Filomat 39:6 (2025), 1869–1882
https://doi.org/10.2298/FIL2506869J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Almost hyperbolic Ricci solitons on $(LCS)_n$-manifolds

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**Abstract.** The purpose of this paper is to investigate $(LCS)_n$-manifolds whose metric tensors satisfy in almost hyperbolic Ricci solitons. We show that $(LCS)_n$-manifolds admit in almost hyperbolic Ricci solitons under some conditions. Also, we give some examples of almost hyperbolic Ricci solitons on $(LCS)_n$-manifolds.

## 1. Introduction

Shaikh [20] in 2003, introduced Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold or LCSM) as a generalization of Lorentzian para-Sasakian manifolds [15, 17]. Also, he proved the existence and applications of LCSMs in general relativity and cosmology. After that, many studies have been done. For instance see [14, 22–24] and the references therein.

On the other hand, geometric flows and geometric solitons are important topics in differential geometry and physics. One of them is Ricci flow which a special solution to it is called Ricci soliton ( or RS) and introduced by Hamilton [13]. Let $(M, g)$ be a pseudo-Riemannian manifold with the Ricci tensor $S$. A RS [5] on $(M, g)$ is a triplet $(g, \zeta, \gamma)$ such that

$$\mathcal{L}_\zeta g + 2S + 2\gamma g = 0, \tag{1}$$

for some vector field $\zeta$ and constant $\gamma$ where $\mathcal{L}_\zeta$ denotes the Lie derivative along $V$. If $\gamma$ is a function on $M$, then Ricci soliton becomes almost RS.

Another one of these geometric flows is the hyperbolic geometric flow which is given by

$$\frac{\partial^2}{\partial t^2} g = -2S, \qquad g(0) = g_0, \qquad \frac{\partial g}{\partial t}(0) = k_0. \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 53C25; Secondary 53D15, 53E20.
*Keywords*. $(LCS)_n$-manifolds, Hyperbolic Ricci soliton, Einstein manifolds.
Received: 12 October 2024; Accepted: 13 October 2024
Communicated by Ljubica Velimirović
The third author acknowledge the grant of the Ministry of Science, Technological Development and Innovation of Serbia 451-03-137/2025-03/ 200124 for carrying out this research.
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Here $k_0$ denotes a symmetric $(0,2)$-tensor (2).
Suppose that $(M^n, g(t))$ is a solution to (2) on $(M, g_0)$. The self-similar solution of (2) [2, 11] is

$$S(g_0) + \lambda \mathcal{L}_X g_0 + (\mathcal{L}_X \circ \mathcal{L}_X) g_0 = \mu g_0, \tag{3}$$

for $\lambda$ and $\mu$ as constants. In this state, the metric $g_0$ is called a hyperbolic Ricci soliton (or HRS). We represent it as $(M, g_0, X, \lambda, \mu)$ (shortly $(g_0, X, \lambda, \mu)$). If $\lambda = \frac{1}{2}$ and $X$ is a 2-Killing [8, 18], i.e., $(\mathcal{L}_X \circ \mathcal{L}_X) g_0 = 0$ then a HRS becomes a RS. The HRS is called a gradient HRS (or GHRS) whenever $X = \nabla\phi$ for a function $\phi$ on $M$.

A 4-tuple $(g, V, \lambda, \mu)$ on $M$ is declared an almost HRS (or AHRS) if it satisfies the equation

$$S + \lambda \mathcal{L}_V g + (\mathcal{L}_V \circ \mathcal{L}_V) g = \mu g, \tag{4}$$

for some functions $\lambda$ and $\mu$ and vector field $V$.

According to the works mentioned above, we consider AHRSs on LCSMs. We give two examples of AHRSs on a LCSMs. The paper is structured as follows: The subsequent section will introduce fundamental concepts and formulas of LCSMs. This will be followed by the presentation of our main results and their corresponding proofs in Section 3. The final section will feature examples of AHRS on LCSMs.

## 2. Preliminaries

Suppose $(M^n, g)$ is a Lorentzain manifold. In this section, vector fields $\zeta_1$, $\zeta_2$ and $\zeta_3$ on $M$ are arbitrary unless otherwise stated. A vector $0 \neq v \in T_pM$ is called timelike when $g_p(v, v) < 0$. Assume that vector field $\nu$ admits $g(\zeta_1, \nu) = A(\zeta_2)$. $\nu$ is called concircular if

$$(\nabla_{\zeta_1} A)(\zeta_2) = \alpha\{g(\zeta_1, \zeta_2) + \omega(\zeta_1)A(\zeta_2)\} \tag{5}$$

for some non-zero function $\alpha$, and some closed 1-form $\omega$ where $\nabla$ is connection corresponding to $g$. If $\nu$ is unit then it is anointed the characteristic vector field (briefly characteristic) of $M$.
Let $\xi$ be the characteristic of $M$, then $g(\xi, \xi) = -1$, and there is a non-vanishing 1-form $\vartheta$ as follows

$$g(\zeta_1, \xi) = \vartheta(\zeta_1). \tag{6}$$

We conclude the following identity

$$(\nabla_{\zeta_1} \vartheta)(\zeta_2) = \alpha\{g(\zeta_1, \zeta_2) + \vartheta(\zeta_1)\vartheta(\zeta_2)\}, \quad \alpha \neq 0, \tag{7}$$

where $\alpha$ satisfies

$$\nabla_{\zeta_1} \alpha = \zeta_1(\alpha) = \rho\vartheta(\zeta_1), \tag{8}$$

where $\rho = -\xi(\alpha)$. From (7) we have

$$\nabla_{\zeta_1} \xi = \alpha(\zeta_1 + \vartheta(\zeta_1)\xi). \tag{9}$$

We consider $\phi\zeta_1 = \frac{1}{\alpha}\nabla_{\zeta_1}\xi$, then

$$\phi\zeta_1 = \zeta_1 + \vartheta(\zeta_1)\xi. \tag{10}$$

The tensor $\phi$ is called the structure tensor. A Lorentzian concircular structure manifold is the Lorentzian manifold $M$ together with $\xi$, $\vartheta$, and $\phi$. The LP-Sasakian structure is obtained when $\alpha = -1$. In a LCSM the following identities are true:

$$\vartheta(\xi) = -1, \ \phi(\xi) = 0, \quad \vartheta \circ \phi = 0, \tag{11}$$

$$g(\phi\zeta_1, \phi\zeta_2) = g(\zeta_1, \zeta_2) + \vartheta(\zeta_1)\vartheta(\zeta_2). \tag{12}$$

Applying (9) and (11), the Riemannian curvature tensor $R$ admit

$$R(\nu_1,\nu_2)\xi = (\alpha^2-\rho)[\vartheta(\nu_2)\nu_1 - \vartheta(\nu_1)\nu_2], \tag{13}$$
$$R(\nu_1,\xi)\nu_2 = (\alpha^2-\rho)[g(\nu_1,\nu_2)\xi - \vartheta(\nu_2)\nu_1], \tag{14}$$
$$\vartheta(R(\nu_1,\nu_2)Z) = (\alpha^2-\rho)[g(\nu_2,Z)\vartheta(\nu_1) - g(\nu_1,Z)\vartheta(\nu_2)]. \tag{15}$$

Also, for the Ricci tensor $S$ of a LCSM $M$, we have

$$S(\nu_1,\xi) = (n-1)(\alpha^2-\rho)\vartheta(\nu_1). \tag{16}$$

## 3. Main results and their proofs

In this section, vector fields $\zeta_1,\dots,\zeta_k, U, \zeta, \nu_1,\nu_2,\nu_3$ and $Z$ on $M$ are arbitrary unless otherwise stated. A LCSM is said to $\vartheta$-Einstein if

$$S = b\vartheta\otimes\vartheta + ag,$$

for some smooth maps $b$ and $a$. Let $M$ be a LCSM. Assume that $M$ admit AHRS (4) with $V = f\xi$ where $f$ is a function on $M$ and $\xi$ is structure vector field. Applying (9), it follows that

$$\begin{aligned}\mathcal{L}_{f\xi}g(\nu_1,\nu_2) &= g(\nabla_{\nu_1}f\xi,\nu_2) + g(\nu_1,\nabla_{\nu_2}f\xi)\\ &= (\nu_1 f)\vartheta(\nu_2) + f\alpha g(\nu_1+\vartheta(\nu_1)\xi,\nu_2) + (\nu_2 f)\vartheta(\nu_1)\\ &\quad + f\alpha g(\nu_1,\nu_2+\vartheta(\nu_2)\xi)\\ &= (\nu_1 f)\vartheta(\nu_2) + (\nu_2 f)\vartheta(\nu_1) + 2\alpha f(g(\nu_1,\nu_2)+\vartheta(\nu_1)\vartheta(\nu_2)),\end{aligned} \tag{17}$$

hence

$$\begin{aligned}&(\mathcal{L}_{f\xi}(\mathcal{L}_{f\xi}g))(\nu_1,\nu_2)\\ &= f\xi(\mathcal{L}_{f\xi}g(\nu_1,\nu_2)) - \mathcal{L}_{f\xi}g(\mathcal{L}_{f\xi}\nu_1,\nu_2) - \mathcal{L}_{f\xi}g(\nu_1,\mathcal{L}_{f\xi}\nu_2)\\ &= f\xi\left((\nu_1 f)\vartheta(\nu_2) + (\nu_2 f)\vartheta(\nu_1) + 2\alpha f(g(\nu_1,\nu_2)+\vartheta(\nu_1)\vartheta(\nu_2))\right)\\ &\quad -\Big(((\mathcal{L}_{f\xi}\nu_1)f)\vartheta(\nu_2) + (\nu_2 f)\vartheta(\mathcal{L}_{f\xi}\nu_1) + 2\alpha f(g(\mathcal{L}_{f\xi}\nu_1,\nu_2) + \vartheta(\mathcal{L}_{f\xi}\nu_1)\vartheta(\nu_2))\Big)\\ &\quad -\Big((\nu_1 f)\vartheta(\mathcal{L}_{f\xi}\nu_2) + ((\mathcal{L}_{f\xi}\nu_2)f)\vartheta(\nu_1) + 2\alpha f(g(\nu_1,\mathcal{L}_{f\xi}\nu_2) + \vartheta(\nu_1)\vartheta(\mathcal{L}_{f\xi}\nu_2))\Big).\end{aligned} \tag{18}$$

Plugging $V = f\xi$ and (18) in (4), we conclude

$$\begin{aligned}&S(\nu_1,\nu_2) + \lambda\left((\nu_1 f)\vartheta(\nu_2) + (\nu_2 f)\vartheta(\nu_1) + 2\alpha f(g(\nu_1,\nu_2)+\vartheta(\nu_1)\vartheta(\nu_2))\right)\\ &+f\xi\left((\nu_1 f)\vartheta(\nu_2) + (\nu_2 f)\vartheta(\nu_1) + 2\alpha f(g(\nu_1,\nu_2)+\vartheta(\nu_1)\vartheta(\nu_2))\right)\\ &-\Big(((\mathcal{L}_{f\xi}\nu_1)f)\vartheta(\nu_2) + (\nu_2 f)\vartheta(\mathcal{L}_{f\xi}\nu_1) + 2\alpha f(g(\mathcal{L}_{f\xi}\nu_1,\nu_2) + \vartheta(\mathcal{L}_{f\xi}\nu_1)\vartheta(\nu_2))\Big)\\ &-\Big((\nu_1 f)\vartheta(\mathcal{L}_{f\xi}\nu_2) + ((\mathcal{L}_{f\xi}\nu_2)f)\vartheta(\nu_1) + 2\alpha f(g(\nu_1,\mathcal{L}_{f\xi}\nu_2) + \vartheta(\nu_1)\vartheta(\mathcal{L}_{f\xi}\nu_2))\Big)\\ &-\mu g(\nu_1,\nu_2) = 0,\end{aligned} \tag{19}$$

We plug $\nu_1=\nu_2=\xi$ in (19) and applying (16) and (11) to obtain

$$(\alpha^2-\rho)(n-1) + 2f\xi(\xi(f)) + 2\lambda\xi(f) + 4(\xi(f))^2 - \mu = 0. \tag{20}$$

So, the following theorem is concluded.

**Theorem 3.1.** *Suppose that $(M,g,\phi,\xi,\vartheta)$ is a LCSM. If $M$ satisfies an AHRS $(g,f\xi,\lambda,\mu)$ for some smooth map $f$ on $M$, then the relation (20) is true.*

Now, assume that $M$ is an $\vartheta$-Einstein LCSM and $V = \xi$. Then, there are two maps $b$ and $a$ on $M$ so that $S = ag + b\vartheta \otimes \vartheta$. From (17), we obtain

$$\mathcal{L}_\xi g(v_1, v_2) = 2\alpha(g(v_1, v_2) + \vartheta(v_1)\vartheta(v_2)), \tag{21}$$

and

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_\xi g))(v_1, v_2) &= 2\alpha\xi\left(g(v_1, v_2) + \vartheta(v_1)\vartheta(v_2)\right)\\ &+2\xi(\alpha)\left(g(v_1, v_2) + \vartheta(v_1)\vartheta(v_2)\right)\\ &-2\alpha\left(g(\mathcal{L}_\xi v_1, v_2) + \vartheta(\mathcal{L}_\xi v_1)\vartheta(v_2)\right)\\ &-2\alpha\left(g(v_1, \mathcal{L}_\xi v_2) + \vartheta(v_1)\vartheta(\mathcal{L}_\xi v_2)\right).\end{aligned} \tag{22}$$

We get

$$\begin{aligned}g(\mathcal{L}_\xi v_1, v_2) &= g([\xi, v_1], v_2)\\ &= g(\nabla_\xi v_1 - \nabla_{v_1}\xi, v_2)\\ &= g(\nabla_\xi v_1 - \alpha(v_1 + \vartheta(v_1)\xi), v_2)\\ &= g(\nabla_\xi v_1, v_2) - \alpha g(v_1, v_2) - \alpha\vartheta(v_1)\vartheta(v_2).\end{aligned} \tag{23}$$

Similarly

$$g(v_1, \mathcal{L}_\xi v_2) = g(v_1, \nabla_\xi v_2) - \alpha g(v_1, v_2) - \alpha\vartheta(v_1)\vartheta(v_2).$$

Then

$$g(\mathcal{L}_\xi v_1, v_2) + g(v_1, \mathcal{L}_\xi v_2) = \xi(g(v_1, v_2)) - 2\alpha(g(v_1, v_2) + \vartheta(v_1)\vartheta(v_2)). \tag{24}$$

Since $\nabla_\xi \xi = \alpha(\xi + \vartheta(\xi)\xi) = 0$, using (23), we have

$$\vartheta(\mathcal{L}_\xi v_1) = g(\mathcal{L}_\xi v_1, \xi) = g(\nabla_\xi v_1, \xi) = \xi(g(v_1, \xi)) = \xi(\vartheta(v_1)),$$

similarly

$$\vartheta(\mathcal{L}_\xi v_2) = \xi(\vartheta(v_2)).$$

Thus

$$\vartheta(\mathcal{L}_\xi v_1)\vartheta(v_2) + \vartheta(v_1)\vartheta(\mathcal{L}_\xi v_2) = \xi(\vartheta(v_1))\vartheta(v_2) + \vartheta(v_1)\xi(\vartheta(v_2)) = \xi(\vartheta(v_1)\vartheta(v_2)). \tag{25}$$

Therefore, applying (24) and (25) in (22), we conclude

$$(\mathcal{L}_\xi(\mathcal{L}_\xi g))(v_1, v_2) = (4\alpha^2 + 2\xi(\alpha))(g(v_1, v_2) + \vartheta(v_1)\vartheta(v_2)). \tag{26}$$

Using (21) and (26), we infer

$$\begin{aligned}&S + \lambda\mathcal{L}_\xi g + (\mathcal{L}_\xi \circ \mathcal{L}_\xi)g - \mu g\\ &= ag + b\vartheta \otimes \vartheta + 2\alpha\lambda g + 2\alpha\lambda\vartheta \otimes \vartheta + (4\alpha^2 + 2\xi(\alpha))g + (4\alpha^2 + 2\xi(\alpha))\vartheta \otimes \vartheta - \mu g\\ &= (a + 2\alpha\lambda + 4\alpha^2 + 2\xi(\alpha) - \mu)g + (b + 2\alpha\lambda + 4\alpha^2 + 2\xi(\alpha))\vartheta \otimes \vartheta.\end{aligned}$$

The last equation yields $M$ admits an AHRS $(g, \xi, \lambda, \mu)$.

So, the following theorem is concluded

**Theorem 3.2.** *Assume that $M$ is an $\vartheta$-Einstein LCSM, that is, there are two maps $b$ and $a$ on $M$ such that $S = ag + b\vartheta \otimes \vartheta$.Then manifold $M$ satisfies an AHRS $(g, \xi, -\frac{b+4\alpha^2-2\rho}{2\alpha}, a - b)$.*

For any $(0,k)$-tensor $T$, $k \geq 1$, and symmetric $(0,2)$-tensor $B$, we give endomorphism $\zeta \wedge_B U$ by

$$(\zeta \wedge_B U)Z = B(U,Z)\zeta - B(\zeta,Z)U,$$

and

$$\begin{aligned}((\zeta \wedge_B U).T)(\zeta_1,\cdots,\zeta_k) &= -T((\zeta \wedge_B U)\zeta_1,\zeta_2,\cdots,\zeta_k)\\ &\quad -T(\zeta_1,(\zeta \wedge_B U)\zeta_2,\cdots,\zeta_k) - \cdots\\ &\quad -T(\zeta_1,\zeta_2,\cdots,(\zeta \wedge_B U)\zeta_k)\end{aligned}$$

for all vector fields $\zeta, U, Z, \zeta_1, \cdots, \zeta_k$. Also, we define $R.T$ and $\Gamma((B,T)$ as follow

$$\begin{aligned}(R(\zeta,U).T)(\zeta_1,\cdots,\zeta_k) &= -TR(\zeta,U)\zeta_1,\zeta_2,\cdots,\zeta_k)\\ &\quad -T(\zeta_1,R(\zeta,U)\zeta_2,\cdots,\zeta_k) - \cdots\\ &\quad -T(\zeta_1,\zeta_2,\cdots,R(\zeta,U)\zeta_k),\end{aligned}$$

and

$$\Gamma(B,T)(\zeta_1,\cdots,\zeta_k;\zeta,U)) = ((\zeta \wedge_B U).T)(\zeta_1,\cdots,\zeta_k).$$

Now assume that a LCSM $(M^n, g)$ satisfying the condition $R.S = f_S\Gamma(g,S)$ with $f_S \neq \alpha^2 - \rho$, that is $M$ is Ricci-pseudosymmetric. From [25], it follows that

$$S = (n-1)(\alpha^2 - \rho)g.$$

So, the next corollary is obtained by using Theorem 3.2.

**Corollary 3.3.** *Let $M$ be a LCSM satisfy the condition $R.S = f_S\Gamma(g,S)$ where $f_S \neq \alpha^2 - \rho$. Then $M$ admits an AHRS $(g, \xi, -\frac{2\alpha^2-\rho}{\alpha}, (n-1)(\alpha^2-\rho))$.*

**Definition 3.4.** *Let $M$ be a LCSM. The concircular curvature tensor $C$ on $M$ is given by*

$$C(v_1,v_2)v_3 = R(v_1,v_2)v_3 - \frac{r}{n(n-1)}\left(g(v_2,v_3)v_1 - g(v_1,v_3)v_2\right). \tag{27}$$

*Also, a LCSM $M$ is called concircular Ricci pesudosymmetric if $C$ satisfies*

$$(C(v_1,v_2).S)(v_3,U) = L_S\Gamma(g,S)(v_3,U;v_1,v_2), \tag{28}$$

*where $L_S$ is some function on on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$.*

Now consider a LCSM $M$ is concircular Ricci pesudosymmetric with $L_S \neq \alpha^2 - \rho - \frac{r}{n(n-1)}$. Then by using (28), it follows that

$$\begin{aligned}&S(C(v_1,v_2)v_3,U) + S(v_3,C(v_1,v_2)U)\\ &= L_S\left[g(v_2,v_3)S(v_1,U) - g(v_1,v_3)S(v_2,U) + g(v_2,U)S(v_1,v_3) - g(v_1,U)S(v_2,v_3)\right].\end{aligned} \tag{29}$$

Inserting $v_3 = v_2 = \xi$ in (29) and applying (27), (13), and (15), we obtain

$$S(v_1,U) = -(n-1)(\alpha^2-\rho)^2\left(g(v_1,U) + 2\vartheta(v_1)\vartheta(U)\right) \tag{30}$$

Therefore, the next corollary is obtained.

**Corollary 3.5.** *Let $M$ be a concircular Ricci pesudosymmetric LCSM with $L_S \neq \alpha^2 - \rho - \frac{r}{n(n-1)}$. Then manifold $M$ satisfies an AHRS $(g, \xi, -\frac{(2-n)(\alpha^2-\rho)+\alpha^2}{\alpha}, (n-1)(\alpha^2-\rho))$.*

It is known that every three-dimensional $(LCS)_3$-manifold is an $\vartheta$-Einstein manifold and $S$ is determined by [27]

$$S = (\frac{r}{2} + (\rho - \alpha^2))g + (\frac{r}{2} + 3(\rho - \alpha^2))\vartheta \otimes \vartheta.$$

So one can stat that:

**Corollary 3.6.** *Then every three-dimensional $(LCS)_3$-manifold satisfies an AHRS $(g, \xi, -\frac{b+4\alpha^2-2\rho}{2\alpha}, a-b)$ where $a = \frac{r}{2} - (\alpha^2 - \rho)$ and $b = \frac{r}{2} - 3(\alpha^2 - \rho)$.*

A non-flat LCSM $(M^n, g)$ $(n \geq 3)$ is termed as generalized weakly symmetric manifold, if its Riemannain curvature tensor $\bar{R}$ is non-zero and

$$\begin{aligned}(\nabla_\nu \bar{R})(\nu_1, \nu_2, \nu_3, \nu_4) &= \theta_1(\nu)\bar{R}(\nu_1, \nu_2, \nu_3, \nu_4) + \theta_2(\nu_1)\bar{R}(\nu, \nu_2, \nu_3, \nu_4) \\ &+\theta_2(\nu_2)\bar{R}(\nu_1, \nu, \nu_3, \nu_4) + \theta_3(\nu_3)\bar{R}(\nu_1, \nu_2, \nu, \nu_4) \\ &+\theta_3(\nu_4)\bar{R}(\nu_1, \nu_2, \nu_3, \nu) + \theta_4(\nu)G(\nu_1, \nu_2, \nu_3, \nu_4) \\ &+\theta_5(\nu_1)G(\nu, \nu_2, \nu_3, \nu_4) + \theta_5(\nu_2)G(\nu_1, \nu, \nu_3, \nu_4) \\ &+\theta_6(\nu_3)G(\nu_1, \nu_2, \nu, \nu_4) + \theta_6(\nu_4)(\nu_1, \nu_2, \nu_3, \nu),\end{aligned}$$

where

$$G(\nu_1, \nu_2, \nu_3, \nu_4) = g(\nu_2, \nu_3)g(\nu_1, \nu_4) - g(\nu_1, \nu_3)g(\nu_2, \nu_4),$$

and $\theta_i, i = 1, \cdots, 6$ are non-zero 1-forms.
Now assume that $(M^n, g)$ be a generalized weakly symmetric LCSM. Then from [3], we arrive at

$$\begin{aligned}S &= \left[\alpha^2 - \rho - (n-2)\frac{(\rho - \alpha^2)\alpha - \theta_5(\xi)}{\alpha + \theta_5(\xi)}\right]g \\ &-\frac{(n-2)[(\rho - \alpha^2)\theta_3(\xi) + \theta_6(\xi)]}{\alpha + \theta_5(\xi)}\vartheta \otimes \vartheta.\end{aligned}$$

This implies that:

**Corollary 3.7.** *Let $(M^n, g)$ be a generalized weakly symmetric LCSM. Then M admits an AHRS $(g, \xi, -\frac{b+4\alpha^2-2\rho}{2\alpha}, a-b)$ where $a = \alpha^2 - \rho + (n-2)\frac{(\alpha^2-\rho)\alpha-\theta_5(\xi)}{\alpha+\theta_5(\xi)}$ and $b = -\frac{(n-2)[(\alpha^2-\rho)\theta_3(\xi)+\theta_6(\xi)]}{\alpha+\theta_5(\xi)}$.*

A manifold is called generalized weakly Ricci symmetric if

$$\begin{aligned}(\nabla_\zeta S)(\nu_1, \nu_2) &= \omega_1(\zeta)S(\nu_1, \nu_2) + \omega_2(\nu_1)S(\zeta, \nu_2) + \omega_3(\nu_2)S(\nu_1, \zeta) \\ &+\omega_4(\zeta)g(\nu_1, \nu_2) + \omega_5(\nu_1)g(\zeta, \nu_2) + \omega_6(\nu_2)g(\nu_1, \zeta),\end{aligned}$$

where $\omega_i, i = 1, \cdots, 6$ are non-zero 1-forms.
Now assume that $(M^n, g)$ be a non-flat generalized weakly Ricci symmetric LCSM. Then from [3] we have

$$\begin{aligned}S &= \left[\frac{(1-n)\alpha(\rho - \alpha^2) - \omega_5(\xi)}{\alpha + \omega_2(\xi)}\right]g \\ &-\frac{(1-n)(\rho - \alpha^2)\omega_2(\xi) + \omega_5(\xi)}{\alpha + \omega_2(\xi)}\vartheta \otimes \vartheta.\end{aligned}$$

Thus we get:

**Corollary 3.8.** *Let $(M^n, g)$ be a non-flat generalized weakly Ricci symmetric LCSM. Then M admits an AHRS $(g, \xi, -\frac{b+4\alpha^2-2\rho}{2\alpha}, a-b)$ where $a = \frac{(n-1)\alpha(\alpha^2-\rho)-\omega_5(\xi)}{\alpha+\omega_2(\xi)}$ and $b = -\frac{(n-1)(\alpha^2-\rho)\omega_2(\xi)+\omega_5(\xi)}{\alpha+\omega_2(\xi)}$.*

For a manifold $M^n$, the Weyl conformal curvature tensor $\zeta$ is provided by

$$\begin{aligned}
&\zeta(\zeta, v_1)v_2 \\
&= R(\zeta, v_1)v_2 - \frac{1}{n-2}\{S(v_1, v_2)\zeta - S(\zeta, v_2)v_1 + g(v_1, v_2)Q\xi - g(\zeta, v_2)Qv_1\} \\
&+\frac{r}{(n-1)(n-2)}\{g(v_1, v_2)\zeta - g(\zeta, v_2)v_1\}.
\end{aligned}$$

where $r$ is the scalar curvature of the manifold.
Now, assume that a LCSM satisfies in condition $C(\xi, Z).\zeta = 0$ and $r \neq n(1-n)(\rho - \alpha^2)$. Then from [28] we have

$$S = \left[\alpha^2 - \rho + \frac{r}{n(1-n)}\right] g + \left[\alpha^2 - \rho - \frac{r}{n(1-n)}\right] \vartheta \otimes \vartheta.$$

So one can stat that:

**Corollary 3.9.** *Let $(M^n, g)$ be a LCSM satisfies in condition $C(\xi, Z).\zeta = 0$ and $r \neq n(1-n)(\rho - \alpha^2)$. Then manifold $M$ admits an AHRS $(g, \xi, -\frac{b+4\alpha^2-2\rho}{2\alpha}, a-b)$ where $a = \alpha^2 - \rho - \frac{r}{n(n-1)}$ and $b = \alpha^2 - \rho + \frac{r}{n(n-1)}$.*

Shaikh and Ahmad [21] introduced CL-curvature tensor field as

$$\begin{aligned}
\mathcal{A}(v_1, v_2)v_3 \quad = \quad & R(v_1, v_2)v_3 - \frac{1}{n-2}[\{S(v_2, v_3)v_1 - S(v_1, v_3)v_2\} \\
&+\{S(v_2, v_3)\vartheta(v_1) - S(v_1, v_3)\vartheta(v_2)\}\xi] \\
&+\frac{\alpha^2 - \rho}{n-2}[\{g(v_2, v_3)v_1 - g(v_1, v_3)v_2\} \\
&+(n-1)\{g(v_2, v_3)\vartheta(v_1) - g(v_1, v_3)\vartheta(v_2)\}\xi].
\end{aligned}$$

A LCSM $M$ is called CL-flat if $\mathcal{A} = 0$. If $r \neq n(1-n)(\rho - \alpha^2)$ then from [21], we have

$$S = \left[-\alpha^2 + \rho + \frac{r}{n-1}\right] g + \left[-n(\alpha^2 - \rho) + \frac{r}{n-1}\right] \vartheta \otimes \vartheta.$$

So, the following result is obtained.

**Corollary 3.10.** *Let $(M^n, g)$ be a CL-flat LCSM and $r \neq n(n-1)(\alpha^2 - \rho)$. Then $M$ admits an AHRS $(g, \xi, -\frac{b+4\alpha^2-2\rho}{2\alpha}, a-b)$ where $a = \rho - \alpha^2 + \frac{r}{n-1}$ and $b = n(\rho - \alpha^2) + \frac{r}{n-1}$.*

The pseudo projective curvature tensor $P$ is given by

$$\begin{aligned}
P(v_1, v_2)v_3 \quad = \quad & aR(v_1, v_2)v_3 + b[S(v_2, v_3)v_1 - S(v_1, v_3)v_2] \\
&-\frac{r}{n}[\frac{a}{n-1} + b][g(v_2, v_3)v_1 - g(v_1, v_3)v_2],
\end{aligned}$$

where $a$ and $b$ are non-zero constants.
Now assume that a LCSM satisfies in condition $R(v_1, v_2).P = 0$, then from [25], we deduce

$$S = (1-n)(\rho - \alpha^2)g + \frac{b}{a}\left[-r - n(1-n)\right] \vartheta \otimes \vartheta.$$

Therefore we conclude:

**Corollary 3.11.** *Suppose that $(M^n, g)$ is a LCSM satisfies in condition $R(X, Y).P = 0$. Then $M$ admits an AHRS $(g, \xi, -\frac{k_2+4\alpha^2-2\rho}{2\alpha}, k_1 - k_2)$ where $k_1 = (1-n)(\rho - \alpha^2)$ and $k_2 = \frac{b}{a}\left[n(n-1) - r\right]$.*

Also, if $P = 0$ then

$$S = \left[\frac{r}{n}(\frac{-a}{(1-n)b} + 1) - \frac{a}{b}(\alpha^2 - \rho)\right] g + \left[\frac{r}{n}(\frac{a}{(n-1)b} + 1) - (n + \frac{a}{b} - 1)(\alpha^2 - \rho)\right] \vartheta \otimes \vartheta.$$

Thus, the following corollary is obtained.

**Corollary 3.12.** *Let $(M^n, g)$ be a pseudo Projectively flat LCSM, that is manifold satisfies in condition $P = 0$. Then manifold $M$ satisfies an almost HRS $(g, \xi, -\frac{k_2+4\alpha^2-2\rho}{2\alpha}, k_1 - k_2)$ where $k_1 = \left[\frac{r}{n}(\frac{-a}{(1-n)b} + 1) - \frac{a}{b}(\alpha^2 - \rho)\right]$ and $k_2 = \left[\frac{r}{n}(\frac{-a}{(1-n)b} + 1) - (\frac{a}{b} + n - 1)(\alpha^2 - \rho)\right]$.*

The C-Bochner curvature tensor [16, 32] is given by

$$\begin{aligned}\mathcal{B}(\nu_1, \nu_2)\nu_3 = {} & R(\nu_1, \nu_2)\nu_3 + \frac{1}{n+3}[S(\nu_1, \nu_3)\nu_2 - S(\nu_2, \nu_3)\nu_1 + g(\nu_1, \nu_3)Q\nu_2 \\ & -g(\nu_2, \nu_3)Q\nu_1 + S(\phi\nu_1, \nu_3)\phi\nu_2 - S(\phi\nu_2, \nu_3)\phi\nu_1 \\ & +g(\phi\nu_1, \nu_3)Q\phi\nu_2 - g(\phi\nu_2, \nu_3)\phi\nu_1 + 2S(\phi\nu_1, \nu_2)\phi\nu_3 \\ & +2g(\phi\nu_1, \nu_2)Q\phi\nu_3 - S(\nu_1, \nu_3)\vartheta(\nu_2)\xi \\ & +S(\nu_2, \nu_3)\vartheta(\nu_1)\xi - \vartheta(\nu_1)\vartheta(\nu_3)Q\nu_2 + \vartheta(\nu_2)\vartheta(\nu_3)Q\nu_1] \\ & -\frac{n+p-1}{3+n}[g(\phi\nu_1, \nu_3)\nu_2 - g(\phi\nu_2, \nu_3)\phi\nu_1 + 2g(\phi\nu_1, \nu_2)\phi\nu_3] \\ & -\frac{p-4}{n+3}[g(\nu_1, \nu_3)\nu_2 - g(\nu_2, \nu_3)\nu_1] \\ & +\frac{p}{n+3}[g(\nu_1, \nu_3)\vartheta(\nu_2)\xi - g(\nu_2, \nu_3)\vartheta(\nu_1)\xi + \vartheta(\nu_1)\vartheta(\nu_3)\nu_2 - \vartheta(\nu_2)\vartheta(\nu_3)\nu_1].\end{aligned}$$

where $p = \frac{n+r-1}{n+1}$. If $\mathcal{B}(\nu_1, \nu_2)\nu_3 = 0$, then we say that the manifold is C-Bochner flat.
Now assume that a LCSM $M$ is C-Bochner flat. Then from [32], we get

$$S = \left[2(\alpha^2 - \rho) - (1 + \frac{r}{n+1})\right] g + \left[(3-n)(\alpha^2 - \rho) - (1 + \frac{r}{n+1})\right] \vartheta \otimes \vartheta.$$

This yields:

**Corollary 3.13.** *Let $(M^n, g)$ be a C-Bochner flat LCSM. Then manifold $M$ satisfies an AHRS $(g, \xi, -\frac{b+4\alpha^2-2\rho}{2\alpha}, (n-1))(\alpha^2 - \rho))$ where $b = (3-n)(\alpha^2 - \rho) - (1 + \frac{r}{n+1})$.*

The quasi-conformal curvature tensor $C$ is determined by

$$\begin{aligned}C(\nu_1, \nu_2)\nu_3 = {} & aR(\nu_1, \nu_2)\nu_3 + b[S(\nu_2, \nu_3)\nu_1 - S(\nu_1, \nu_3)\nu_2 + g(\nu_2, \nu_3)Q\nu_1 - g(\nu_1, \nu_3)Q\nu_2] \\ & -\frac{r}{n}[\frac{a}{n-1} + 2b][g(\nu_2, \nu_3)\nu_1 - g(\nu_1, \nu_3)\nu_2].\end{aligned}$$

where $a$ and $b$ are non-zero constants.
Now assume that a LCSM is quasi-conformal flat, that is, satisfies in condition $C(\nu_1, \nu_2)\nu_3 = 0$, then from [1], we obtain

$$S = \left[\frac{r}{nb}(\frac{-a}{(1-n)} + 2b) - (n + \frac{a}{b} - 1)(\alpha^2 - \rho)\right] g + \left[\frac{r}{nb}(\frac{-a}{(1-n)} + 2b) + (\frac{a}{b} - 2(1-n))(\rho - \alpha^2)\right] \vartheta \otimes \vartheta.$$

Thus, we can the following assertion:

**Corollary 3.14.** *Suposse that $(M^n, g)$ is a quasi-conformal flat LCSM. Then manifold M satisfies an almost HRS $(g, \xi, -\frac{k_2+4\alpha^2-2\rho}{2\alpha}, k_1-k_2)$ where $k_1 = \frac{r}{nb}(\frac{a}{(n-1)}+2b)+(\frac{a}{b}+n-1)(\rho-\alpha^2)$ and $k_2 = \frac{r}{nb}(\frac{a}{(n-1)}+2b)+(\frac{a}{b}-2(1-n))(\rho-\alpha^2)$.*

The B-tensor $C$ is defined as

$$B = aS + brg$$

where $b$ and $a$ are non-zero constants.
Now let an LCSM is B-pseudosymmetric, that is, satisfies in condition $(R(\zeta_1, \zeta_2).B)Z(Z, \zeta) = L_B\Gamma(g, B)(Z, \zeta; \zeta_1, \zeta_2)$. If $L_B \neq \alpha^2 - \rho$, then from [4], we have

$$S = (n-1)(\alpha^2 - \rho)g.$$

So one can stat that:

**Corollary 3.15.** *Suposse that $(M^n, g)$ is a quasi-conformal flat LCSM. Then M admits an AHRS $(g, \xi, -\frac{2\alpha^2-\rho}{\alpha}, (n-1)(\alpha^2-\rho))$.*

An n-dimensional LCSM is called C-pseudosymmetric if $R(\zeta_1, \zeta_2).C = L_C(\zeta_1 \wedge_g \zeta_2).Q)$ where $L_C$ is some map on $U_C = \{x \in M : C + \frac{r}{n(1-n)}G \neq 0 \,\text{at}\, x\}$ where $G$ is the $(0, 4)$-tensor is determined by $G(\zeta_1, \zeta_2, Z, U) = g((\zeta_1 \wedge_g \zeta_2)Z, U)$. Now assume that a LCSM is C-pseudosymmetric, then from [4], we can write

$$S = \left[\frac{r}{n} - (1-n)(\alpha^2 - \rho - \frac{r}{n(n-1)})\right]g.$$

Then, one can stat that:

**Corollary 3.16.** *Let $(M^n, g)$ be a C-pseudosymmetric LCSM. Then M admits an AHRS $(g, \xi, -\frac{2\alpha^2-\rho}{\alpha}, \frac{r}{n}+(n-1)(\alpha^2-\rho+\frac{r}{n(1-n)})$.*

**Definition 3.17.** *A conformal Killing vector field (or CKVF) V is a vector field such that*

$$\mathcal{L}_V g = 2hg, \tag{31}$$

*for some smooth function h. The CKVF V is called Killing, homothetic, and proper when $h = 0$, h is a constant, and h is not constant, respectively.*

Let $V$ is a CKVF and satisfies in (31). Then

$$\begin{aligned} ((\mathcal{L}_V \circ \mathcal{L}_V)g)(v_1, v_2) &= V(\mathcal{L}_V g(v_1, v_2)) - \mathcal{L}_V g(\mathcal{L}_V v_1, v_2) - \mathcal{L}_V g(v_1, \mathcal{L}_V v_2) \\ &= V(2hg(v_1, v_2)) - 2hg(\mathcal{L}_V v_1, v_2) - 2hg(v_1, \mathcal{L}_V v_2) \\ &= 2V(h)g(v_1, v_2) + 2h\mathcal{L}_V g(v_1, v_2) \\ &= (2V(h) + 4h^2)g(v_1, v_2). \end{aligned} \tag{32}$$

By inserting (32) in the equation (4), we have

$$S(v_1, v_2) + 2h\lambda g(v_1, v_2) + (2V(h) + 4h^2)g(v_1, v_2) - \mu g(v_1, v_2) = 0. \tag{33}$$

Replacing $v_2$ by $\xi$ in (33), one gets

$$((n-1)(\alpha^2 - \rho) + 2h\lambda + 2V(h) + 4h^2 - \mu)\vartheta(v_1) = 0.$$

Since $\zeta_1$ is optional, we have the next theorem.

**Theorem 3.18.** *If the metric g of a LCSM admits the AHRS $(g, V, \lambda, \mu)$ where V is CKVF, that is $\mathcal{L}_V g = 2hg$ then M is Einstein and*

$$(n-1)(\alpha^2 - \rho) + 2h\lambda + 2V(h) + 4h^2 - \mu = 0. \tag{34}$$

**Definition 3.19.** *A torse-forming vector field $V$ ( or TFVF) [30] on pseudo-Riemannian manifold $(M, g)$ is a vector field such that*

$$\nabla_\zeta V = h\zeta + \sigma(\zeta)V, \tag{35}$$

*for smooth function $h$ and a 1-form $\sigma$. The TFVF becomes concircular [7, 29], concurrent [19, 31], parallel, and torqued [6], if $\sigma$ vanishes identically, $\sigma$ vanishes identically and $h = 1$, $h = \sigma = 0$, and $\sigma(V) = 0$, respectively.*

Let $(g, V, \lambda, \mu)$ be an AHRS on a LCSM where $V$ is a TFVF and admits (35). Then

$$\mathcal{L}_V g(\nu_1, \nu_2) = 2hg(\nu_1, \nu_2) + \sigma(\nu_1)g(V, \nu_2) + \sigma(\nu_2)g(V, \nu_1) \tag{36}$$

and

$$\begin{aligned}(\mathcal{L}_V(\mathcal{L}_V g))(\nu_1, \nu_2) &= V\left(2hg(\nu_1, \nu_2) + \sigma(\nu_1)g(V, \nu_2) + \sigma(\nu_2)g(V, \nu_1)\right)\\ &\quad -2hg(\mathcal{L}_V\nu_1, \nu_2) - \sigma(\mathcal{L}_V\nu_1)g(V, \nu_2) - \sigma(\nu_2)g(V, \mathcal{L}_V\nu_1)\\ &\quad -2hg(\nu_1, \mathcal{L}_V\nu_2) - \sigma(\nu_1)g(V, \mathcal{L}_V\nu_2) - \sigma(\mathcal{L}_V\nu_2)g(V, \nu_1).\end{aligned}$$

On the other hand,

$$g(\mathcal{L}_V\nu_1, \nu_2) = g(\nabla_V\nu_1, \nu_2) - hg(\nu_1, \nu_2) - \sigma(\nu_2)g(V, \nu_1), \tag{37}$$

similarly

$$g(\nu_1, \mathcal{L}_V\nu_2) = g(\nu_1, \nabla_V\nu_2) - hg(\nu_1, \nu_2) - \sigma(\nu_1)g(V, \nu_2). \tag{38}$$

Thus

$$g(\mathcal{L}_V\nu_1, \nu_2) + g(\nu_1, \mathcal{L}_V\nu_2) = V(g(\nu_1, \nu_2)) - 2hg(\nu_1, \nu_2) - \sigma(\nu_2)g(V, \nu_1) - \sigma(\nu_1)g(V, \nu_2). \tag{39}$$

Also, we have

$$\begin{aligned}\sigma(\mathcal{L}_V\nu_1) &= \sigma(\nabla_V\nu_1 - \nabla_{\nu_1}V) = \sigma(\nabla_V\nu_1 - h\nu_1 - \sigma(\nu_1)V)\\ &= \sigma(\nabla_V\nu_1) - h\sigma(\nu_1) - \sigma(\nu_1)\sigma(V),\end{aligned}$$

similarly

$$\sigma(\mathcal{L}_V\nu_2) = \sigma(\nabla_V\nu_2) - h\sigma(\nu_2) - \sigma(\nu_2)\sigma(V).$$

Hence,

$$\begin{aligned}(\mathcal{L}_V(\mathcal{L}_V g))(\nu_1, \nu_2) &= (2V(h) + 4h^2)g(\nu_1, \nu_2) + V\left(\sigma(\nu_1)g(V, \nu_2) + \sigma(\nu_2)g(V, \nu_1)\right)\\ &\quad +4h\sigma(\nu_1)g(V, \nu_2) + 4h\sigma(\nu_2)g(V, \nu_1)\\ &\quad -\sigma(\nabla_V\nu_1)g(V, \nu_2) + \sigma(\nu_1)\sigma(V)g(V, \nu_2) - \sigma(\nu_2)g(\nabla_V\nu_1, V)\\ &\quad +\sigma(\nu_2)\sigma(V)g(V, \nu_1) - \sigma(\nabla_V\nu_2)g(V, \nu_1) - \sigma(\nu_1)g(\nabla_V\nu_2, V)\\ &\quad +2\sigma(\nu_1)\sigma(\nu_2)|V|^2.\end{aligned} \tag{40}$$

Inserting $\nu_1 = \nu_2 = \xi$ in (36) and (40), we infer

$$\mathcal{L}_V g(\xi, \xi) = -2h + 2\sigma(\xi)\vartheta(V) \tag{41}$$

and

$$\begin{aligned}(\mathcal{L}_V(\mathcal{L}_V g))(\xi, \xi) &= -(2V(h) + 4h^2) + V\left(\sigma(\xi)\vartheta(V) + \sigma(\xi)\vartheta(V)\right)\\ &\quad +4h\sigma(\xi)\vartheta(V) + 4h\sigma(\xi)\vartheta(V)\\ &\quad -2\sigma(\nabla_V\xi)\vartheta(V) + 2\sigma(\xi)\sigma(V)\vartheta(V) - 2\sigma(\xi)g(\nabla_V\xi, V)\\ &\quad +2\sigma(\xi)\sigma(\xi)|V|^2\\ &= -(2V(h) + 4h^2) + 2V\left(\sigma(\xi)\vartheta(V)\right) + 8h\sigma(\xi)\vartheta(V)\\ &\quad +2\sigma(\xi)\sigma(V)\vartheta(V) - 2\alpha\sigma(V)\vartheta(V) - 4\alpha(\vartheta(V))^2\sigma(\xi)\\ &\quad -2\alpha\sigma(\xi)|V|^2 + 2\sigma(\xi)\sigma(\xi)|V|^2.\end{aligned} \tag{42}$$

Applying (41) and (42) into (4), we arrive at

$$\begin{aligned}&-(n-1)(\alpha^2-\rho)+\mu+\lambda(-2h+2\sigma(\xi)\vartheta(V))-(2V(h)+4h^2)\\&+2V(\sigma(\xi)\vartheta(V))+8h\omega(\xi)\vartheta(V)+2\sigma(\xi)\sigma(V)\vartheta(V)-2\alpha\sigma(V)\vartheta(V)\\&-4\alpha(\vartheta(V))^2\sigma(\xi)-2\alpha\sigma(\xi)|V|^2+2\sigma(\xi)\sigma(\xi)|V|^2.\end{aligned} \tag{43}$$

Hence, we can state:

**Theorem 3.20.** *If the metric $g$ of a LCSM admits the AHRS $(g, V, \lambda, \mu)$ where $V$ is TFVF and satisfied in (35), then the equation (43) holds.*

**Corollary 3.21.** *If the metric $g$ of a LCSM admits the AHRS $(g, V, \lambda, \mu)$ where $V$ is concircular vector filed, that is, $\nabla_{\zeta_1} V = hX$, then $(n-1)(\alpha^2-\rho)+\mu+2h\lambda+2V(h)+4h^2 = 0$.*

In [12], the authors using two $(0, 2)$ tensor fields, have defined bi-conformal vector fields. Then De et al. in [10] defined Ricci bi-conformal vector fields as follows.

**Definition 3.22.** *A vector field $U$ on a pseudo-Riemannian manifold $(M, g)$ is declared to be a Ricci bi-conformal vector field if the following equations are satisfying*

$$\mathcal{L}_U g = fg + hS, \tag{44}$$

*and*

$$\mathcal{L}_U S = fS + hg, \tag{45}$$

*for some non-zero smooth functions $f$ and $h$.*

Now assume that a LCSM $(M^n, g)$ admits AHRS $(g, V, \lambda, \mu)$ and $V$ satisfies in (44) and (45). We have

$$\mathcal{L}_V(\mathcal{L}_V g) = (f^2+h^2+V(f))g + (2fh+V(h))S. \tag{46}$$

Inserting (44) and (46) in (4), we get

$$(1+\lambda h+2fh+V(h))S(\zeta_1,\zeta_2) + (\lambda f-\mu+f^2+h^2+V(f))g(\zeta_1,\zeta_2) = 0. \tag{47}$$

Putting $\zeta_1 = \zeta_2 = \xi$ in (47), we find

$$(1+\lambda h+2fh+V(h))(n-1)(\alpha^2-\rho) + (\lambda f-\mu+f^2+h^2+V(f)) = 0, \tag{48}$$

and

$$\Big(1+\lambda h+2fh+V(h)\Big)\Big(S(X,Y)-(n-1)(\alpha^2-\rho)g(\zeta_1,\zeta_2)\Big) = 0. \tag{49}$$

Set $A = 1+\lambda h+2fh+V(h)$ and $B = \lambda\alpha-\mu+\alpha^2+h^2+V(\alpha)$. Taking the Lie derivative of (47) and applying (44) and (45), we conclude

$$(fA+hB+V(A))S(X,Y) + (fB+hA+V(B))B(\zeta_1,\zeta_2) = 0. \tag{50}$$

Putting $\zeta_1 = \zeta_2 = \xi$ in (50), we arrive at

$$(fA+hB+V(A))(n-1)(\alpha^2-\rho) + fB+hA+V(B) = 0. \tag{51}$$

Applying (48) and (51), we obtain

$$A\Big[h(1-(n-1)^2(\rho-\alpha^2)^2)-(1-n)V(\rho-\alpha^2)\Big] = 0.$$

If $A \neq 0$ then equation (49) implies that manifold is a Einstein manifold and $r = (n-1)(\alpha^2-\rho)$ where $\rho-\alpha^2$ is a constant. If $A = 0$ then $B = 0$. Thereore, we can write:

**Theorem 3.23.** *Suppose that the metric $g$ of a LCSM admits the AHRS $(g, V, \lambda, \mu)$ where $V$ satisfies in (44) and (45). Then manifold $M$ is a Einstein manifold and $r = (1-n)(\rho-\alpha^2)$ or $\lambda = -\frac{1}{h}(1+2fh+V(h))$ and $\mu = -\frac{f}{h}(1+V(h))-f^2+h^2+V(f)$.*

## 4. Examples

Now, we give some examples of LCSM admit AHRSs.

**Example 4.1.** *We denote the standard coordinates of* $\mathbb{R}^3$ *by* $(x, y, z)$ *and consider* $M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0\}$ *and*

$$v_1 = e^z (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}), \qquad v_2 = e^z \frac{\partial}{\partial y}, \qquad v_3 = e^{2z}\frac{\partial}{\partial z}.$$

*The metric* $g$ *determined as follows*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

*The structure* $(\phi, \xi, \vartheta)$ *on* $M$ *is determined by*

$$\xi = v_3, \quad \vartheta(\zeta) = g(\zeta, v_3), \quad \phi = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

*Note the relations (11) and (12) hold. We get*

$$[v_1, v_2] = -e^z v_2, \;\; [v_1, v_3] = -e^{2z} v_1, \;\; [v_2, v_3] = -e^{2z} v_2$$

*and*

$$\nabla_{v_i} v_j = \begin{pmatrix} -e^{2z} v_3 & 0 & -e^{2z} v_1 \\ e^z v_2 & -e^{2z} v_3 - e^z v_1 & -e^{2z} v_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

*The identity* $\nabla_\zeta \xi = -e^{2z}(\zeta + \vartheta(\zeta)\xi)$ *is true, then we get a* $(LCS)_3$*-manifold with* $\alpha = -e^{2z}$*. Then* $\rho = 2e^{4z}$*. The nonvanishing elements of the curvature tensor are:*

$$\begin{aligned} &R(v_1, v_2)v_1 = -e^{2z}(e^{2z} - 1)v_2, \; R(v_1, v_2)v_2 = (e^{4z} - e^{2z})v_1, \\ &R(v_1, v_3)v_1 = -e^{4z} v_3, \; R(v_2, v_3)v_2 = -e^{3z}(e^z v_3 + v_1), \\ &R(v_1, v_3)v_3 = -ee^{4z} v_1, \; R(v_2, v_3)v_3 = -e^{4z} v_2, \end{aligned}$$

*Hence, we obtain*

$$S = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{2z} & 0 \\ 0 & 0 & 2e^{4z} \end{pmatrix} = e^{2z} g + (2e^{4z} + e^{2z})\vartheta \otimes \vartheta.$$

*We have* $\mathcal{L}_\xi g = -2e^{2z}(g + \vartheta \otimes \vartheta)$ *and* $(\mathcal{L}_\xi \circ \mathcal{L}_\xi)g = 0$*. Therefore* $(g, \xi, \lambda = -(e^{2z} + \frac{1}{2}), \mu = -2e^{4z})$ *is an AHRS on manifold* $M$*.*

**Example 4.2.** *Let* $M = \{(y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5 | y_5 > 0\}$ *and*

$$\gamma_1 = y_3\frac{\partial}{\partial y_1}, \qquad \gamma_2 = y_3\frac{\partial}{\partial y_2}, \qquad \gamma_3 = y_3\frac{\partial}{\partial y_3}, \qquad \gamma_4 = y_3\frac{\partial}{\partial y_4}, \qquad \gamma_5 = y_3\frac{\partial}{\partial y_5}.$$

*We consider the metric* $g$ *as* $g_{11} = g_{22} = g_{33} = g_{44} = 1$, $g_{55} = -1$ *and other component of* $g$ *are zero with respect to* $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$*. Structure* $(\phi, \xi, \vartheta)$ *on* $M$ *defined as follows* $\xi = \gamma_5$, $\quad \vartheta(\zeta) = g(\zeta, \gamma_5)$ *and*

$$\phi(\gamma_1) = -\gamma_2, \quad \phi(\gamma_2) = -\gamma_1, \quad \phi(\gamma_3) = -\gamma_4, \quad \phi(\gamma_4) = -\gamma_1, \quad \phi(\gamma_5) = -0,$$

*The relations (11) and (12) are true. Hence, $(\phi, \xi, \vartheta, g)$ is an almost contact structure on manifold. We get $[\gamma_i, \gamma_5] = -[\gamma_5, \gamma_i] = -\gamma_i$ for $i = 1, 2, 3, 4$ and other brackets are equal to zero. Also*

$$\nabla_{\gamma_i}\gamma_j = \begin{pmatrix} -\gamma_5 & 0 & 0 & 0 & -\gamma_1 \\ 0 & -\gamma_5 & 0 & 0 & -\gamma_2 \\ 0 & 0 & -\gamma_5 & 0 & -\gamma_3 \\ 0 & 0 & 0 & -\gamma_5 & -\gamma_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

*The relation $\nabla_\zeta \xi = -(\zeta + \vartheta(\zeta)\xi)$ is true, thus we get a $(LCS)_5$-manifold with $\alpha = -1$. Also, we obtain*

$$\begin{aligned} &R(\gamma_1, \gamma_2)\gamma_1 = -\gamma_2,\ R(\gamma_1, \gamma_2)\gamma_2 = \gamma_1,\ R(\gamma_1, \gamma_3)\gamma_1 = -\gamma_3, \\ &R(\gamma_1, \gamma_3)\gamma_3 = \gamma_1,\ R(\gamma_1, \gamma_4)\gamma_1 = -\gamma_4,\ R(\gamma_1, \gamma_4)\gamma_4 = \gamma_1, \\ &R(\gamma_1, \gamma_5)\gamma_1 = -\gamma_5,\ R(\gamma_1, \gamma_5)\gamma_5 = -\gamma_1,\ R(\gamma_2, \gamma_3)\gamma_2 = -\gamma_3, \\ &R(\gamma_2, \gamma_3)\gamma_3 = \gamma_2,\ R(\gamma_2, \gamma_4)\gamma_2 = -\gamma_4,\ R(\gamma_2, \gamma_4)\gamma_4 = \gamma_2, \\ &R(\gamma_2, \gamma_5)\gamma_2 = -\gamma_5,\ R(\gamma_2, \gamma_5)\gamma_5 = -\gamma_2,\ R(\gamma_3, \gamma_4)\gamma_3 = -\gamma_4, \\ &R(\gamma_3, \gamma_4)\gamma_4 = \gamma_3,\ R(\gamma_3, \gamma_5)\gamma_3 = -\gamma_5,\ R(\gamma_3, \gamma_5)\gamma_5 = -\gamma_3, \\ &R(\gamma_4, \gamma_5)\gamma_4 = -\gamma_5,\ R(\gamma_4, \gamma_5)\gamma_5 = -\gamma_4. \end{aligned}$$

*Hence, we get*

$$S = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} = 4g - 8\vartheta \otimes \vartheta.$$

*We have $\mathcal{L}_\xi g = -2(g + \vartheta \otimes \vartheta)$ and $(\mathcal{L}_\xi \circ \mathcal{L}_\xi)g = 4(g + \vartheta \otimes \vartheta)$. Then $(g, \xi, \lambda = -2, \mu = 12)$ is a HRS on manifold M.*

## References


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