) $\|x\|_{(\Psi,m,N)} = 0$.

Proof. Assume that (1) holds. By definition of $\|x\|_{(\Psi,m,N)}$ we have $0 \le \|x\|_{(\Psi,m,N)} \le \epsilon$, for every $\epsilon > 0$. So we conclude that $\|x\|_{(\Psi,m,N)} = 0$. Suppose, on the contrary, that there exists $\epsilon_0 > 0$ such that $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\frac{|x_k|}{\epsilon_0}\Big) > 1$. By Lemma 2.1 we have $\|x\|_{(\Psi,m,N)} \ge \epsilon_0 > 0$. As a consequence, we conclude that $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\frac{|x_k|}{\epsilon}\Big) \le 1$ for every $\epsilon > 0$. □

Lemma 2.3. *Let Ψ be a Young function, $\psi \in G_\psi$, $m \in \mathbb{Z}$ and $N \in \omega$. Then the following statements are equivalent:*
(1) $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\alpha|x_k|\Big) = 0$ *for every $\alpha > 0$.*
(2) $\|x\|_{(\Psi,m,N)} = 0$

Proof. Suppose that (1) holds. As before, we can obtain $\|x\|_{(\Psi,m,N)} = 0$. Now, suppose that (2) holds. Take an arbitrary $0 < \epsilon \le 1$. Since Ψ is a convex function, we have

$$\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\alpha|x_k|\Big) \le \epsilon\left(\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\frac{\alpha|x_k|}{\epsilon}\Big)\right) \le \epsilon.$$

Since $0 < \epsilon \le 1$ is arbitrary, we can conclude that $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\alpha|x_k|\Big) = 0$.
□

Proposition 2.4. *Let Ψ be a Young function and $\psi \in G_\psi$, the mapping $\|\cdot\|_{\ell_{\psi,\Psi}}$ defines a norm on $\ell_{\psi,\Psi}(\mathbb{R})$. Moreover, $(\ell_{\psi,\Psi}(\mathbb{R}), \|\cdot\|_{\ell_{\psi,\Psi}})$ is a Banach space.*

Proof. It is easy to prove that $\|x\|_{\ell_{\psi,\Psi}} \ge 0$ and $\|\alpha x\|_{\ell_{\psi,\Psi}} = |\alpha|\|x\|_{\ell_{\psi,\Psi}}$, for every $x \in \ell_{\psi,\Psi}(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Now, we will prove $\|x\|_{\ell_{\psi,\Psi}} = 0$ if and only if $x = 0$. If $x = 0$, then we have $\|x\|_{\ell_{\psi,\Psi}} = 0$. Let $\|x\|_{\ell_{\psi,\Psi}} = 0$, then $\|x\|_{(\Psi,m,N)} = 0$ for every $m \in \mathbb{Z}$ and $N \in \omega$. By Lemma 2.3 we have $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\alpha|x_k|\Big) = 0$. In fact, α, $|S_{m,N}|$, and Ψ are positive values, we have $x_k = 0$ for every $k \in S_{m,N}$. Since $m \in \mathbb{Z}$ and $N \in \omega$ are arbitrary, we have $x = 0$.

Next, we will prove $\|x_1 + x_2\|_{\ell_{\psi,\Psi}} \le \|x_1\|_{\ell_{\psi,\Psi}} + \|x_2\|_{\ell_{\psi,\Psi}}$, for every $x_1, x_2 \in \ell_{\psi,\Psi}(\mathbb{R})$. Let $x_1 = (x_k^{(1)})_{k=1}^\infty$ and $x_2 = (x_k^{(2)})_{k=1}^\infty$ be elements of $\ell_{\psi,\Psi}(\mathbb{R})$. For any $m \in \mathbb{Z}$ and $N \in \omega$, we have

$$\begin{aligned}\sum_{k\in S_{m,N}}\Psi\Big(\frac{|x_k^{(1)}) + (x_k^{(2)})|}{\|x_1\|_{(\Psi,m,N)} + \|x_2\|_{(\Psi,m,N)}}\Big) &\le \sum_{k\in S_{m,N}}\Phi\Big(\sum_{i=1}^{2}\frac{\|x_i\|_{(\Psi,m,N)}}{\|x_1\|_{(\Psi,m,N)} + \|x_2\|_{(\Psi,m,N)}}\frac{|x_k^{(i)}|}{\|x_i\|_{(\Psi,m,N)}}\Big)\\ &\le \sum_{i=1}^{2}\left(\frac{\|x_i\|_{(\Psi,m,N)}}{\|x_1\|_{(\Psi,m,N)} + \|x_2\|_{(\Psi,m,N)}}\sum_{k\in S_{m,N}}\Psi\Big(\frac{|x_k^{(i)}|}{\|x_i\|_{(\Psi,m,N)}}\Big)\right)\\ &\le |S_{m,N}|\end{aligned}$$

or $\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}}\Psi\Big(\frac{|(x_k^{(1)})+x_k^{(2)})|}{\|x_1\|_{(\Psi,m,N)}+\|x_2\|_{(\Psi,m,N)}}\Big) \le 1$. By definition of $\|x_1 + x_2\|_{(\Psi,m,N)}$, we have

$$\|x_1 + x_2\|_{(\Psi,m,N)} \le \|x_1\|_{(\Psi,m,N)} + \|x_2\|_{(\Psi,m,N)}.$$

Furthermore, we have

$$\psi(2N + 1)\|x_1 + x_2\|_{(\Psi,m,N)} \le \psi(2N + 1)\|x_1\|_{(\Psi,m,N)} + \psi(2N + 1)\|x_2\|_{(\Psi,m,N)}.$$

By taking supremum over $m \in \mathbb{Z}$ and $N \in \omega$, we conclude that $\|x_1 + x_2\|_{\ell_{\psi,\Psi}} \le \|x_1\|_{\ell_{\psi,\Psi}} + \|x_2\|_{\ell_{\psi,\Psi}}$, for every $x_1, x_2 \in \ell_{\psi,\Psi}(\mathbb{R})$. □

Lemma 2.5. *[15, 17, 21] Suppose that* Ψ *is a Young function and* Ψ^{-1} *denotes its inverse, which is given by* $\Psi^{-1}(s) := \inf\{r \geq 0 : \Psi(r) > s\}$ *for every* $s \geq 0$*. Then the followings hold:*
(1) $\Psi^{-1}(0) = 0$.
(2) $\Psi^{-1}(s_1) \leq \Psi^{-1}(s_2)$ *for* $s_1 \leq s_2$.
(3) $\Psi(\Psi^{-1}(s)) \leq s \leq \Psi^{-1}(\Psi(s))$ *for* $0 \leq s < \infty$.
(4) *If, for some constants* $C_1, C_2 > 0$*, we have* $\Psi_2^{-1}(s) \leq C_1\Psi_1^{-1}(C_2 s)$*, then* $\Psi_1(\frac{t}{C_1}) \leq C_2\Psi_2(t)$ *for* $t = \Psi_2^{-1}(s)$.

Lemma 2.6. *Let* Ψ *be a Young function and* $\psi \in G_\psi$*. For* $m \in \mathbb{Z}$ *and* $N_0 \in \omega$*, let* ξ^{m,N_0} *be the characteristics sequence given by*

$$\xi^{m,N_0} := \begin{cases} 1, \text{if} k \in S_{m,N_0} \\ 0, \textit{otherwise} \end{cases}$$

then $\|\xi^{m,N_0}\|_{(\Psi,m,N)} = \frac{1}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\right)}$.

Proof. Let

$$A_{\Psi,m,N} := \Big\{b > 0 : \Psi\Big(\frac{1}{b}\Big) \leq \frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big\}$$

and

$$B_{\Psi,m,N} := \Big\{r > 0 : \Psi(r) > \frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big\}.$$

Observe that,

$$\begin{aligned} \|\xi^{m_0,N_0}\|_{(\Psi,m,N)} &:= \inf\Big\{b > 0 : \frac{1}{|S_{m.N}|}\sum_{k \in S_{m,N}} \Psi\Big(\frac{|\xi^{m,N_0}|}{b}\Big) \leq 1\Big\} \\ &= \inf\Big\{b > 0 : \frac{|S_{m,N_0} \cap S_{m,N}|}{|S_{m,N}|}\Psi\Big(\frac{1}{b}\Big) \leq 1\Big\} \\ &= \inf\Big\{b > 0 : \Psi\Big(\frac{1}{b}\Big) \leq \frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big\} \\ &= \inf A_{\Psi,m,N}. \end{aligned}$$

Meanwhile, $\Psi^{-1}\Big(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big) = \inf\Big\{r \geq 0 : \Psi(r) > \frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big\} = \inf B_{\Psi,m,N}$. Let $b = \frac{1}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\right)}$, by Lemma 2.5 (3) we have

$$\Psi\Big(\frac{1}{b}\Big) = \Psi\left(\Psi^{-1}\Big(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big)\right) \leq \frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}.$$

By definition of $\|\cdot\|_{(\Psi,m,N)}$, we have $\|\xi^{m,N_0}\|_{(\Psi,m,N)} \leq \frac{1}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\right)}$. Suppose, on the contrary, i.e., $\|\xi^{m,N_0}\|_{(\Psi,m,N)} < \frac{1}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\right)}$, then $\frac{1}{\|\xi^{m,N_0}\|_{(\Psi,m,N)}} > \Psi^{-1}\Big(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big)$.

By definition of $\Psi^{-1}\Big(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big)$, there exists a $r_1 \in B_{\Psi,m,N}$ such that

$$\frac{1}{\|\xi^{m,N_0}\|_{(\Psi,m,N)}} > r_1 \geq \Psi^{-1}\Big(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\Big).$$

Since $r_1 \in B_{\Psi,m,N}$ we obtain $\frac{1}{r_1} \notin A_{\Psi,m,N}$. So we can conclude that $\frac{1}{r_1} \leq \|\xi^{m,N_0}\|_{(\Psi,m,N)}$. As a consequence, we obtain $\|\xi^{m,N_0}\|_{(\Psi,m,N)} = \frac{1}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\right)}$, as desired. □

Proposition 2.7. *Let Ψ be a Young function and $\phi \in G_{\psi}$. For $m \in \mathbb{Z}$ and $N_0 \in \omega$, let ξ^{m,N_0} be the characteristics sequence given by*

$$\xi^{m,N_0} := \begin{cases} 1, \text{if} k \in S_{m,N_0} \\ 0, otherwise \end{cases}$$

then $\|\xi^{m,N_0}\|_{\ell_{\psi,\Psi}} = \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}$.

Proof. Take arbitrary $m \in \mathbb{Z}$ and $N_0 \in \omega$. By Lemma 2.6 we have

$$\begin{aligned} \|\xi^{m,N_0}\|_{\ell_{\psi,\Psi}} &= \sup_{m\in\mathbb{Z},N\in\omega} \|\xi^{m,N_0}\|_{(\Psi,m,N)} \\ &= \sup_{m\in\mathbb{Z},N\in\omega} \frac{\psi(2N+1)}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0}\cap S_{m,N}|}\right)} \\ &\geq \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}. \end{aligned}$$

Now, we will prove $\|\xi^{m,N_0}\|_{\ell_{\psi,\Psi}} \leq \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}$.

Case I : For $N \geq N_0$, write $N = N_0 + N_1$ for $N_1 \in \omega$. Observe that,

$$\begin{aligned} \frac{\psi(2N_0+1)}{\Psi^{-1}(1)} &= \frac{\psi((2N_0+1)+0)}{\Psi^{-1}\left(\frac{(2N_0+1)+0}{2N_0+1}\right)} \\ &\geq \frac{\psi(2(N_0+N_1)+1)}{\Psi^{-1}\left(\frac{2(N_0+N_1)+1}{2N_0+1}\right)} \\ &= \frac{\psi(2N+1)}{\Psi^{-1}\left(\frac{2N+1}{2N_0+1}\right)} \\ &= \frac{\psi(2N+1)}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0}\cap S_{m,N}|}\right)}. \end{aligned}$$

So we get, $\frac{\psi(2N+1)}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0}\cap S_{m,N}|}\right)} \leq \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}$.

Case II : for $N < N_0$, we have $\psi(2N+1) \leq \psi(2N_0+1)$. Its show that

$$\frac{\psi(2N+1)}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0}\cap S_{m,N}|}\right)} = \frac{\psi(2N+1)}{\Psi^{-1}\left(\frac{2N+1}{2N+1}\right)} \leq \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}.$$

Since Case I and II are true for arbitrary $m \in \mathbb{Z}, N \in \omega$, we have

$$\begin{aligned} \|\xi^{m,N_0}\|_{\ell_{\phi,\Phi}} &= \sup_{m\in\mathbb{Z},N\in\omega} \|\xi^{m,N_0}\|_{(\Psi,m,N)} \\ &= \sup_{m\in\mathbb{Z},N\in\omega} \frac{\psi(2N+1)}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0}\cap S_{m,N}|}\right)} \\ &\leq \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}, \end{aligned}$$

as desired. □

Now we come to inclusion properties of (strong) Discrete Orlicz–Morrey spaces in the following.

Theorem 2.8. *Let Ψ_1, Ψ_2 be Young functions such that $\Psi_1 \prec \Psi_2$ and $\psi_1, \psi_2 \in G_\psi$. Then the following statements are equivalent:*
(1) $\psi_1 \leq \psi_2$ *(on $2\omega + 1$).*
(2) $\ell_{\psi_2,\Psi_2}(\mathbb{R}) \subseteq \ell_{\psi_1,\Psi_1}(\mathbb{R})$.
(3) *There exists a constant $C > 0$ such that*

$$\|x\|_{\ell_{\psi_1,\Psi_1}} \leq C\|x\|_{\ell_{\psi_2,\Psi_2}},$$

for every $x \in \ell_{\psi_2,\Psi_2}(\mathbb{R})$.

Proof. Let us first prove that (1) implies (2). Let $x \in \ell_{\psi_2,\Psi_2}(\mathbb{R})$. Recall that $\Psi_1 \prec \Psi_2$ and $\psi_1 \leq \psi_2$ means that there exist constants $C_1, C_2 > 0$ such that $\Psi_1(t) \leq \Psi_2(C_1 t)$ every $t > 0$ and $\psi_1(2N+1) \leq C_2\psi_2(2N+1)$ for every $N \in \omega$. For every $m \in \mathbb{Z}$ and $N \in \omega$, we have

$$\begin{aligned}
\frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi_1\Big(\frac{|x_k|}{C_1\|x\|_{(\Psi_2,m,N)}}\Big) &\leq \frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi_2\Big(\frac{C_1|x_k|}{C_1\|x\|_{(\Psi_2,m,N)}}\Big)\\
&= \frac{1}{|S_{m,N}|}\sum_{k\in S_{m,N}} \Psi_2\Big(\frac{|x_k|}{\|x\|_{(\Psi_2,m,N)}}\Big)\\
&\leq 1.
\end{aligned}$$

By definition of $\|x\|_{(\Psi_1,m,N)}$, we have $\|x\|_{(\Psi_1,m,N)} \leq C_2\|x\|_{(\Psi_2,m,N)}$. Furthermore, we have

$$\begin{aligned}
\|x\|_{\ell_{\psi_1,\Psi_1}} &:= \sup_{m\in\mathbb{Z},N\in\omega} \psi_1(2N+1)\|x\|_{(\Psi_1,m,N)}\\
&\leq \sup_{m\in\mathbb{Z},N\in\omega} C_1C_2\psi_2(2N+1)\|x\|_{(\Psi_2,m,N)}\\
&= C_1C_2\|x\|_{\ell_{\psi_2,\Psi_2}}.
\end{aligned}$$

This proves that $\ell_{\psi_2,\Psi_2}(\mathbb{R}) \subseteq \ell_{\psi_1,\Psi_1}(\mathbb{R})$.

Next, since $(\ell_{\psi_2,\Psi_2}(\mathbb{R}), \ell_{\psi_1,\Psi_1}(\mathbb{R}))$ is a Banach pair, it follows from [11, Lemma 3.3] that (2) and (3) are equivalent. It thus remains to show that (3) implies (1).

Assume that (3) holds. Let $m_0 \in \mathbb{Z}$ and $N \in \omega$. By Proposition 2.7, we have

$$\frac{\psi_1(2N_0+1)}{\Psi_1^{-1}(1)} = \|\xi^{m,N_0}\|_{\ell_{\psi_1,\Psi_1}} \leq C\|\xi^{m,N_0}\|_{\ell_{\psi_2,\Psi_2}} = \frac{C\psi_2(2N_0+1)}{\Psi_2^{-1}(1)},$$

whence $\psi_1(2N_0+1) \leq \frac{C\Psi_1^{-1}(1)}{\Psi_2^{-1}(1)}\psi_2(2N_0+1)$. Since $N_0 \in \omega$ is arbitrary, we conclude that $\psi_1(2N+1) \leq C_1\psi_2(2N+1)$ for every $N \in \omega$, where $C_1 = \frac{C\Psi_1^{-1}(1)}{\Psi_2^{-1}(1)}$. □

3. Inclusion properties of weak Discrete Orlicz–Morrey spaces

We have discussed inclusion properties of Discrete Orlicz–Morrey spaces. Now we come to discuss inclusion properties of weak Discrete Orlicz–Morrey spaces. First we give definition of weak Discrete Orlicz–Morrey spaces and some lemmas in the following.

Let Ψ be a Young function and $\psi \in G_\psi$. The weak Discrete Orlicz–Morrey spaces $w\ell_{\psi,\Psi}(\mathbb{R})$ is the set of all sequences $x = (x_k)_{k=1}^\infty$ taking values in $\mathbb{R}$ such that

$$\|x\|_{w\ell_{\psi,\Psi}} := \sup_{m\in\mathbb{Z},N\in\omega} \psi(2N+1)\|x\|^w_{(\Psi,m,N)} < \infty,$$

where

$$\|x\|^w_{(\Psi,m,N)} = \inf\left\{b > 0 : \sup_{t>0} \Psi(t)\left|\left\{k \in S_{m,N} : \frac{|x_k|}{b} > t\right\}\right| \leq 1\right\}$$

is finite. In next section, we will show the weak Discrete Orlicz–Morrey spaces $w\ell_{\psi,\Psi}(\mathbb{R})$ is a quasi Banach space with respect to $\|x\|_{w\ell_{\psi,\Psi}}$.

For $\psi(2N+1) := 2N+1$, the space $w\ell_{\psi,\Psi}(\mathbb{R})$ is the weak Discrete Orlicz space $w\ell_{\Psi}(\mathbb{R})$. Meanwhile, for $\Psi(t) = t^p$, the space $w\ell_{\psi,\Psi}(\mathbb{R})$ reduces to the generalized Discrete Morrey space $w\ell^p_{\psi}(\mathbb{R})$.

The relation between $w\ell_{\psi,\Psi}(\mathbb{R})$ and $\ell_{\psi,\Psi}(\mathbb{R})$ is presented in the following lemma. (We leave the proof to the reader.)

Lemma 3.1. *Let Ψ be a Young function and $\psi \in G_{\psi}$. Then $\ell_{\psi,\Psi}(\mathbb{R}) \subseteq w\ell_{\psi,\Psi}(\mathbb{R})$ with*

$$\|x\|_{w\ell_{\psi,\Psi}(\mathbb{R})} \leq \|x\|_{\ell_{\psi,\Psi}(\mathbb{R})}$$

for every $x \in \ell_{\psi,\Psi}(\mathbb{R})$.

The following is an analog of Lemma 2.5.

Lemma 3.2. *Let Ψ be a Young function and $\psi \in G_{\psi}$. For $m \in \mathbb{Z}$ and $N_0 \in \omega$, let ξ^{m,N_0} be the characteristics sequence given by*

$$\xi^{m,N_0} := \begin{cases} 1, \text{if} k \in S_{m,N_0} \\ 0, otherwise \end{cases}$$

then $\|\xi^{m,N_0}\|^w_{(\Psi,m,N)} = \frac{1}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\right)}$.

The following proposition indicates that the characteristic sequence also obtained in Discrete weak Orlicz–Morrey spaces.

Proposition 3.3. *Let Ψ be a Young function and $\psi \in G_{\psi}$. For $m \in \mathbb{Z}$ and $N_0 \in \omega$, let ξ^{m,N_0} be the characteristics sequence given by*

$$\xi^{m,N_0} := \begin{cases} 1, \text{if} k \in S_{m,N_0} \\ 0, otherwise \end{cases}$$

then we have $\|\xi^{m,N_0}\|_{w\ell_{\psi,\Psi}} = \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}$.

Proof. Take arbitrary $m \in \mathbb{Z}$ and $N_0 \in \omega$. By Lemma 3.2 we have

$$\begin{aligned} \|\xi^{m,N_0}\|_{w\ell_{\psi,\Psi}} &= \sup_{m\in\mathbb{Z}, N\in\omega} \|\xi^{m,N_0}\|^w_{(\Psi,m,N)} \\ &= \sup_{m\in\mathbb{Z}, N\in\omega} \frac{\psi(2N+1)}{\Psi^{-1}\left(\frac{|S_{m,N}|}{|S_{m,N_0} \cap S_{m,N}|}\right)} \\ &\geq \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}. \end{aligned}$$

On the other hand, by Proposition 2.7 and Lemma 3.1 we have $\|\xi^{m_0,N_0}\|_{w\ell_{\psi,\Psi}} \leq \frac{\psi(2N_0+1)}{\Psi^{-1}(1)}$, as desired. □

Now we come to inclusion properties of weak Discrete Orlicz–Morrey spaces in the following.

Theorem 3.4. *Let Ψ_1, Ψ_2 be Young functions such that $\Psi_1 \prec \Psi_2$ and $\psi_1, \psi_2 \in G_\psi$. Then the following statements are equivalent:*(1) $\psi_1 \preceq \psi_2$ *(on $2\omega + 1$).*
(2) $w\ell_{\psi_2,\Psi_2}(\mathbb{R}) \subseteq w\ell_{\psi_1,\Psi_1}(\mathbb{R})$.
(3) *There exists a constant $C > 0$ such that*

$$\|x\|_{w\ell_{\psi_1,\Psi_1}} \leq C\|x\|_{w\ell_{\psi_2,\Psi_2}},$$

for every $x \in w\ell_{\psi_2,\Psi_2}(\mathbb{R})$.

Proof. Let us first prove that (1) implies (2). Let $x \in \ell_{\psi_2,\Psi_2}(\mathbb{R})$. Recall that $\Psi_1 \prec \Psi_2$ and $\psi_1 \preceq \psi_2$ means that there exist constants $C_1, C_2 > 0$ such that $\Psi_1(t) \leq \Psi_2(C_1 t)$ for every $t > 0$ and $\psi_1(2N+1) \leq C_2\psi_2(2N+1)$ for every $N \in \omega$. For $m \in \mathbb{Z}$ and $N \in \omega$, Let

$$A_{\Psi_1,M,N} = \Big\{b > 0 : \sup_{t>0} \frac{\Psi_1(t)\Big|\{k \in S_{m,N} : \frac{|x_k|}{b} > t\}\Big|}{|S_{m,N}|} \leq 1\Big\}$$

and

$$\begin{aligned} A_{\Psi_2,m,N} &= \Big\{b > 0 : \sup_{t>0} \frac{\Psi_2(C_1 t)\Big|\{k \in S_{m,N} : \frac{|x_k|}{b} > t\}\Big|}{|S_{m,N}|} \leq 1\Big\} \\ &= \Big\{b > 0 : \sup_{t_1>0} \frac{\Psi_2(t_1)\Big|\{x \in S_{m,N} : \frac{|x_k|}{b} > \frac{t_1}{C_1}\}\Big|}{|S_{m,N}|} \leq 1\Big\} \\ &= \Big\{b > 0 : \sup_{t_1>0} \frac{\Psi_2(t_1)\Big|\{k \in S_{m,N} : \frac{|C_1 x_k|}{b} > t_1\}\Big|}{|S_{m,N}|} \leq 1\Big\}. \end{aligned}$$

Then $\|C_1 x\|^{w}_{(\Psi_2,m,N)} = \inf A_{\Psi_2,m,N}$. Observe that, for arbitrary $b \in A_{\Psi_2,m,N}$ and $t > 0$, we have (by setting $t_1 = C_1 t$)

$$\begin{aligned} \frac{\Psi_1(t)\Big|\{k \in S_{m,N} : \frac{|x_k|}{b} > t\}\Big|}{|S_{m,N}|} &\leq \frac{\Psi_2(C_1 t)\Big|\{k \in S_{m,N} : \frac{|x_k|}{b} > t\}\Big|}{|S_{m,N}|} \\ &= \frac{\Psi_2(t_1)\Big|\{k \in S_{m,N} : \frac{|C_1 x_k|}{b} > t_1\}\Big|}{|S_{m,N}|} \\ &\leq \sup_{t_1>0} \frac{\Psi_2(t_1)\Big|\{k \in S_{m,N} : \frac{|C_1 x_k|}{b} > t_1\}\Big|}{|S_{m,N}|} \\ &\leq 1. \end{aligned}$$

Since $t > 0$ is arbitrary, we have $\sup_{t>0} \frac{\Psi_1(t)\Big|\Big\{k \in S_{m,N} : \frac{|x_k|}{b} > t\Big\}\Big|}{|S_{m,N}|} \leq 1$. Hence it follows that $b \in A_{\Psi_1,m,N}$, and so we conclude that $A_{\Psi_2,m,N} \subseteq A_{\Psi_1,m,N}$. Accordingly, we obtain

$$\|x\|^{w}_{(\Psi_1,m,N)} = \inf A_{\Psi_1,m,N} \leq \inf A_{\Psi_2,m,N} = \|C_1 x\|^{w}_{(\Psi_2,m,N)} = C_1\|x\|^{w}_{(\Psi_2,m,N)}.$$

Furthermore, we have

$$\begin{aligned} \|x\|_{w\ell_{\psi_1,\Psi_1}} &:= \sup_{m\in\mathbb{Z},N\in\omega} \psi_1(2N+1)\|x\|^{w}_{(\Psi_1,m,N)} \\ &\leq \sup_{m\in\mathbb{Z},N\in\omega} C_1 C_2 \psi_2(2N+1)\|x\|^{w}_{(\Psi_2,m,N)} \\ &= C_1 C_2 \|x\|_{w\ell_{\psi_2,\Psi_2}}. \end{aligned}$$

This proves that $w\ell_{\psi_2,\Psi_2}(\mathbb{R}) \subseteq w\ell_{\psi_1,\Psi_1}(\mathbb{R})$.

As mentioned in [18, Appendix G], we know that Lemma 3.3 in [11] still holds for quasi-Banach spaces, so (2) and (3) are equivalent.

Now, we will show that (3) implies (1). Assume that (3) holds. Let $m \in \mathbb{Z}$ and $N_0 \in \omega$. By Proposition 3.3, we have we have

$$\frac{\psi_1(2N_0+1)}{\Psi_1^{-1}(1)} = \|\xi^{m,N_0}\|_{w\ell_{\psi_1,\Psi_1}} \leq C\|\xi^{m,N_0}\|_{w\ell_{\psi_2,\Psi_2}} = \frac{C\psi_2(2N_0+1)}{\Psi_2^{-1}(1)},$$

whence $\psi_1(2N_0+1) \leq \frac{C\Psi_1^{-1}(1)}{\Psi_2^{-1}(1)}\psi_2(2N_0+1)$. Since $N_0 \in \omega$ is arbitrary, we conclude that $\psi_1(2N+1) \leq C_1\psi_2(2N+1)$ for every $N \in \omega$, where $C_1 = \frac{C\Psi_1^{-1}(1)}{\Psi_2^{-1}(1)}$. $\square$

4. Concluding Remarks

We have shown the inclusion property of (strong) Discrete Orlicz– spaces and of weak Discrete Orlicz–Morrey spaces. Both use the norm of the characteristic sequences in $\mathbb{R}$. As our final conclusion, we can states that the inclusion property of (strong) Discrete Orlicz–Morrey spaces are equivalent to that of weak Discrete Orlicz–Morrey spaces.

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