Filomat 39:6 (2025), 1831–1841
https://doi.org/10.2298/FIL2506831K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On convergence of McShane type integral and Radon measure

Hemanta Kalita[a,*], Hemen Bharali[b], Puja Dey Sarkar[b]

[a]*Mathematics Division, School of Advanced Sciences and Languages, VIT Bhopal University, Bhopal, Madhya Pradesh, India*
[b]*Department of Mathematics, Assam Don Bosco University, Tapesia, Sonapur, Assam, India*

Abstract. McShane integrals are generalized Riemann type integrals. In this paper, monotone convergence of μ-McShane integrable function is discussed. Further we introduce μ-equi-integrability, and μ-uniformly absolutely continuity on a complete metric space, endowed with a Radon measure μ and a family of cells that satisfies the Vitali covering theorem with respect to μ. We find several convergence theorems based on μ-equi-integrability, and μ-uniformly absolutely continuity. Finally, we establish a necessary and sufficient condition for the sequence of μ-McShane integrable functions to be μ-equi-integrable with respect to μ-uniformly absolutely continuity.

1. Introduction and preliminaries

For a great majority of mathematicians, the Lebesgue integral is considered the official or standard integral in the field. To comprehend Lebesgue integral, a substantial knowledge of measure theory is required. Lebesgue integral becomes complex due to the abstract nature of measure theory. McShane established a Riemann-type integral in the late 1960s and showed that the McShane integral is equivalent to the Lebesgue integral. Being a Riemann-type integral, it is more user-friendly to work with than the Lebesgue integral. Measures and σ-algebras are also excluded from his integral. McShane integral has undergone several extensions. Gordon [3] discussed several properties of McShane integrals. Additionally several convergence properties are also included in his book. Gordon [4] introduced and developed the properties of McShane integral for the case in which the function has values in a Banach space. One can read [1] for strongly McShane integrable functions and the representation theorem. In [10], Paluga et al. defined the McShane integral of a function with values in a topological vector space (TVS). It is demonstrated in this study that when the space under discussion is Banach, the TVS-version and the Banach-version of the McShane integral are equal. Kurzweil et al. [7] discussed the idea of equi-integrability, a convergence theorem for McShane integrable sequences of functions is demonstrated to be equal to the Vitali convergence theorem. One can see [5, 8, 9] and their references therein for different convergence results of different integrals.

2020 *Mathematics Subject Classification*. Primary 26A39, 28A12.
Keywords. μ-McShane integral, monotone convergence theorem, μ-equi-integrability, μ-uniformly absolutely continuity.
Received: 04 October 2024; Accepted: 12 December 2024
Communicated by Dragan S. Djordjević
* Corresponding author: Hemanta Kalita
Email addresses: hemanta30kalita@gmail.com (Hemanta Kalita), hemen.bharali@dbuniversity.ac.in (Hemen Bharali), pujadeysarkar8@gmail.com (Puja Dey Sarkar)
ORCID iDs: https://orcid.org/0000-0002-9798-6608 (Hemanta Kalita), https://orcid.org/0000-0002-4800-9536 (Hemen Bharali)

A Henstock-Kurzweil type integral is studied by Corrao et al.[2] on a complete metric measure space **X**, which is equipped with a Radon measure μ and a family of cells that satisfy the Vitali covering theorem in relation to μ. Using a family of cells that fulfils the Vitali covering theorem with respect to Radon measure μ, H. Kalita et al. [6] introduced a McShane type integral on a complete metric space. Certain basic characteristics of such integrals are examined, including the Saks-Henstock type lemma in terms of additive functions. The relationship between the Lebesgueintegrals and μ-McShane integral is established.

The work of Kurzweil et al. [7] of McShane equi-integrability and Vitali's convergence theorem motivated us to investigate: several convergence theorem of μ-McShane integrable sequences of functions based on the concept of equi-integrability, that is equivalent to the Vitali convergence theorem.

The manuscript is organized as follows: in Section 2 the basic concepts and terminology are introduced together with some definitions and results of μ-McShane integrals. In Section 3, we discuss monotone convergence theorem for μ-McShane integrable functions on a complete metric space **X**. Several convergence theorems are presented with μ-equi-integrability, and μ-uniformly continuity. In Section 4, for each sequence of μ-McShane integrable functions, we provide a necessary and sufficient condition of μ-equi-integrability with respect to μ-uniformly absolutely continuity.

2. Preliminaries

Let $(\mathbf{X}, d)$ be a metric space. The diameter of a non-empty subset A of **X** is $diam(A) = \sup\{d(x, y) : x, y \in A\}$. For each element $x \in \mathbf{X}$ and for each A & B non empty subsets of **X**, the distance from x to A and distance between A & B are defined as

$$d(x, A) = \inf\{d(x, y) : y \in A\} \text{ and } d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Definition 2.1. *A sequence (x_n) of elements of **X** is said to be a fundamental or Cauchy sequence if for any $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq k$. It is well known that every Cauchy sequence may not convergent.*

Example 2.2. *Let $\mathbf{X} = \mathbb{Q}$ be set of rational numbers and $d(x, y) = |x - y|$ for all $x, y \in \mathbf{X}$ be the usual metric on **X**. Consider a sequence (x_n) represented in decimal system such that $x_n = 1.a_1a_2....a_n$ is the largest rational number satisfying $x_n^2 < \sqrt{2}$. Then we have the following sequence of rational numbers: $x_1 = 1.4$, $x_2 = 1.41$, $x_3 = 1.414$, $x_4 = 1.4142, ...$ Then $d(x_m, x_n) \to 0$ as $n \to \infty$. So (x_n) is a Cauchy sequence and converges to $\sqrt{2} \notin \mathbb{Q}$.*

Definition 2.3. *A metric space $(\mathbf{X}, d)$ is said to be complete if and only if every Cauchy sequence of elements of **X** converges to some elements of **X** in the space.*

Throughout our work, we assume $\mathbf{X} = (\mathbf{X}, d)$ to be a complete metric space. Suppose that **C** is an arbitrary set of subsets of **X**. The smallest σ-algebra $\sigma(\mathbf{C})$ containing **C** is called the σ-algebra generated by **C**. Let $\mathcal{M}$ be a σ-algebra of subsets of a set **X**. Recall a positive function $\mu : \mathcal{M} \to [0, +\infty]$ is called a measure if

1. $\mu(\emptyset) = 0$;
2. $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum\limits_{j=1}^{\infty} \mu(A_j)$ for every sequences $\{A_j\}_j$ of pointwise disjoint sets from $\mathcal{M}$.

Then $(\mathbf{X}, \mathcal{M}, \mu)$ is called a measure space. We denote a family of μ-measurable functions by $\mathcal{M}_\mu$. Suppose U is the Borel σ-algebra of **X**. Recall that a measure μ is called a Radon measure if μ is a Borel measure with the following properties:

1. $\mu(\mathbf{K}) < \infty$ for every compact set $\mathbf{K} \subset \mathbf{X}$.
2. $\mu(V) = \sup\{\mu(K) : K \subset V, \mathbf{K} \text{ is compact}\}$ for every open set $\mathbf{V} \subset \mathbf{X}$;
3. $\mu(A) = \inf\{\mu(\mathbf{V}) : A \subset \mathbf{V}, \mathbf{V} \text{ is open}\}$ for every $A \subset \mathbf{X}$.

Let λ be a signed measure defined on the σ-algebra of all μ-measurable subsets of $\mathbf{X}$. Then λ is called absolutely continuous with respect to μ if $\mu(E) = 0$ implies $|\lambda|(E) = 0$ for each μ-measurable subset E of $\mathbf{X}$. It is denoted by $\lambda << \mu$. In the entire work, we consider μ a non-atomic Radon measure and $\mathbf{D}$ is a family of non-empty closed subsets of $\mathbf{X}$. For $E \subset \mathbf{X}$, we denote the indicator function, diameter, interior and the boundary of E by χ_E, $diam(E)$, E^0 and ∂E, respectively. Throughout the article we denote $d(x, E)$, the distance from x to E. Recall that $\mathbf{Q}_1,\ \mathbf{Q}_2 \in \mathbf{D}$ are said to be non overlapping if interiors of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are disjoint. A finite collection $\left\{\mathbf{Q}_1,\ \mathbf{Q}_2, ..., \mathbf{Q}_m\right\}$ of pairwise non-overlapping elements of $\mathbf{D}$ is a division of $\mathbf{Q}$ if $\bigcup_{i=1}^{m} \mathbf{Q}_i = \mathbf{Q}$. Let $\mathbf{G}$ be a sub family of $\mathbf{D}$, then $\mathbf{G}$ is called a fine cover of $E \subset \mathbf{X}$ if

$$\inf\left\{\text{diam}(\mathbf{Q}) :\ \mathbf{Q} \in \mathbf{G},\ x \in \mathbf{Q}\right\} = 0 \text{ for each } x \in E.$$

Definition 2.4. *[2, Definition 2.14] We say $\mathbf{D}$ is a μ-Vitali family if for each subset E of $\mathbf{X}$ and for each subfamily $\mathbf{G}$ of $\mathbf{D}$ that is a fine cover of E, there exists a countable system $\left\{\mathbf{Q}_1, \mathbf{Q}_2, .., \mathbf{Q}_j, ..\right\}$ of pairwise non-overlapping elements of $\mathbf{G}$ such that $\mu(E \setminus \cup \mathbf{Q}_j) = 0$.*

Definition 2.5. *[2] Let $\mathbf{F}$ be a μ-Vitali family. We say $\mathbf{F}$ is a family of μ-cells if it satisfies the following conditions:*

(a) Given $\mathbf{Q} \in \mathbf{F}$ and a constant $\delta > 0$, there exists a division $\left\{\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_m\right\}$ of $\mathbf{Q}$, such that $diam(\mathbf{Q}_i) < \delta$, for $i = 1, 2, .., m$;

(b) Given $A,\ \mathbf{Q} \in \mathbf{F}$ and $A \subset \mathbf{Q}$, there exists a division $\left\{\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_m\right\}$ of $\mathbf{Q}$, such that $A = \mathbf{Q}_1$;

(c) $\mu(\partial\mathbf{Q}) = 0$ for each $\mathbf{Q} \in \mathbf{F}$.

3. μ-McShane integral with Radon measure

In this section, we recall μ- McShane integral with respect to a Radon measure. We recall that, a gauge on a set $\mathbf{Q}$ is any positive real function ν defined on $\mathbf{Q}$.

Definition 3.1. *[6] Let $\mathbf{Q} \in \mathbf{F}$, let $E \subset \mathbf{Q}$ and ν be a gauge on $\mathbf{Q}$. A collection $\mathbf{P} = \left\{(x_i, \mathbf{Q}_i)\right\}_{i=1}^{m}$ of finite ordered pairs of points and cells is said to be:*

1. *a free partition of $\mathbf{Q}$ if $\left\{\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_m\right\}$ is a division of $\mathbf{Q}$ and $x_i \in \mathbf{Q}$ for $i = 1, 2, ..., m$;*
2. *a free partial partition on $\mathbf{Q}$ if $\left\{\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_m\right\}$ is a subsystem of a division of $\mathbf{Q}$ and $x_i \in \mathbf{Q}$ for $i = 1, 2, ..., m$;*
3. *ν-fine if $diam(\mathbf{Q}_i) < \nu(x_i)$ for $i = 1, 2, ..., m$;*
4. *E-tagged if the points $x_1, x_2, x_3, ..., x_m$ belongs to E.*

The following Cousin's type lemma addresses the existence of ν-fine free partitions of a given cell $\mathbf{Q}$.

Lemma 3.2. *[6] If ν is a gauge on $\mathbf{Q}$, then there exists a ν-fine free partition of $\mathbf{Q}$.*

Let $f : \mathbf{Q} \to \mathbb{R}$ be a given function. If $P = \left\{(x_i, \mathbf{Q}_i)\right\}_{i=1}^{m}$ is any partition of $\mathbf{Q} \in \mathbf{F}$, we define the Riemann sum as $\mathbf{S}(f, P) = \sum_{i=1}^{m} f(x_i)\mu(\mathbf{Q}_i)$. Next we recall the definition of μ-McShane integral on $\mathbf{Q}$ as follows:

Definition 3.3. *[6] A function $f : \mathbf{Q} \to \mathbb{R}$ is called μ-McShane integrable on a cell $\mathbf{Q}$ with respect to μ if there exists a real number l such that for each $\epsilon > 0$ there is a gauge ν on $\mathbf{Q}$ so that $|S(f, P) - l| < \epsilon$ whenever P is a free tagged partition of $\mathbf{Q}$ that is ν-fine.*

We write $l = \int_{\mathbf{Q}} f d\mu$. Throughout the work, we consider $M_\mu(\mathbf{Q})$ to be a family of μ-McShane integrable functions on $\mathbf{Q}$. Clearly the number l is unique. Additionally, it is known that every μ-McShane integrable function is μ-Henstock-Kurzweil integrable on a cell $\mathbf{Q}$ and that the integrals are equal. Few simple properties of μ-McShane integrals are as follows.

Theorem 3.4. *[6]*

1. *Let f, g are μ-McShane integrable on $\mathbf{Q}$ then $f + g$ are μ-McShane integrable on $\mathbf{Q}$, and $\int_{\mathbf{Q}}(f + g)d\mu = \int_{\mathbf{Q}} f d\mu + \int_{\mathbf{Q}} g d\mu$.*
2. *If f is μ-McShane integrable on $\mathbf{Q}$ and $k \in \mathbb{R}$, then kf is also μ-McShane integrable on $\mathbf{Q}$ and $\int_{\mathbf{Q}} kf d\mu = k\int_{\mathbf{Q}} f d\mu$.*
3. *If f is μ-McShane integrable on $\mathbf{Q}$ and $f(x) \geq 0$ for each $x \in \mathbf{Q}$, then $\int_{\mathbf{Q}} f d\mu \geq 0$.*

Corollary 3.5. *[6] Let f, g are μ-McShane integrable functions on $\mathbf{Q}$. If $f \geq g$ for each $x \in \mathbf{Q}$, then $\int_{\mathbf{Q}} f d\mu \geq \int_{\mathbf{Q}} g d\mu$.*

Theorem 3.6. *[6]* (The Cauchy Criterion) A function $f : \mathbf{Q} \to \mathbb{R}$ is μ- McShane integrable on $\mathbf{Q}$ if and only if for each $\epsilon > 0$ there exists a gauge ν on $\mathbf{Q}$ such that $|S(f, P_1) - S(f, P_2| < \epsilon$ for each pair ν-fine free partitions P_1 and P_2 of $\mathbf{Q}$.

Theorem 3.7. *If f is μ-McShane integrable, and if A is a subcell of $\mathbf{Q}$, then $\int_A f d\mu = \int_{\mathbf{Q}} f\chi_A d\mu$.*

Theorem 3.8. *[6, Theorem 4.4] A function $f : \mathbf{Q} \to \mathbb{R}$ is μ-McShane integrable on $\mathbf{Q}$ if and only if there exists an additive cell function π defined on the family of all subcells of $\mathbf{Q}$ such that for each $\epsilon > 0$ there exists a gauge ν on $\mathbf{Q}$ with*

$$\sum_{(x_i, \mathbf{Q}_i) \in P} |\pi(\mathbf{Q}_i) - f(x_i)\mu(\mathbf{Q}_i)| < \epsilon$$

for each ν-fine free tagged partition P of $\mathbf{Q}$. In this situation π is the indefinite μ-McShane integral of f on $\mathbf{Q}$

Theorem 3.9. (Saks-Henstock Lemma) A function $f : \mathbf{Q} \to \mathbb{R}$ is μ-McShane integrable on $\mathbf{Q}$ if and only if there exists an additive cell function π defined on the family of all subcells of $\mathbf{Q}$ such that for each $\epsilon > 0$ there exists a gauge ν on $\mathbf{Q}$ with

$$\sum_{(x_i, \mathbf{Q}_i) \in P} \left| \pi(\mathbf{Q}_i) - f(x_i)\mu(\mathbf{Q}_i) \right| < \epsilon,$$

for each ν-fine free tagged partition P of $\mathbf{Q}$.

Theorem 3.10. *Let $f : \mathbf{Q} \to \mathbb{R}$ be μ-McShane integrable on $\mathbf{Q}$, then $|f|$ is also μ-McShane integrable on $\mathbf{Q}$.*

Theorem 3.11. *[6, Theorem 4.6] If $f : \mathbf{Q} \to \mathbb{R}$ is μ-McShane integrable then f is Lebesgue integrable.*

4. Convergence of μ-McShane Integral

We have already seen the Monotone Convergence Theorem and the Dominated Convergence Theorem for the Lebesgue integral (see [3]). Monotone convergence theorem for McShane integral ([3, Theorem 10.10], Convergence of McShane integrable sequences of functions based on the concept of equi-integrability is proved and it is shown that this theorem is equivalent to the Vitali convergence theorem in [7]. In this Section, we discuss the monotone convergence theorem of μ-McShane integral. We further discuss the convergence theorem of μ-McShane integral based on the concept of μ-equi-integrability on $\mathbf{Q}$. We start the Section with the following theorem.

Theorem 4.1. *Let $\{f_k\}$ be a non-decreasing sequence of μ-McShane integrable function on a cell* $\mathbf{Q}$ *and* $f = \lim_k f_k$. *If* $f = \lim_{k\to\infty}(\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu < \infty$ *then* f *is* μ*-McShane integrable on* $\mathbf{Q}$ *and* $(\mu\text{-}M)\int_{\mathbf{Q}} f d\mu = \lim_{k\to\infty}(\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu$.

Proof. Let $\{f_k\}$ be a non-decreasing sequence. Since $\{(\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu\}_k$ is bounded on $\mathbf{Q}$. So $\{(\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu\}_k$ converges to a real number $A \in \mathbb{R}$. Then given $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that $k \geq K$ we have $0 \leq A - (\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu \leq \epsilon$. From the Theorem 3.8 there exists an additive function π on the subcells of $\mathbf{Q}$ such that for all $\epsilon > 0$, there exists a guage ν on $\mathbf{Q}$ with

$$\sum_{(x_i,\mathbf{Q}_i)\in P} |\pi(\mathbf{Q}_i) - f_k(x_i)\mu(\mathbf{Q}_i)| < \frac{\epsilon}{2^k}$$

for each ν-fine free tagged partition P of $\mathbf{Q}$ and $\pi(\mathbf{Q}_i) = (\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu$. Since $\lim_{k\to\infty}(\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu = (\mu\text{-}M)\int_{\mathbf{Q}} f d\mu$, so for each $x \in \mathbf{Q}$ there exists a natural number $n(x) \geq K$ such that.

$$|f(x) - f_k(x)| < \epsilon$$

whenever $k \geq k(x) \geq K$. If $\nu(x) = \nu_{k(x)}$ for $x \in \mathbf{Q}$, then ν is the gauge of $\mathbf{Q}$. Suppose $P = \{(\mathbf{Q}_1, x_1), (\mathbf{Q}_2, x_2), ..., (\mathbf{Q}_k,\ x_k)\}$ be a free tagged partition in $\mathbf{Q}$, Then we have

$$\sum_{i=1}^{k} |f(x_i) - f_{k(x_i)}|\mu(\mathbf{Q}_i) < \epsilon\mu(\mathbf{Q}).$$

Also,

$$\left|\sum_{i=1}^{k} f_{k(x_i)}(x_i)\mu(\mathbf{Q}_i) - \sum_{i=1}^{k} f_{k(x_i)} d\mu\right| \leq \sum_{i=1}^{k}\left|f_{k(x_i)}(x_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M)\int_{Q} f_{k(x_i)} d\mu\right| < \epsilon.$$

Next from the hypothesis $f = \lim_k f_k$. This implies that $\{f_k\}$ is a pointwise bounded sequence of functions. Hence,

$$\begin{aligned}(\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu &= \sum_{i=1}^{k}(\mu\text{-}M)\int_{\mathbf{Q}_i} f_k d\mu \\ &\leq \sum_{i=1}^{k} f_{k(x_i)} d\mu.\end{aligned}$$

Also,

$$0 \leq A - \sum_{i=1}^{k}(\mu\text{-}M)\int_{\mathbf{Q}_i} f_{k(x_i)} d\mu \leq (\mu\text{-}M)\int_{\mathbf{Q}} f_k d\mu < \epsilon.$$

Finally we have,

$$\begin{aligned}|S(f, P) - A| &\leq |\sum_{i=1}^{k} f(x_i)\mu(\mathbf{Q}_i) - \sum_{i=1}^{k} f_{k(x_i)}(x_i)\mu(\mathbf{Q}_i)| \\ &\quad + |\sum_{i=1}^{k} f_{k(x_i)}(x_i)\mu(\mathbf{Q}_i) - \sum_{i=1}^{k}(\mu\text{-}M)\int_{\mathbf{Q}_i} f_{n(x_i)} d\mu| \\ &\quad + |\sum_{i=1}^{k}(\mu\text{-}M)\int_{\mathbf{Q}_i} f_{k(x_i)} d\mu - A| \\ &< \epsilon\mu(\mathbf{Q}) + \epsilon + \epsilon.\end{aligned}$$

Since ϵ is arbitrary, f is μ-McShane integrable on $\mathbf{Q}$ where $A = (\mu\text{-}M)\int_{\mathbf{Q}} f d\mu$. □

Theorem 4.2. *(Egoroff's Theorem) Let $E \subset \mathbf{Q}$ be a measurable cell. Let $\{f_k\}$ be a measurable sequence of μ-McShane integrable functions in E. If $\{f_k\}$ converges pointwise a.e. on E to a function f, then for each $\epsilon > 0$ there exists a measurable subcell $H \subseteq E$ such that $\mu(E \setminus H) < \epsilon$ and $\{f_k\}$ converges uniformly to f in H.*

Proof. The proof is very standard. So we have ommitted the proof. □

Next we define μ-equi-integrability as follows:

Definition 4.3. *A family of μ-measurable cell function $\mathcal{M}_\mu$ of $f : \mathbf{Q} \to \mathbb{R}$ is called μ-equi-integrable if for $f \in \mathcal{M}_\mu$ is μ-McShane integrable and for every $\epsilon > 0$ there is a gauge ν such that for any $f \in \mathcal{M}_\mu$,*

$$|\sum_i f(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M)\int_{\mathbf{Q}} f d\mu| < \epsilon$$

hold for each ν-fine free tagged partition $P = \{(\mathbf{Q}_i, x_i)\}_{i=1}^k$ of $\mathbf{Q}$ and $k \in \mathbb{N}$.

Next, we discuss a necessary and sufficient condition of μ-equi-integrability.

Theorem 4.4. *A family $\mathcal{M}_\mu$ is μ-equi-integrable if and only if for every $\epsilon > 0$ there exists a gauge ν such that*

$$\left|\sum_i f(t_i)\mu(\mathbf{Q}_i) - \sum_j f(s_j)\mu(\mathbf{S}_j)\right| < \epsilon$$

for every ν-fine free tagged partitions $\{(\mathbf{Q}_i, t_i)\}$ and $\{(\mathbf{S}_j, s_j)\}$ of $\mathbf{Q}$ and $f \in \mathcal{M}_\mu$.

Proof. Let us consider a family $\mathcal{M}_\mu$ of μ-equi-integrable function. Then by definition of μ-equi-integrability: for every $\epsilon > 0$ there is a gauge ν such that for any $f \in \mathcal{M}_\mu$,

$$|\sum_i f(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M)\int_{\mathbf{Q}} f d\mu| < \frac{\epsilon}{2}$$

hold for each ν-fine free tagged partition $P = \{(\mathbf{Q}_i, x_i)\}_{i=1}^k$ of $\mathbf{Q}$ and $k \in \mathbb{N}$. Let us consider another ν-fine free tagged partition $P' = \{(\mathbf{S}_j, x_j)\}_{j=1}^m$ of $\mathbf{Q}$ and $m \in \mathbb{N}$. Then $\left|\sum\limits_j f(s_j)\mu(\mathbf{S}_j) - (\mu\text{-}M)\int_{\mathbf{S}} f d\mu\right| < \frac{\epsilon}{2}$. So,

$$\begin{aligned}
&\left|\sum_i f(t_i)\mu(\mathbf{Q}_i) - \sum_j f(s_j)\mu(\mathbf{S}_j)\right| \\
&= \left|\sum_i f(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M)\int_{\mathbf{Q}} f d\mu + (\mu\text{-}M)\int_{\mathbf{Q}} f d\mu - \sum_j f(s_j)\mu(\mathbf{S}_j)\right| \\
&\leq |\sum_i f(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M)\int_{Q} f d\mu| + \left|\sum_j f(s_j)\mu(\mathbf{S}_j) - (\mu\text{-}M)\int_{\mathbf{S}} f d\mu\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Conversely, for each $k \in \mathbb{N}$, let ν_k be a gauge on $\mathbf{Q}$ such that

$$\left|\sum_i f(t_i)\mu(\mathbf{Q}_i) - \sum_j f(s_j)\mu(\mathbf{S}_j)\right| < \frac{1}{k}$$

for every ν_k-fine free tagged partitions $\{(\mathbf{Q}_i, t_i)\}$ and $\{(\mathbf{S}_j, s_j)\}$ of $\mathbf{Q}$ and $f \in \mathcal{M}_\mu$. Let $\Delta_k(x) = \min\{\nu_1(x), \nu_2(x), ..., \nu_k(x)\}$ be a gauge on $\mathbf{Q}$. By [6, Lemma 3.2], there exists a Δ_k-fine free partition P_k of $\mathbf{Q}$, for every $k \in \mathbb{N}$. Let $\epsilon > 0$ be a given and choose a positive natural number N such that $\frac{1}{N} < \frac{\epsilon}{2}$. If m and n are positive natural number ($n < m$) such that $n \geq N$, then $\left|\sum\limits_n f(t_n)\mu(\mathbf{Q}'_n) - \sum\limits_m f(t_m)\mu(\mathbf{Q}'_m)\right| < \frac{\epsilon}{2}$ whenever

$\{(\mathbf{Q}'_n, t_n)\}$ and $\{(\mathbf{Q}'_m, t_m)\}$ are Δ_k fine free tagged partitions of $\mathbf{Q}$. Consequently $\left\{ \sum_n f(t_n)\mu(\mathbf{Q}'_n) \right\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers and hence $A = \lim_n \Big(f(t_n)\mu(\mathbf{Q}'_n) \Big)$. Clearly $\left| \sum_n f(t_n)\mu(\mathbf{Q}'_n) - A \right| < \frac{\epsilon}{2}$ for each $n \geq N$. Let P be a Δ_k-fine free tagged partitions on $\mathbf{Q}$, then

$$\begin{aligned} &\left| \sum_i f(t_i)\mu(\mathbf{Q}_i) - A \right| \\ &\leq \left| \sum_i f(t_i)\mu(\mathbf{Q}_i) - \sum_n f(t_n)\mu(\mathbf{Q}'_n) \right| + \left| \sum_n f(t_n)\mu(\mathbf{Q}'_n) - A \right| < \epsilon. \end{aligned}$$

So, $f \in \mathcal{M}$ is μ-McShane integrable and $A = (\mu\text{-}M) \int_{\mathbf{Q}} f d\mu$. □

If the sequence f_k is μ- equi-integrable on $\mathbf{Q}$ then the Theorem 4.1 can be stated as follows:

Theorem 4.5. *Let $\{f_k\}$ be a μ-equi-integrable,non-decreasing sequence of μ-McShane integrable function on a cell $\mathbf{Q}$ and $f = \lim_k f_k$. If $f = \lim_{k\to\infty} (\mu\text{-}M) \int_{\mathbf{Q}} f_k d\mu < \infty$ then f is μ-McShane integrable on $\mathbf{Q}$ and $(\mu\text{-}M) \int_{\mathbf{Q}} f d\mu = \lim_{k\to\infty} (\mu\text{-}M) \int_{\mathbf{Q}} f_k d\mu$*

Proof. From Definition 4.3 we have that for each $\epsilon > 0$ there exists a free tagged partition P of $\mathbf{Q}$ such that

$$|S(f_k, P) - (\mu\text{-}M) \int_{\mathbf{Q}} f_k d\mu| < \epsilon$$

hold for each ν-fine free tagged partition $P = \{(\mathbf{Q}_i, x_i)\}_{i=1}^{k}$ of $\mathbf{Q}$ and $k \in \mathbb{N}$. If we fix P, then $\lim_{k\to\infty} f_k(x) = f(x)$, and for $u_0 \in \mathbf{N}$ we have

$$|S(f_k, P) - S(f_u, P)| < \epsilon$$

for all $k, u > u_0$. This implies $\left|(\mu\text{-}M) \int_{\mathbf{Q}} f_k d\mu - (\mu\text{-}M) \int_{\mathbf{Q}} f_u d\mu\right| < \epsilon$. Therefore $\left\{(\mu\text{-}M) \int_{\mathbf{Q}} f_k d\mu\right\}$ is Cauchy and $\lim_{k\to\infty} (\mu\text{-}M) \int_{\mathbf{Q}} f_k d\mu = L \in \mathbf{R}$ exists. This implies that

$$|(\mu\text{-}M) \int_{\mathbf{Q}} f_k d\mu - L| < \epsilon$$

For $u_1 \in \mathbf{N}$ with $k > m_1$. Let any ν-fine free tagged partition P of $\mathbf{Q}$ be $\{(\mathbf{Q}, t)\}$. Since $\lim_{k\to\infty} f_k(x) = f(x)$ then there exists $u_2 > u_1$ such that $|S(f_{u_2}, P_{\mathbf{Q}}) - S(f, P)| < \epsilon$.

In this case $|S(f, P_{\mathbf{Q}}) - L| < \epsilon$. Proceeding this way, using mathematical induction we can find f is μ-McShane intergal on $\mathbf{Q}$ and $(\mu\text{-}M) \int_{\mathbf{Q}} f d\mu = \lim_{k\to\infty} (\mu\text{-}M) \int_{\mathbf{Q}} f_k d\mu$.
□

We now deduce the convergence theorem of μ-McShane integrals under the conditions of uniformly continuous, uniform convergence and pointwise boundedness. We define μ-uniformly absolutely continuous μ-McShane integrable on $\mathbf{Q}$ as follows:

Definition 4.6. *Let $\mathcal{M}_\mu$ be a family of μ-McShane integrable function $f : \mathbf{Q} \to \mathbb{R}$. If for every $\epsilon > 0$ there is a $\delta > 0$ such that for $\tau \subset \mathbf{Q}$ with $\mu(\tau) < \delta$ and $\left|(\mu\text{-}M) \int_\tau f\right| < \epsilon$. Then $\mathcal{M}_\mu$ is called μ-uniformly absolutely continuous.*

Following theorems are based on μ-uniformly absolutely continuous functions.

Theorem 4.7. *Let $\{f_k\}$ be a sequence of μ-McShane integrable functions defined on $\mathbf{Q}$ and let $F_k(x) = \int_E f_k$ for each k. Then $\{F_k\}$ is μ-uniformly absolutely continuous with respect to radon measure on $\mathbf{Q}$ whenever $E \subset \mathbf{Q}$.*

Proof. Let $\{F_k\}$ be a μ-uniformly absolutely continuous on $\mathbf{Q}$ and let $\epsilon > 0$. By hypothesis, there exists a $\delta > 0$ such that $\Big|\sum_{i=1}^{q}(F_k(\mathbf{Q}_i)\Big| < \epsilon$ for all k whenever $\Big\{\mathbf{Q}_i : 1 \leq i \leq q\Big\}$ is finite collection of cells in $\mathbf{Q}$ with $\sum_{i=1}^{q}\mu(\mathbf{Q}_i) < \delta$. Let $E \subset \mathbf{Q}$ with $\mu(E) < \delta$. Fix n. Choose $\beta > 0$ so that $(\mu\text{-}M)\int_E |f_k| < \epsilon$ whenever $E \subset \mathbf{Q}$ with $\mu(E) < \beta$. Next an analogous to [3, Theorem 1.13], we can find a finite collection $\Big\{\mathbf{Q}_i : 1 \leq i \leq q\Big\}$ of disjoint cells in $\mathbf{Q}$ such that $\mathbf{Q} = \bigcup_{i=1}^{q}\mathbf{Q}_i$ satisfies $\mu(\mathbf{Q}\Delta E) < \min\Big\{\beta, \delta - \mu(E)\}$. Since $\mu(\mathbf{Q}) < \delta$ and $\mathbf{Q} \subseteq E \cup (\mathbf{Q}\Delta E)$ so,

$$\begin{aligned}
\Big|(\mu\text{-}M)\int_E f_n\Big| &= \Big|(\mu\text{-}M)\int_{\mathbf{Q}} f_k + (\mu\text{-}M)\int_{E\setminus\mathbf{Q}} f_k - (\mu\text{-}M)\int_{\mathbf{Q}\setminus E} f_k\Big| \\
&\leq \Big|(\mu\text{-}M)\int_{\mathbf{Q}} f_k\Big| + (\mu\text{-}M)\int_{E\setminus\mathbf{Q}} |f_k| + (\mu\text{-}M)\int_{\mathbf{Q}\setminus E} |f_k| \\
&= \Big|\sum_{i=1}^{q} F_k(\mathbf{Q})\Big| + (\mu\text{-}M)\int_{\mathbf{Q}\Delta E} |f_k| \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Hence $\Big\{(\mu\text{-}M)\int f_k\Big\}$ is μ-uniformly absolutely continuous on $\mathbf{Q}$ with respect to the measure μ. Hence the complete proof. □

Theorem 4.8. (Relaxed Vitali convergence) Assume that a sequence of μ-McShane integrable functions $f_k : \mathbf{Q} \to \mathbb{R}$, $n \in \mathbb{N}$ is given such that f_k is μ-convergence to f. Let $\Big\{f_k : k \in \mathbb{N}\Big\}$ is μ-uniformly absolutely continuous then f is μ-McShane integrable and $\lim_{k\to\infty}(\mu\text{-}M)\int_{\mathbf{Q}} f_k = (\mu\text{-}M)\int_{\mathbf{Q}} f$.

Proof. By Theorem 4.7, for $\epsilon > 0$, there exists $\delta > 0$ such that $(\mu - M)\int_E |f_k| < \epsilon$ for all k whenever $E \subset \mathbf{Q}$ with $\mu(E) < \delta$. If $\mu(E) < \delta$, by Fatou's Lemma $(\mu\text{-}M)\int_E |f| \leq \lim_{k\to\infty}\inf(\mu\text{-}M)\int_E |f_k| \leq \varepsilon$. By Egoroff's Theorem, there exist a measurable cell $E \subset \mathbf{Q}$ and a positive integer N such that $|f_k(x) - f(x)| < \epsilon \ \forall\ n \geq N$ for all $x \in E$ and $\mu(\mathbf{Q}') < \delta$ where $\mathbf{Q}' = \mathbf{Q} \setminus E$. Consequently for each $n \geq N$,

$$\begin{aligned}
&\Big|(\mu\text{-}M)\int_{\mathbf{Q}} f_k - (\mu\text{-}M)\int_{\mathbf{Q}} f\Big| \\
&\leq (\mu\text{-}M)\int_{\mathbf{Q}} |f_k - f\Big| \\
&\leq (\mu\text{-}M)\int_E |f_k - f| + (\mu\text{-}M)\int_{\mathbf{Q}'} |f_k| + (\mu\text{-}M)\int_{\mathbf{Q}'} |f| \\
&< \epsilon\mu(\mathbf{Q}) + \epsilon + \epsilon.
\end{aligned}$$

So, $(\mu\text{-}M)\int_{\mathbf{Q}} f = \lim_{k\to\infty}(\mu\text{-}M)\int_{\mathbf{Q}} f_k$. □

Corollary 4.9. *Let $f_k : \mathbf{Q} \to \mathbb{R}$, $k \in \mathbb{N}$ be a given sequence of μ-McShane integrable functions that converges to f pointwise in $\mathbf{Q}$. If the set $\{f_k : k \in \mathbb{N}\}$ is μ-uniformly absolutely continuous then the function f is μ-McShane integrable and $\lim_{k\to\infty}(\mu\text{-}M)\int_{\mathbf{Q}} f_k = (\mu\text{-}M)\int_{\mathbf{Q}} f$.*

5. Main result

In this Section, we give a necessary and sufficient condition of μ-equi-integrability of sequence of μ-McShane integrable function with respect to μ-uniformly absolutely continuity. We start this Section with the following Proposition.

Proposition 5.1. *Consider $f_k : \mathbf{Q} \to \mathbb{R}$, $k \in \mathbb{N}$ be a sequence of μ-McShane integrable functions such that*

1. *$f_k(t) \to f(t)$ for $t \in \mathbf{Q}$;*
2. *The set $\{f_k : k \in \mathbb{N}\}$ is μ-uniformly absolutely continuous.*

Then the set $\{f_k : k \in \mathbb{N}\}$ is μ-equi-integrable.

Proof. Let $f_k(t) \to f(t)$ for $t \in \mathbf{Q}$. By Egoroff's Theorem, for every $i \in \mathbb{N}$ we can find a measurable subcell $E_i \subset \mathbf{Q}$ such that $\mu(\mathbf{Q} \setminus E_i) < \epsilon = \frac{1}{i}$ whenever $E_i \subset E_{i+1}$ and $f_k(t) \to f(t)$ uniformly for $t \in E_i$. That is for $\epsilon > 0$, and $t \in E_i$ we can find $k,\ K \in \mathbb{N}$ we have

$$|f_k(t) - f(t)| < \epsilon \text{ for } k > K. \tag{1}$$

Consider $\mathfrak{Q} = \mathbf{Q} \setminus \bigcup\limits_{i=1}^{\infty} E_i$. Then $\mu(\mathfrak{Q}) \le \mu(\mathbf{Q} \setminus E_i) < \frac{1}{i}$ gives $\mu(\mathfrak{Q}) = 0$. Let us consider for $k \in \mathbb{N}$ and $t \in \mathfrak{Q}$, $f_k(t) = f(t) = 0$. Since set $\{f_k : k \in \mathbb{N}\}$ is μ-uniformly absolutely continuous, for $\epsilon > 0$, there is a $i \in \mathbb{N}$ such that

$$(\mu\text{-}M) \int_{\mathbf{Q}\setminus E_i} |f_k| < \epsilon \text{ for all } k \in \mathbb{N}. \tag{2}$$

By Eqn. (1) and Eqn. (2),

$$\begin{aligned}
(\mu\text{-}M) \int_{\mathbf{Q}} |f_k - f_s| &= (\mu\text{-}M) \int_{E_i} |f_k - f_s| + (\mu\text{-}M) \int_{\mathbf{Q}\setminus E_i} |f_k - f_s| \\
&\le (\mu\text{-}M) \int_{E_i} |f_k - f| + (\mu\text{-}M) \int_{E_i} |f - f_s| \\
&+ (\mu\text{-}M) \int_{\mathbf{Q}\setminus E_i} |f_k| + (\mu\text{-}M) \int_{\mathbf{Q}\setminus E_i} |f_s| \\
&< 2\epsilon\mu(E_i) + 2\epsilon \\
&\le 2\epsilon\Big[\mu(\mathbf{Q}_i) + 1\Big].
\end{aligned}$$

So, $\{f_k\}$ is a Cauchy sequence of μ-McShane integrable functions and

$$\lim_{k\to\infty} (\mu\text{-}M) \int_{\mathbf{Q}} |f_k - f| = 0 \text{ for } k > K. \tag{3}$$

By definition of μ-McShane integral, there exists a gauge ν_1 on $\mathbf{Q}$ such that $|\sum\limits_i f(t_i)\mu(\mathbf{Q}) - (\mu\text{-}M) \int_{\mathbf{Q}} f| < \epsilon$. Further, there exists a gauge ν_2 such that $\Big|\sum\limits_i f_k(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M) \int_{\mathbf{Q}} f_k\Big| < \epsilon$ whenever ν_2 is a free tagged partition of $\mathbf{Q}$ whenever $k \le K$, K given by Eqn. (3). Similarly for any $j \in \mathbb{N}$ we have a gauge ν_j such that $\Big|\sum\limits_i f_k(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M) \int_{\mathbf{Q}} f_k\Big| < \frac{\epsilon}{2^j}$ for every ν_j fine free tagged partitions of $\mathbf{Q}$ and $k \le K_j$.

If $k > K$, using Saks-Henstock type Lemma for μ-McShane integrals we can find $\Big|\sum\limits_i f_k(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M) \int_{\mathbf{Q}} f_k\Big| < \epsilon$. Combining both cases of $k \le K$ and $k > K$, we have $\Big|\sum\limits_i f_k(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M) \int_{\mathbf{Q}} f_k\Big| < \epsilon$ for all $k \in \mathbb{N}$. Hence the set $\{f_k : k \in \mathbb{N}\}$ is μ-equi-integrable. □

Proposition 5.2. *Let $f_k : \mathbf{Q} \to \mathbf{R}$ be μ-McShane integrable function such that*

1. *$f_k(t) \to f(t)$ for $t \in \mathbf{Q}$;*
2. *The set $\{f_k : k \in \mathbb{N}\}$ is μ-equi-integrable.*

Then for every $\epsilon > 0$ there is a $\beta > 0$ such that for any finite class $\{\mathbf{Q}_j : j = 1, .., p\}$ of disjoint subcells of $\mathbf{Q}$ with $\sum\limits_j \mu(\mathbf{Q}_j) < \beta$ we $\Big|\sum\limits_j (\mu\text{-}M) \int_{\mathbf{Q}_j} f_k\Big| < \epsilon$, $k \in \mathbb{N}$.

Proof. The proof follows from Saks-Henstock Lemma of μ-McShane integrable functions. □

Theorem 5.3. *Let $f_k : \mathbf{Q} \to \mathbb{R}$, $k \in \mathbb{N}$ are μ-McShane integrable function such that*

1. $f_k(t) \to f(t)$ *for* $t \in \mathbf{Q}$;
2. *The set* $\{f_k : k \in \mathbb{N}\}$ *is μ-equi-integrable.*

Then $f_k.\chi_E$, $k \in \mathbb{N}$ is μ-equi-integrable when $E \subset \mathbf{Q}$.

Proof. Let $\epsilon > 0$ be given and let $\beta > 0$. Consider $E \subset \mathbf{Q}$. Then we can find $\mathfrak{E} \subset \mathbf{Q}$ closed, $\mathfrak{G} \subset \mathbf{Q}$ open such that $\mathfrak{F} \subset E \subset \mathfrak{G}$ with $\mu(\mathfrak{G} \setminus \mathfrak{F})) < \beta$. Let $\left\{(U_l, u_l)\right\}$ and $\left\{(V_m, v_m)\right\}$ are ν-fine free tagged partitions of $\mathbf{Q}$. If $u_l \in E$ then $U_l \subset \mathfrak{G}$; $\mathfrak{F} \subset int \bigcup_{u_l \in \mathfrak{F}} U_l$ and if $v_m \in E$ then $V_m \subset \mathfrak{G}$; $\mathfrak{F} \subset int \bigcup_{v_m \in \mathfrak{F}} V_m$. By Thereom 4.4, we have $\left|\sum_{l, u_l \in E} f_k(u_l)\mu(U_l) - \sum_{m, v_m \in E} f_k(m)\mu(V_m)\right| \leq \epsilon$. Therefore, $\left|\sum_l f_k(u_l)\chi_E(u_l)\mu(U_l) - \sum_m f_k(v_k)\chi_E(v_m)\mu(V_m)\right| \leq \epsilon$. □

Proposition 5.4. *Assume $f_k : \mathbf{Q} \to \mathbb{R}$, $k \in \mathbb{N}$ are μ-McShane integrable functions such that*

1. $f_k(t) \to f(t)$ *for* $t \in \mathbf{Q}$;
2. *The set* $\{f_k : k \in \mathbb{N}\}$ *is μ-equi-integrable.*

Then for every $\epsilon > 0$ there exists an $\beta > 0$ such that if $E \subset \mathbf{Q}$ with $\mu(E) < \beta$ then

$$\left|(\mu\text{-}M)\int_{\mathbf{Q}} f_k.\chi_E\right| = \left|(\mu\text{-}M)\int_E f_k\right| \leq 3\epsilon \text{ for every } k \in \mathbb{N}.$$

Proof. Consider $\epsilon > 0$ be given and $\beta > 0$. Let consider a measurable subcell E of $\mathbf{Q}$ with $\mu(E) < \beta$. Then we can find an open cell $\mathfrak{G} \subset \mathbf{Q}$ so that $E \subset \mathfrak{G}$ and $\mu(\mathfrak{G}) < \beta$. From our assumption $\{f_k : k \in \mathbb{N}\}$ is μ-equi-integrable. Now by the definition of μ-equi-integrability: for every $\epsilon > 0$ there is a gauge ν such that for any $\mathcal{M}$,

$$\left|\sum_i f(t_i)\mu(\mathbf{Q}_i) - (\mu\text{-}M)\int_{\mathbf{Q}} f d\mu\right| < \epsilon$$

hold for each ν-fine free tagged partition $P = \{(\mathbf{Q}_i, x_i)\}_{i=1}^n$ of $\mathbf{Q}$ and $n \in \mathbb{N}$. By Proposition 5.4, $(\mu\text{-}M)\int_{\mathbf{Q}} f_k.\chi_E$, $k \in \mathbb{N}$ exist and

$$\left|\sum_m f_k(v_m)\chi_E(v_m)\mu(V_m) - (\mu\text{-}M)\int_{\mathbf{Q}} f.\chi_E\right| \leq \epsilon$$

whenever $V_m \subset \mathfrak{G} \subset \mathbf{Q}$ and $\{(V_m, v_m)\}$ is any ν-fine free tagged partition of $\mathbf{Q}$. If $v_m \in E \subset \mathfrak{G}$ then $V_m \subset \mathfrak{G}$ and $\sum_{m, v_m \in E} \mu(V_m) \leq \beta$. As $\{(V_m, v_m) : v_m \in E\}$ is ν-fine free tagged partition, using Saks-Henstock Lemma of μ-McShane integral we have $\left|\sum_{m, v_m \in E}\left[f_k(v_m)\mu(V_m) - (\mu\text{-}M)\int_{V_m} f_k\right]\right| \leq \epsilon$. By Proposition 5.2, we have $\left|\sum_{m, v_m \in E}(\mu\text{-}M)\int_{V_m} f_k\right| \leq \epsilon$. So,

$$\begin{aligned}
\left|(\mu\text{-}M)\int_E f\right| &\leq \epsilon + \left|\sum_{m, v_m \in E} f_k(v_m)\mu(V_m)\right| \\
&\leq \epsilon + \left|\sum_{m, v_m \in E}\left[f_k(v_m)\mu(V_m) - (\mu\text{-}M)\int_{V_m} f_k\right]\right| + \left|\sum_{m, v_m \in E}(\mu\text{-}M)\int_{V_m} f_k\right| \\
&\leq \epsilon + 2\epsilon = 3\epsilon.
\end{aligned}$$

□

Finally, we state our necessary and sufficient condition of μ-equi-integrability with respect of μ-uniformly absolutely continuous as follows:

Theorem 5.5. *Let $f_k : \mathbf{Q} \to \mathbb{R}$, $k \in \mathbb{N}$ be μ-McShane integrable functions such that $f_k(t) \to f(t)$ for $t \in \mathbf{Q}$. Then the set $\{f_k : k \in \mathbb{N}\}$ forms an μ-equi-integrable sequence if and only if $\{f_k : k \in \mathbb{N}\}$ is μ-uniformly absolutely continuous. Consequently f is μ-uniformly absolutely continuous.*

Proof. Let $f_k : \mathbf{Q} \to \mathbb{R}$, $k \in \mathbb{N}$ be μ-McShane integrable functions such that $f_k(t) \to f(t)$ for $t \in \mathbf{Q}$. Let $\{f_k : k \in \mathbb{N}\}$ forms an μ-uniformly absolutely continuous sequence of μ-McShane integrable functions. By Proposition 5.1, the set $\{f_k : k \in \mathbb{N}\}$ is μ-equi-integrable.

Conversely, let $\{f_k : k \in \mathbb{N}\}$ is μ-equi-integrable where $f_k : \mathbf{Q} \to \mathbb{R}$ is μ-McShane integrable in nature with $f_k \to f(t)$ for $t \in \mathbf{Q}$. By Proposition 5.4, for every $\epsilon > 0$ there exists a $\beta > 0$ such that if $E \subset \mathbf{Q}$ with $\mu(E) < \beta$, we have

$$\left|(\mu\text{-}M) \int_{\mathbf{Q}} f_k.\chi_E\right| = \left|(\mu\text{-}M) \int_E f_k\right| \leq \epsilon \text{ for every } k \in \mathbb{N}.$$

□

6. Conclusion

With the help of the definition and properties of μ-equi-integrability, and μ-uniformly absolutely continuity, discussed on a complete metric space, endowned with a Radon measure μ and a family of cells that satisfies the Vitali covering theorem with respect to μ, we have explored several convergence results. A necessary and sufficient condition of μ-equi-integrability of sequence of μ-McShane integrable function with respect to μ-uniformly absolutely continuity has been presented.

Acknowledgments

The authors thank to the reviewers for their constructive suggestions for better presentation of the article.

Declaration

Funding: Not Applicable, the research is not supported by any funding agency.
Conflict of Interest/Competing interests: The authors declare that there is no conflicts of interest.
Availability of data and material: The article does not contain any data for analysis.
Code Availability: Not Applicable.
Authors Contributions: All the authors have equal contribution for the preparation of the article.

References

[1] T. P. Becerra, H. Kalita, J. A. E. Reyna, Vector-valued McShane and strong McShane integrals with the Alexiewicz norm and its Riesz representation theorems, Bollettino del. Mat. Italiana, In press, 2024.
[2] G. Corrao, D. Bongiorno, *An Henstock–Kurzweil Type Integral on a Measure Metric Space*, Universita Degli Studi Di Palermo Palermo, Italy, 2013
[3] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron and Henstock*, American Mathematical Society, 1994.
[4] R. A. Gordon, The McShane Integral of Banach-valued functions, Illinois J. Math. 34 (1990), 557–567.
[5] H. Kalita, Ravi P. Agarwal, B. Hazarika, Convergence of ap-Henstock-Kurzweil integral on locally compact spaces, Czec. M. Journal, (2023),1–19.
[6] H. Kalita, H. Bharali, S. Kalita, A McShane type integral on complete metric measure space, Filomat, 38(15), (2024), 1–9.
[7] J. Kurzweil, S. Schwabik, McShane equi-integrability and Vitali's convergence theorem, Math. Bohemica, 129(2), (2004), 141–157.
[8] E. Mema, Equiintegrability and Controlled Convergence for the Henstock-Kurzweil Integral, Int. Math. Forum, 8(19), (2013), 913–919.
[9] J. M. Park, H. Deok, J. H. Yoon, B. M. Kim, The convergence theorems for the AP-integral, J. Chung. Math. Soc., 1(12),(1999), 113–118.
[10] R. N. Paluga, S. R. Canoy Jr., On the McShane integral in topological vector spaces, Chamchuri J. Math., 2(2), (2010), 45–56.