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Certain mathematical formulas for moments of Geometric distribution by means of the Apostol-Bernoulli polynomials

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Abstract. Even though the first and second order moments of Geometric distribution are easy to compute, the higher order moments of this distribution are very complicated to derive and calculate. There is an immense need for statisticians to calculate these moments manually or programmatically using statistical softwares. The aim of this paper is to identify certain mathematical formulas. For derivation of the moments, factorial moments, and moment generating function of Geometric distribution, we apply the generating functions for the Apostol-Bernoulli number and polynomials. These newly derived moment formulas are linked to Stirling numbers and other special functions. In addition, when we apply the z-transform to the probability distribution of Geometric distribution, we obtain some new computational formulas that can help calculating the higher order factorial moments of Geometric distribution. These new formulas give alternative way of calculating them without dealing with the cumbersome derivations moments and moment generating function of this distribution.

1. Introduction

The Geometric distribution gives the probability that the first occurrence of success in k independent trials, each with success probability p. This makes it very widely used in statistics and modeling because it raised in many real-world phenomena that involves two outcomes (success and failure) and waiting the trial to get the first success. Some real-world examples can be given as coin tossing, sports applications, feedback from customers, number of supporters of a law, number of faulty products manufactured, number of bugs in a code, a teacher examining test records, throwing darts at a dartboard, number of network failures, etc (*cf.* [5]-[27]).

Let $\mathbb{N} = \{1, 2, \ldots\}$. Let X be the Bernoulli trials needed to get one success, p be the probability of success on each trial, then the k-th trial of the first success is described as Geometric distribution and its probability distribution given as follows:

$$P_G(X = k) := f_G(k) = q^{k-1}p, \tag{1}$$

where $k = 1, 2, 3, \ldots,$ $p + q = 1$ (*cf.* [11, p. 49], see also [5, p. 98], [8]). The moments of the Geometric distribution is defined as follows:

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$$\mathbf{E}(X^n) = \mu_n = \sum_{k=1}^{\infty} k^n f_G(k),$$

where μ_n depends on p and q with $p + q = 1$ (*cf.* [11], [5], [8]). Using the above formula for μ_n, we have the following known formula:

$$\mu_n = \frac{p}{q}\mathbf{Li}_{-n}(q), \tag{2}$$

where $q = 1 - p$ and $\mathbf{Li}_z(u)$ denotes the polylogarithm function

$$\mathbf{Li}_z(u) = \sum_{k=1}^{\infty} \frac{u^k}{k^z}$$

($z = a + ib$, a and b real numbers, ($i = \sqrt{-1}$) when $|u| < 1$; $a > 1$ when $|u| = 1$). This function is reduces to the ordinary natural logarithm when $z = 1$ (*cf.* [25],[26, p.200, Eq. 40]; and see also the references cited in each of these earlier works). Modification of the formula (2) was also studied in [28].

The first and second order moments are used to calculate the expected value and variance of the distribution. Letting $n = 1$ yields the expected value of the Geometric distribution with parameter p:

$$\mathbf{E}(X) = \mu_1 = \frac{1}{p}$$

and letting $n = 2$, $\mathbf{E}(X^2) = \mu_2$ helps deriving the well-known formula for variance of this distribution:

$$Var(X) = \mu_2 - (\mu_1)^2 = \frac{1-p}{p^2}.$$

Because $\mathbf{E}(X^n) = \mu_n$ is cumbersome especially for $n > 2$, in this work, we seek for new derivational formulas for the the higher order moments of Geometric distribution by using the Apostol-Bernoulli numbers and polynomials, and also Stirling numbers. These numbers and polynomials are briefly summarized next.

The first kind Stirling numbers arise in the fields of permutation and combinatorics and are frequently used in these fields. These numbers are known to count according to the number of cycles of permutations. The Stirling numbers of the first kind $S_1(m, v)$ are defined by the following generating function:

$$\frac{(\log(1+z))^v}{v!} = \sum_{m=0}^{\infty} S_1(m, v)\frac{z^m}{m!}, \tag{3}$$

where $v \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Some basic properties of these numbers are given as follows:

These numbers have the following recurrence relation:

$$S_1(m + 1, v) = -mS_1(m, v) + S_1(m, v - 1) \tag{4}$$

and

$$z_{(v)} = \sum_{d=0}^{v} S_1(v, d)z^d \tag{5}$$

where $z_{(v)}$ denotes the falling factorial:

$$z_{(v)} = z(z - 1)(z - 2)\cdots(z - v + 1),$$

($v \in \mathbb{N}$) and $z_{(0)} = 1$. By using (4), we have $S_1(0,0) = 1$ and if $v > m$, then $S_1(m,v) = 0$. Thus, few values of the numbers $S_1(v,m)$ are given by

$v\backslash m$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	−1	1	0	0	0
3	0	2	−3	1	0	0
4	0	−6	11	−6	1	0
5	0	24	−50	35	−10	1

(*cf.* [7], [9], [10], [12], [24], [26]; and see also the references cited in each of these earlier works).

Let

$$\exp(z) := e^z.$$

The Stirling numbers of the second kind, which are the number of partitions of a set of m elements into v nonempty subsets, are defined by

$$\frac{(\exp(z)-1)^v}{v!} = \sum_{m=0}^{\infty} S_2(m,v)\frac{z^m}{m!}, \tag{6}$$

and

$$z^v = \sum_{d=0}^{v} S_2(v,d) z_{(d)}, \tag{7}$$

where $v \in \mathbb{N}_0$. Using (7), we have

$$S_2(m,m) = S_2(0,0) = 1$$

if $v > m$ then

$$S_2(m,v) = 0$$

and

$m\backslash v$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

(*cf.* [7], [9], [10], [12], [24], [26]).

The Stirling numbers play an important role in probability theory, especially in moment calculation formulas, partition theory, etc. (*cf.* [9], [10], [26]).

The Apostol-Bernoulli polynomials are defined by the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{B}_n(s;t)\frac{z^n}{n!} = \frac{z\exp(zs)}{t\exp(z)-1} \tag{8}$$

(*cf.* [1]).

Remark 1.1. *Subsequently, in a rather frequently-cited sequel to [1] by Srivastava [23], these mild generalizations of the classical Bernoulli numbers and polynomials were applied, in conjunction with Lerch's functional equation for the generalized zeta function, in the study these numbers and polynomials in rational arguments. In fact, it this 2000 paper of Srivatava [23] that has led to the flooding of the literature by a large number of papers on these and related numbers and polynomials.*

Setting $s = 0$ in (8) yields the the Apostol-Bernoulli numbers:

$$\mathcal{B}_n(t) = \mathcal{B}_n(0; t). \tag{9}$$

Also, setting $s = 1$ into (8), the following relations are obtained:

$$\mathcal{B}_1(1; t) = \frac{1 + \mathcal{B}_1(t)}{t}$$

and for $n \geq 2$,

$$\mathcal{B}_n(1; t) = \frac{1}{t}\mathcal{B}_n(t), \tag{10}$$

(*cf.* [1]).

Using (8), Apostol [1] gave the following calculation formula for the polynomials $\mathcal{B}_n(s; t)$:

$$\mathcal{B}_n(s; t) = \sum_{j=0}^{n} \binom{n}{j} s^{n-j} \mathcal{B}_j(t), \tag{11}$$

where $\mathcal{B}_0(t) = 0$ and, for $n \geq 1$ and $t \neq 1$,

$$\mathcal{B}_n(t) = \frac{nt}{(t-1)^n} \sum_{v=1}^{n-1} (-1)^v v! t^{v-1} (t-1)^{n-1-v} S_2(n-1, v) \tag{12}$$

(*cf.* [1]). Putting $n = 1, 2, \ldots, 4$ in (11), we have the following some values of the polynomials $\mathcal{B}_n(s; t)$, which are used in the following sections:

$$\begin{aligned}
\mathcal{B}_1(s; t) &= \frac{1}{t-1}, \\
\mathcal{B}_2(s; t) &= \frac{s}{t-1} - \frac{2t}{(t-1)^2}, \\
\mathcal{B}_3(s; t) &= \frac{3s^2}{t-1} - \frac{6ts}{(t-1)^2} + \frac{3t(t+1)}{(t-1)^3}, \\
\mathcal{B}_4(s; t) &= \frac{4s^3}{t-1} - \frac{12ts^2}{(t-1)^2} + \frac{12t(t+1)s}{(t-1)^3} - \frac{4t\left(t^2+4t+1\right)}{(t-1)^4},
\end{aligned}$$

(*cf.* [1], [12], [24], [26]).

It is well known that the Apostol-Bernoulli numbers and polynomials have many applications in different areas. They are related to zeta type functions including the polylogarithm function. There are also many applicable relations between the Apostol-Bernoulli numbers, the Apostol-Bernoulli polynomials, and the Eulerian number $E(n, k)$, which is the number of permutations of the numbers from 1 to n such that exactly k element is greater than the previous element. Incidentally, Eulerian numbers are related to the Stirling numbers of the second kind.

In this paper, our motivation is to find more easy and efficient formulas that can help calculating the moments, factorial moments, and moment generating function of Geometric distribution. To achieve this goal, we used commonly used mathematical tools, namely Apostol-Bernoulli number and polynomials, Stirling numbers and other special functions.

In Section 2, the moment generating function of Geometric distribution is represented Apostol-Bernoulli polynomials. Similarly, the moments of this distribution is also represented in terms of Apostol-Bernoulli polynomials. Also, the moments are represented by the second kind Stirling numbers.

In Section 3, we give z-transform for factorial moments. Specifically, we find computation formula factorial moments of the geometric random variable with parameter p using z-transform and higher order derivatives of the probability generating function.

In Section 4, we compare the new formulas with the traditional ways of calculating the moments of Geometric distribution to show the simplicity of the use of our new formulas.

In the last section, we give conclusion section.

2. Moment generating function of the Geometric distribution related to the Apostol-Bernoulli numbers and polynomials

In this section, an alternative formula for moment generating function of the Geometric distribution is derived, and interestingly these formulas are actually linked to the Apostol-Bernoulli numbers and polynomials and other special functions associated with the Stirling numbers. We also give, direct computation formula for the moments of Geometric distribution.

First, we derived a formula of the moment generating distribution of Geometric distribution in terms of the Apostol-Bernoulli polynomials, which is given in below theorem.

Theorem 2.1. *Let $p + q = 1$ and $M_X(u)$ be the moment generating function of the X geometric random variable with parameter p. Then we have*

$$M_X(u) = -p \sum_{n=0}^{\infty} \frac{\mathcal{B}_{n+1}(1;q)}{(n+1)!} u^n. \tag{13}$$

Proof. The moment generating function for Geometric distribution:

$$M_X(u) = \mathbf{E}(e^{uX}) = p \exp(u) \sum_{k=1}^{\infty} q^{k-1} \exp((k-1)u).$$

By using geometric series expansion for $\left|q \exp(u)\right| < 1$, we have

$$M_X(u) = -\frac{p \exp(u)}{q \exp(u) - 1} \tag{14}$$

(*cf.* [5, p.236]). We give a relation between (8) and (14) by the following equation:

$$M_X(u) = -p \sum_{n=0}^{\infty} \mathcal{B}_n(1;q) \frac{u^{n-1}}{n!}. \tag{15}$$

After some calculations in the previous equation, proof of theorem is completed. □

Next, we obtain the moment formula of Geometric distribution in terms of the Apostol-Bernoulli polynomilas, which is given in below theorem.

Theorem 2.2. *Let $p + q = 1$ and $j \in \mathbb{N}_0$. Let μ_j be jth moment of the X geometric random variable with parameter p. Then we have*

$$\mu_j = -\frac{p\mathcal{B}_{j+1}(1;q)}{j+1}. \tag{16}$$

Proof. For $j \in \mathbb{N}$, if we take j times derivative of the moment generating function $M_X(u)$ in (15) for $u = 0$, we obtain the following moments of geometric random variable with parameter p:

$$\mu_j = \mathbf{E}(X^j) = \frac{d^j}{du^j} \{M_X(u)\}|_{u=0} .$$

After performing some operations on the previous equation, the following results are obtained:

$$\frac{d^j}{du^j} \{M_X(u)\} = -p\frac{d^j}{du^j} \left\{ \sum_{n=0}^{\infty} \mathcal{B}_n(1;q)\frac{u^{n-1}}{n!} \right\}.$$

Combining the above equation with the next equation for $u = 0$:

$$\mu_j = \frac{d^j}{du^j} \{\mathbf{E}(\exp(uX))\}|_{u=0} , \tag{17}$$

the desired result is found. □

The right handside of equations (16) and (17) are equal to each other. This is why, we get:

$$\mu_j = -\frac{p\mathcal{B}_{j+1}(1;q)}{j+1} = \frac{d^j}{du^j} \{\mathbf{E}(\exp(uX))\}|_{u=0} . \tag{18}$$

Combining (18) with (2), we get

$$\sum_{k=1}^{\infty} k^j q^k = -q\frac{\mathcal{B}_{j+1}(1;q)}{j+1}. \tag{19}$$

Remark 2.3. *For $j \in \mathbb{N}_0$, with the aid of (10), equation (19) reduces to the following known formula:*

$$\sum_{k=1}^{\infty} k^j q^k = -\frac{\mathcal{B}_{j+1}(q)}{j+1}$$

(cf. [26, p.200, Eq. 40]).

Note that similar equation was given in [26]; however, the connection between this equation and the moments of Geometric distribution was not stated.

In the following theorem, the novel formula of the moments of Geometric distribution is given in terms of second kind Stirling numbers.

Theorem 2.4. *Let $p + q = 1$ and $j \in \mathbb{N}$. Let μ_j be jth moment of the X geometric random variable with parameter p. Then we have*

$$\mu_j = p \sum_{v=1}^{j} \frac{v! q^{v-1}}{(1-q)^{v+1}} S_2(j, v). \tag{20}$$

Proof. Combining (16) with (12) and (10), after some calculations are done, we get (16). □

Thanks to novel formula for the moments of geometric random variable with parameter p, given in the theorem below, it gives direct calculations without relying on any series formula, differentiation process or special polynomials or numbers.

Theorem 2.5. *Let $p + q = 1$ and $j \in \mathbb{N}$. Let μ_j be jth moment of the X geometric random variable with parameter p. Then we have*

$$\mu_j = \frac{1}{q} \sum_{v=1}^{j} \sum_{d=0}^{v} (-1)^{v-d} \binom{v}{d} \left(\frac{q}{p}\right)^v d^j. \tag{21}$$

Proof. Substituting the following well-known formula of the Stirling numbers of the second kind

$$S_2(j,v) = \frac{1}{v!}\sum_{d=0}^{v}(-1)^{v-d}\binom{v}{d}d^j$$

into equation (20), after making the necessary calculations, we obtain desired result. □

3. z-Transform for factorial moments

We now briefly give the z-transform, which is sometimes called the probability generating function. Let $f(x) = P(X = x)$ be the probability distribution of the nonnegative discrete random variable X; that is, $f(x) = 0$ for $x < 0$. The z-transform of $f(x)$, denoted by $G_x(z)$ is defined by

$$G_x(z) = \mathbf{E}(z^X) = \sum_{n=1}^{\infty} z^n f(n).$$

It is clear that

$$G_x(1) = 1$$

(*cf.* [5, p. 230], [19]).
It is well-known that the higher order derivatives of the probability generating function evaluated at $z = 1$ yields the factorial moments, $\mu_k(X_{(k)})$. That is,

$$\begin{aligned}\mu_k(X_{(k)}) &= \frac{d^k}{dz^k}\left\{G_X(z)\right\}|_{z=1} \\ &= \sum_{n=1}^{\infty} n_{(k)} f(n),\end{aligned}$$

where $X_{(k)}$ denotes the falling factorial. Therefore, combining (5) with the following equation

$$\mu_k(X_{(k)}) = \sum_{n=1}^{\infty} n_{(k)} f(n),$$

we have

$$\mu_k(X_{(k)}) = \sum_{j=0}^{k} S_1(k,j)\sum_{n=1}^{\infty} n^j f(n) \tag{22}$$

(*cf.* [19]).

Theorem 3.1. *Let $p + q = 1$ and $k \in \mathbb{N}$. Let μ_k be kth moment of the X geometric random variable with parameter p. Then we have*

$$\mu_k(X_{(k)}) = -p\sum_{j=0}^{k}\frac{S_1(k,j)\mathcal{B}_{j+1}(1;q)}{j+1}. \tag{23}$$

Proof. By combining (22) with (1), we get the following factorial moments of the geometric random variable X with parameter p:

$$\mu_k(X_{(k)}) = \sum_{j=0}^{k} S_1(k,j)\sum_{n=1}^{\infty} n^j f_G(n)$$

Therefore

$$\mu_k(X_{(k)}) = \sum_{j=0}^{k} S_1(k,j) \sum_{n=1}^{\infty} n^j q^{n-1} p.$$

After some calculations in the above equation, we obtain

$$\mu_k(X_{(k)}) = \frac{p}{q} \sum_{j=0}^{k} S_1(k,j) \sum_{n=1}^{\infty} n^j q^{n}.$$

Combining the previous equation with (19), then proof of theorem is completed. □

It is important to point out that combining (23) with (16), a relation between the jth moments and the kth factorial moments of the geometric random variable is given as follows:

$$\mu_k(X_{(k)}) = \sum_{j=0}^{k} S_1(k,j)\mu_j. \tag{24}$$

4. Calculation of moments

Calculation of the moments are very difficult when using the traditional formula of Geometric distribution given in equation (2) because it involves higher order derivatives on the geometric series expansions. This can be easily seen from below Table 1:

j	$\sum_{k=1}^{\infty} k^j q^k$	$\mu_j = E(X^j) = p \sum_{k=1}^{\infty} k^j q^{k-1}$
0	$\frac{q}{1-q}$	$\mu_0 = \frac{p}{q}\frac{q}{1-q} = 1$
1	$\frac{q}{(1-q)^2}$	$\mu_1 = \frac{p}{q}\frac{q}{(1-q)^2} = \frac{1}{p}$
2	$\frac{q(1+q)}{(1-q)^3}$	$\mu_2 = \frac{p}{q}\frac{q+q^2}{(1-q)^3} = \frac{2-p}{p^2}$
3	$\frac{q(1+4q+q^2)}{(1-q)^4}$	$\mu_3 = \frac{p}{q}\frac{q(1+4q+q^2)}{(1-q)^4} = \frac{p^2-6p+6}{p^3}$
4	$\frac{q(1+11q+11q^2+q^3)}{(1-q)^5}$	$\mu_4 = \frac{p}{q}\frac{q(1+11q+11q^2+q^3)}{(1-q)^5} = \frac{-p^3+14p^2-36p+24}{p^4}$

Table 1: Calculation μ_j by the geometric series

The values in the first column of the Table 1 above were found with the help of geometric series. The jth line is obtained by the derivative of the series in the $(j-1)$th line, which gives the jth moment μ_j. Becaue this is very lengthy and cumbersome, the traditional formula makes it challenging to calculate the higher order moments of Geometric distributıon.

The theorems given in Section 2 shows that there is a one-to-one relationship between the traditional formula and the Apostol-Bernoulli numbers. Therefore, this gives an easier option to calculate the moments of Geometric distribution. In Table 2 below, we illusturate the new formula given in equation (16) with aid of equation (10):

j	$\mathcal{B}_{j+1}(q)$	$\mu_j = -\frac{p\mathcal{B}_{j+1}(1;q)}{(j+1)}$
0	$\mathcal{B}_1(q) = \frac{1}{q-1}$	$\mu_0 = -\frac{p(1+\mathcal{B}_1(q))}{q} = 1$
1	$\mathcal{B}_2(q) = -\frac{2q}{(q-1)^2}$	$\mu_1 = -\frac{p\mathcal{B}_2(q)}{2q} = \frac{1}{p}$
2	$\mathcal{B}_3(q) = \frac{3q^2+3q}{(q-1)^3}$	$\mu_2 = -\frac{p\mathcal{B}_3(q)}{3q} = \frac{2-p}{p^2}$
3	$\mathcal{B}_4(q) = -\frac{4q^3+16q^2+4q}{(q-1)^4}$	$\mu_3 = -\frac{p\mathcal{B}_4(q)}{4q} = \frac{p^2-6p+6}{p^3}$
4	$\mathcal{B}_5(q) = \frac{5q^4+55q^3+55q^2+5q}{(q-1)^5}$	$\mu_4 = -\frac{p\mathcal{B}_5(q)}{5q} = \frac{-p^3+14p^2-36p+24}{p^4}$

Table 2: Calculation μ_j by the Apostol-Bernoulli numbers

Here, the values in the first column of Table 2 shows the Apostol-Bernoulli numbers which are derived from their generating function. With the help these the Apostol-Bernoulli numbers, the moments of Geometric distribution with parameter p is shown in the second column above. It can be clearly seen that our new formula provides an alternative way of calculating the moments of Geometric distribution with more trivial way.

5. Further Remarks and Observations

The Stirling numbers and moments formulas have many useful applications in mathematics and statistics. Therefore, new methods and techniques are needed to find effective and elegant formulas for moments. Among these, the methods that we can describe as interesting are special numbers involving the Apostol-Bernoulli numbers and the Stirling numbers of the first and the second kinds and their generating functions. For example, by using the Stirling numbers, Harris [14] found calculation formula for moments in statistics. Joarder and Mahmood [15] gave a formula for the raw moments of discrete distributions. Benyi and Manago [4] derived a recursive formla for the moments of the Binomial distribution. Knoblauch (2008), Griffiths [13], and Nguyen [18] gave formulas for the raw moments of the Binomial distribution via probabilistic approach. Using the Stirling numbers of the second kind, Bagui and Mehra [2] gave moments of some discrete distributions such as Binomial, Poisson, Geometric, and Negative Binomial.

Unlike the known formulas mentioned above, our formulas include both kinds of Stirling numbers of the first and the second kinds, and most importantly, they include Apostol-Bernoulli numbers and polynomials. Perhaps this new approach has the qualities to make greater contributions to those who study probability and statistical applications.

6. Conclusion

The aim of this article is to investigate explicit computation formulas of moments, moment generating function, and factorial moments of Geometric distribution. The traditional formulas of Geometric distribution for the moments are pretty difficult to calculate because they require taking higher order derivatives on the geometric series expansions. This is there is a tremendous need of finding alternative formulas to calculate the moments of this distribution. For this reason, our goal was to address this need. In this paper, we found a connection between the moments of Geometric distribution and the Apostol-Bernoulli numbers and polynomials and also the Stirling numbers.

We first gave moment generating function of Geometric distribution in terms of the Apostol-Bernoulli numbers and polynomials. By applying generating function of the Apostol Bernoulli polynomials to this function, we gave some new formulas, associated with the Stirling numbers and Apostol-Bernoulli numbers and polynomails, for jth moments of the geometric random variable with parameter p. By using z-transform and higher order derivatives of the probability generating function, we found computation formula factorial moments of the geometric random variable with parameter p.

As a result, we showed that moment calculation can be found much more simply and quickly with our new formulas than the traditional calculation method which depend on derivative of the geometric series many times.

Using this new method, other probability distribution functions can also be examined by considering the statistical properties of other distributions. New relations among the moment generating functions of these distributions, moments, and generating functions containing other special functions may be investigated and studied.

Moment formulas have very important applications in applications of spline curves, including Bezier curves, as well as in applications of statistics. These new formulas may provide significant application potential for these areas.

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