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On pseudo almost periodic solutions for Boussinesq systems on real hyperbolic manifolds

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Abstract. We study the existence, uniqueness and exponential stability of pseudo almost periodic (PAP-) mild solutions for the Boussinesq systems with the initial data in $L^p(\mathbb{H}^d(\mathbb{R}))$-phase space for $p > d$, where $\mathbb{H}^d(\mathbb{R})$ is a real hyperbolic manifold with dimension $d \geqslant 2$. First, we prove the existence and the uniqueness of the bounded mild solutions for the corresponding linear systems by using dispersive and smoothing estimates of the vectorial matrix semigroup. Then, we prove a Massera-type principle to obtain the existence of PAP-mild solutions to the linear systems. Next, using the fixed point arguments, we can pass from the linear systems to the semilinear systems to establish the well-posedness of such solutions. Finally, we will establish the exponential stability of PAP-mild solutions by using Gronwall's inequality.

1. Introduction

We are concerned with the incompressible Boussinesq system on the whole line time-axis $\mathbb{R}$ and on the real hyperbolic space $(\mathbb{H}^d(\mathbb{R}), g)$, where the dimension $d \geqslant 2$ and g is the hyperbolic metric

$$\begin{cases} u_t + (u\cdot\nabla)u - Lu + \nabla p = \kappa\theta h + F & x \in \mathbb{H}^d(\mathbb{R}),\ t \in \mathbb{R}, \\ \operatorname{div} u = 0 & x \in \mathbb{H}^d(\mathbb{R}), \\ \theta_t - \widetilde{L}\theta + (u\cdot\nabla)\theta = f & x \in \mathbb{H}^d(\mathbb{R}),\ t \in \mathbb{R}, \end{cases} \tag{1}$$

where where $L = -(d-1) + \overrightarrow{\Delta}$ is Ebin-Marsden's Laplace operator, $\widetilde{L} = \Delta_g$ is Laplace-Beltrami operator associated with metric g, the constant $\kappa > 0$ is the volume expansion coefficient. The field $h(x,t)$ represents the generalized gravitational field and the constant $\kappa > 0$ is the volume expansion coefficient. The unknowns $u(x,t)$ is the velocity field, $p(x,t)$ is the scalar pressure, and $\theta(x,t)$ is the temperature. The fields $f(x,t)$ and $F(x,t)$ represent the external force. Considering the zero-temperature case, i.e., $\theta = 0$, then Boussinesq system (1) becomes the Navier-Stokes equation.

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We first recall some recent works of the well-posedness and stability for fluid dynamical flows on the real hyperbolic manifold and non-compact manifolds with some bounded and negative conditions of curvatures. In [10], Pierfelice proved the dispersive and smoothing estimates of vectorial heat and Stokes semigroups and, then the author used these estimates to establish establish the well-posedness of mild solutions for Navier-Stokes equations on the hyperbolic manifolds and on non-compact manifolds. The existence and exponential stability of periodic and almost periodic mild solutions for Navier-Stokes equations were obtained by Huy et al. in [3, 4, 6]. Such problems of asymptically almost periodic mild solutions for Navier-Stokes equations on the hyperbolic manifolds were been presented by Xuan et al. in [11, 12] and for Keller-Segel systems in [13]. We recall also some related papers on the existence and stability of weighted pseudo almost periodic mild solutions for Navier-Stokes equations on exterior domains of $\mathbb{R}^n$ was been provided in [7]. Moreover, such problems of periodic and almost periodic mild solutions for Boussinesq systems on Euclidean space $\mathbb{R}^d$ were done in [4, 9, 14]. However, there is no work which provides the well-posedness and asymptotic behaviour of pseudo almost periodic mild solutions for Boussinesq systems in the framework of hyperbolic manifolds.

In this paper, we extend the results obtained in [11, 13] to study the existence, uniqueness and exponential stability of pseudo almost periodic (PAP-) mild solutions for Boussinesq systems (1). We first represent system (1) under the matrix intergral equation. Then, we use the dispersive and smoothing estimates of the matrix semigroup to prove the well-posedness of bounded mild solutions defined on the whole line time-axis $\mathbb{R}$ for the linear Boussinesq systems (see Theorem 3.2 (*i*)). This well-posedness permits us to define the solution operator for linear systems. Then, we prove a Massera-type principle to guarantees the existence of PAP-mild solutions for linear Boussinesq systems (see Theorem 3.2 (*ii*)). Using the results of linear Boussinesq systems and fixed point argument we establish the well-posedness of PAP-mild solutions for the Boussinesq systems (1) (see Theorem 4.1). Finally, we prove an exponential stability for such PAP-mild solutions by using the Gronwall inequality (see Theorem 4.2).

This paper is organized as follows: Section 2 relies on some preliminaries consisting the setting of Boussinesq systems on hyperbolic manifolds and notions of generalized functions; Section 3 gives the well-posedness of PAP-mild solutions for linear Boussinesq systems; Section 4 provides the well-posedness and stability of PAP-mild solutions for Boussinesq systems.

2. Preliminaries

2.1. Boussinesq systems on hyperbolic manifold

Let $(\mathbf{M} =: \mathbb{H}^d(\mathbb{R}), g)$ be a real hyperbolic manifold of dimension $d \geqslant 2$ which is realized as the upper sheet

$$x_0^2 - x_1^2 - x_2^2 ... - x_d^2 = 1 \ (x_0 \geqslant 1),$$

of hyperboloid in $\mathbb{R}^{d+1}$, equipped with the Riemannian metric

$$g := -dx_0^2 + dx_1^2 + ... + dx_d^2.$$

In geodesic polar coordinates, the hyperbolic manifold is

$$\mathbb{H}^d(\mathbb{R}) := \left\{ (\cosh\tau, \omega \sinh\tau),\ \tau \geqslant 0, \omega \in \mathbb{S}^{d-1} \right\}$$

with the metric

$$g := d\tau^2 + (\sinh\tau)^2 d\omega^2$$

where $d\omega^2$ is the canonical metric on the sphere $\mathbb{S}^{d-1}$. A remarkable property on $\mathbf{M}$ is the Ricci curvature tensor : $\mathrm{Ric}_{ij} = -(d-1)g_{ij}$.

For simplicity, we assume that the volume expansion coefficient $\kappa = 1$ and the Boussinesq system becomes

$$\begin{cases} u_t + (u\cdot\nabla)u - Lu + \nabla p &= \theta h + F \qquad x \in \mathbf{M},\ t \in \mathbb{R}, \\ \nabla \cdot u &= 0 \qquad\qquad\quad x \in \mathbf{M}, \\ \theta_t - \widetilde{L}\theta + (u\cdot\nabla)\theta &= f \qquad\qquad\quad x \in \mathbf{M},\ t \in \mathbb{R}, \end{cases} \tag{2}$$

where $L = -(d-1) + \overrightarrow{\Delta}$ is Ebin-Marsden's Laplace operator (see [11] and also [2, 10]), $\widetilde{L} = \Delta_g$ is Laplace-Beltrami operator associating with metric g. Normally, the gravitational field and is not depend on the time, but in this paper we will consider the general case, where $h(x,t) : \mathbf{M} \times \mathbb{R} \to \Gamma(T\mathbf{M})$ is depend on the time. The functions $f : \mathbf{M} \times \mathbb{R} \to \Gamma(T\mathbf{M})$ is given such that $\mathrm{div} f$ represents the reference temperature and $F : \mathbf{M} \times \mathbb{R} \to \Gamma(T\mathbf{M} \otimes T\mathbf{M})$ is a second order tensor fields such that $\mathrm{div} F$ represents the external force. The unknowns are $u(x,t) : \mathbf{M} \times \mathbb{R} \to \Gamma(T\mathbf{M}), p(x,t) : \mathbf{M} \times \mathbb{R} \to \mathbb{R}$ and $\theta(x,t) : \mathbf{M} \times \mathbb{R} \to \mathbb{R}$ representing respectively, the velocity field, the pressure and the temperature of the fluid at point $(x,t) \in \mathbf{M} \times \mathbb{R}$.

Since we have the following identity

$$\mathrm{div}(\theta u) = \theta \mathrm{div} u + (\nabla\theta) \cdot u = (u \cdot \nabla)\theta,$$

the Boussinesq system (2) can be rewritten as

$$\begin{cases} u_t + \mathrm{div}(u \otimes u) - Lu + \nabla p &= \theta h + F \qquad x \in \mathbf{M},\ t \in \mathbb{R}, \\ \nabla \cdot u &= 0 \qquad x \in \mathbf{M}, \\ \theta_t - \widetilde{L}\theta + \mathrm{div}(\theta u) &= f \qquad x \in \mathbf{M},\ t \in \mathbb{R}, \end{cases} \tag{3}$$

We consider system (3) for (u,θ) in the Casterian product space $C_b(\mathbb{R}, L^p(\mathbf{M}; \Gamma(T\mathbf{M}))) \times C_b(\mathbb{R}, L^p(\mathbf{M}; \mathbb{R}))$, where $p > d$. Assume that $h \in C_b(\mathbb{R}, L^\infty(\mathbf{M}, \Gamma(T\mathbf{M})))$ and $F \in C_b(\mathbb{R}, L^p(\mathbf{M}, \Gamma(T\mathbf{M})))$. Therefore, by applying the Kodaira-Hodge operator $\mathbb{P} := I + \mathrm{grad}(-\Delta_g)^{-1}\mathrm{div}$ to the system (3) (similar the same manner for Navier-Stokes equation, in details see [10, 11]), we get

$$\begin{cases} u_t &= Lu + \mathbb{P}(\theta h) + \mathbb{P}\mathrm{div}(-u \otimes u) + \mathbb{P}(F) \qquad x \in \mathbf{M},\ t \in \mathbb{R}, \\ \nabla \cdot u &= 0 \qquad x \in \mathbf{M}, \\ \theta_t &= \widetilde{L}\theta + \mathrm{div}(-\theta u) + f \qquad x \in \mathbf{M},\ t \in \mathbb{R}. \end{cases} \tag{4}$$

We set $\mathcal{A} := \begin{bmatrix} -L & 0 \\ 0 & -\widetilde{L} \end{bmatrix}$ acting on the Cartesian product space $L^p(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^p(\mathbf{M}; \mathbb{R})$. Therefore, using Duhamel's principle in a matrix form, we arrive at the following integral formulation for (4)

$$\begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} = \mathcal{B}\left(\begin{bmatrix} u \\ \theta \end{bmatrix}, \begin{bmatrix} u \\ \theta \end{bmatrix}\right)(t) + \mathcal{H}_h(\theta)(t) + \mathcal{F}\left(\begin{bmatrix} F \\ f \end{bmatrix}\right)(t), \tag{5}$$

where the bilinear, linear-coupling and external forced operators used in the above equation are given respectively by

$$\mathcal{B}\left(\begin{bmatrix} u \\ \theta \end{bmatrix}, \begin{bmatrix} v \\ \xi \end{bmatrix}\right)(t) := -\int_{-\infty}^t e^{-(t-s)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(u \otimes v) \\ u\xi \end{bmatrix}(s) ds, \tag{6}$$

$$\mathcal{H}_h(\theta)(t) := \int_{-\infty}^t e^{-(t-s)\mathcal{A}} \begin{bmatrix} \mathbb{P}(h\theta) \\ 0 \end{bmatrix}(s) ds \tag{7}$$

and

$$\mathcal{F}\left(\begin{bmatrix} F \\ f \end{bmatrix}\right)(t) := \int_{-\infty}^t e^{-(t-s)\mathcal{A}} \begin{bmatrix} \mathbb{P}(F) \\ f \end{bmatrix}(s) ds. \tag{8}$$

2.2. *Pseudo almost periodic functions*

We recall some concepts of generalized functions. For more details we refer the readers to books [1] and references therein. Let X be a Banach space, we denote

$$C_b(\mathbb{R}, X) := \{f : \mathbb{R} \to X \mid f \text{ is continuous on } \mathbb{R} \text{ and } \sup_{t\in\mathbb{R}} \|f(t)\|_X < \infty\}$$

which is a Banach space endowed with the norm $\|f\|_{\infty,X} = \|f\|_{C_b(\mathbb{R},X)} := \sup_{t\in\mathbb{R}} \|f(t)\|_X$.

Definition 2.1. *(AP-function) A function $h \in C_b(\mathbb{R}, X)$ is called almost periodic function if for each $\epsilon > 0$, there exists $l_\epsilon > 0$ such that every interval of length l_ϵ contains at least a number T with the following property*

$$\sup_{t\in\mathbb{R}} \|h(t+T) - h(t)\| < \epsilon.$$

The collection of all almost periodic functions $h : \mathbb{R} \to X$ will be denoted by $AP(\mathbb{R}, X)$ which is a Banach space endowed with the norm $\|h\|_{AP(\mathbb{R},X)} = \sup_{t\in\mathbb{R}} \|h(t)\|_X$.

Definition 2.2. *(PAP-function) A function $f \in C_b(\mathbb{R}, X)$ is called pseudo almost periodic if it can be decomposed as $f = g + \phi$ where $g \in AP(\mathbb{R}, X)$ and ϕ is a bounded continuous function with vanishing mean value i.e*

$$\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \|\phi(t)\|_X dt = 0.$$

We denote the set of all functions with vanishing mean value by $PAP_0(\mathbb{R}, X)$ and the set of all the pseudo almost periodic (PAP-) functions by $PAP(\mathbb{R}, X)$.

We have that $(PAP(\mathbb{R}, X), \|.\|_{\infty,X})$ is a Banach space, where $\|.\|_{\infty,X}$ is the supremum norm (see [1, Theorem 5.9]). We have the following decomposition (see also [1]):

$$PAP(\mathbb{R}, X) = AP(\mathbb{R}, X) \oplus PAP_0(\mathbb{R}, X).$$

The notion of pseudo almost periodic function is a generalisation of the periodic and almost periodic functions. Precisely, we have the following inclusions

$$P(\mathbb{R}, X) \hookrightarrow AP(\mathbb{R}, X) \hookrightarrow PAP(\mathbb{R}, X) \hookrightarrow C_b(\mathbb{R}, X).$$

where $P(\mathbb{R}, X)$ is the space of all continuous and periodic functions from $\mathbb{R}$ to X.
Example.

(*i*) The function $h(t) = \sin t + \sin(\sqrt{2}t)$ is almost periodic but not periodic, $\tilde{h}(t) = \sin t + \sin(\sqrt{2}t) + \dfrac{1}{|t|}$ is pseudo almost periodic but not almost periodic.

(*ii*) Let X be a Banach space and $g \in X - \{0\}$, we have that $f = hg \in AP(\mathbb{R}, X)$ and $\tilde{f} = \tilde{h}g \in PAP(\mathbb{R}, X)$.

3. Wellposedness of PAP-mild solutions for linear systems

We denote and use the norm on the product space $L^p(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^p(\mathbf{M}; \mathbb{R})$ by

$$\left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{L^p\times L^p} = \|u\|_{L^p(\mathbf{M};\Gamma(T\mathbf{M}))} + \|\theta\|_{L^p(\mathbf{M};\mathbb{R})}.$$

For a given vector $(v, \eta) \in L^p(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^p(\mathbf{M}; \mathbb{R})$ we consider the following linear equation

$$\begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} = \mathcal{B}\left(\begin{bmatrix} v \\ \eta \end{bmatrix}, \begin{bmatrix} v \\ \eta \end{bmatrix} \right)(t) + \mathcal{H}_h(\eta)(t) + \mathcal{F}\left(\begin{bmatrix} F \\ f \end{bmatrix} \right)(t), \tag{9}$$

where the operators $\mathcal{B}, \mathcal{H}_h$ and $\mathcal{F}$ given by (6),(7) and (8), respectively.

The dispersive and smoothing estimates of matrix semigroup $e^{-t\mathcal{A}}$ are given in the following lemma.

Lemma 3.1.

(*i*) *(Dispersive estimates) For $t > 0$, and p, q such that $1 \leqslant p \leqslant q \leqslant \infty$, the following dispersive estimates hold:*

$$\left\| e^{-t\mathcal{A}} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{L^q \times L^q} \leqslant [h_d(t)]^{\frac{1}{p}-\frac{1}{q}} e^{-t(\gamma_{p,q})} \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{L^p \times L^p} \tag{10}$$

for all $(u_0, \theta_0) \in L^p(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^p(\mathbf{M}; \mathbb{R})$, where $h_d(t) := C \max\left(\frac{1}{t^{d/2}}, 1\right)$, $\gamma_{p,q} := \frac{\delta_d}{2}\left[\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{8}{q}\left(1 - \frac{1}{p}\right)\right]$ and δ_d are positive constants depending only on d.

(*ii*) *(Smoothing estimates) For p and q such that $1 < p \leqslant q < \infty$ we obtain for all $t > 0$ that*

$$\left\| e^{-t\mathcal{A}} \mathrm{div} \begin{bmatrix} T_0^\sharp \\ U_0^\sharp \end{bmatrix} \right\|_{L^q \times L^q} \leqslant [h_d(t)]^{\frac{1}{p}-\frac{1}{q}+\frac{1}{d}} e^{-t\left(\frac{\gamma_{q,q}+\gamma_{p,q}}{2}\right)} \left\| \begin{bmatrix} T_0^\sharp \\ U_0^\sharp \end{bmatrix} \right\|_{L^p \times L^p} \tag{11}$$

for all tensor $T_0^\sharp \in L^p(\mathbf{M}; \Gamma(T\mathbf{M} \otimes T\mathbf{M}))$ and vector field $U_0 \in L^p(\mathbf{M}; \Gamma(T\mathbf{M}))$.

Proof. We have $e^{-t\mathcal{A}} = \begin{bmatrix} e^{tL} & 0 \\ 0 & e^{t\widetilde{L}} \end{bmatrix}$. Assertion (*i*) is valid since the fact that: for $t > 0$, and p, q such that $1 \leqslant p \leqslant q \leqslant \infty$, the following $L^p - L^q$-dispersive estimates hold (see [10, Theorem 4.1] and its proof):

$$\left\| e^{tL} u_0 \right\|_{L^q} \leqslant [h_d(t)]^{\frac{1}{p}-\frac{1}{q}} e^{-t(d-1+\gamma_{p,q})} \|u_0\|_{L^p} \leqslant [h_d(t)]^{\frac{1}{p}-\frac{1}{q}} e^{-t(\gamma_{p,q})} \|u_0\|_{L^p}$$

for all $u_0 \in L^p(\mathbf{M}; \Gamma(T\mathbf{M}))$ and

$$\left\| e^{t\widetilde{L}} \theta_0 \right\|_{L^q} \leqslant [h_d(t)]^{\frac{1}{p}-\frac{1}{q}} e^{-t(\gamma_{p,q})} \|\theta_0\|_{L^p} \text{ for all } \theta_0 \in L^p(\mathbf{M}; \mathbb{R}),$$

where $h_d(t) := C \max\left(\frac{1}{t^{d/2}}, 1\right)$, $\gamma_{p,q} := \frac{\delta_d}{2}\left[\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{8}{q}\left(1 - \frac{1}{p}\right)\right]$ and δ_d are positive constants depending only on d.

Assertion (*ii*) comes from the following $L^p - L^q$-smoothing estimates: for $1 < p \leqslant q < \infty$ and $t > 0$ we have (see [10, Corollary 4.3] and its proof):

$$\begin{aligned} \left\| e^{tL} \mathrm{div} T_0^\sharp \right\|_{L^q} &\leqslant [h_d(t)]^{\frac{1}{p}-\frac{1}{q}+\frac{1}{d}} e^{-t\left(d-1+\frac{\gamma_{q,q}+\gamma_{p,q}}{2}\right)} \left\| T_0^\sharp \right\|_{L^p} \\ &\leqslant [h_d(t)]^{\frac{1}{p}-\frac{1}{q}+\frac{1}{d}} e^{-t\left(\frac{\gamma_{q,q}+\gamma_{p,q}}{2}\right)} \left\| T_0^\sharp \right\|_{L^p} \end{aligned}$$

and

$$\left\| e^{t\widetilde{L}} \mathrm{div} U_0^\sharp \right\|_{L^q} \leqslant [h_d(t)]^{\frac{1}{p}-\frac{1}{q}+\frac{1}{d}} e^{-t\left(\frac{\gamma_{q,q}+\gamma_{p,q}}{2}\right)} \left\| U_0^\sharp \right\|_{L^p}$$

for all tensor $T_0^\sharp \in L^p(\mathbf{M}; \Gamma(T\mathbf{M} \otimes T\mathbf{M}))$ and vector field $U_0 \in L^p(\mathbf{M}; T\mathbf{M})$. □

Now we state and prove the main results of this section in the following theorem.

Theorem 3.2. *Let $(\mathbf{M}, g)$ be a d-dimensional real hyperbolic manifold with $d \geqslant 2$. For given functions $v \in \mathscr{X} := C_b(\mathbb{R}, L^p(\mathbf{M}; \Gamma(T\mathbf{M})))$, $\eta \in \mathscr{S} := C_b(\mathbb{R}, L^p(\mathbf{M}; \mathbb{R}))$, if the fields $h \in \mathscr{H} := C_b(\mathbb{R}, L^p(\mathbf{M}; \Gamma(T\mathbf{M})))$, $F \in \mathscr{F} := C_b(\mathbb{R}, L^r(\mathbf{M}; \Gamma(T\mathbf{M}))$, $f \in \mathscr{O} := C_b(\mathbb{R}, L^r(\mathbf{M}; \mathbb{R}))$ with $p > d$ and $\frac{dp}{2p+d} < r \leqslant p$, then the following assertions hold*

(*i*) *The linear equation* (9) *has a unique bounded solution $(\hat{u}, \hat{\theta}) \in \mathscr{X} \times \mathscr{S}$. Moreover, we have satisfying*

$$\left\| \begin{bmatrix} \hat{u} \\ \hat{\theta} \end{bmatrix} \right\|_{\mathscr{X} \times \mathscr{S}} \leqslant M_1 \left\| \begin{bmatrix} v \\ \eta \end{bmatrix} \right\|^2_{\mathscr{X} \times \mathscr{S}} + M_2 \|h\|_{\mathscr{H}} \left\| \begin{bmatrix} v \\ \eta \end{bmatrix} \right\|_{\mathscr{X} \times \mathscr{S}} + M_3 \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\mathscr{F} \times \mathscr{O}}. \tag{12}$$

Here the positive constants M_1, M_2 and M_3 do not depend on given functions v, η, h, F and f.

(*ii*) *If the functions* $v(x,\cdot), \eta(x,\cdot), h(x,\cdot), F(x,\cdot)$ *and* $f(x,\cdot)$ *are both pseudo almost periodic, then the mild solution* $(\hat{u}, \hat{\theta})$ *obtained in Assertion* (*i*) *is also pseudo almost periodic.*

Proof. (*i*) Using Lemma 3.1 and the boundedness of the operator $\mathbb{P}$ (see [8]), we have

$$\begin{aligned}
\left\| \begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} \right\|_{L^p\times L^p} &\leqslant \left\| \mathcal{B}\left(\begin{bmatrix} v \\ \eta \end{bmatrix}, \begin{bmatrix} v \\ \eta \end{bmatrix} \right)(t) \right\|_{L^p\times L^p} + \|\mathcal{H}_h(\eta)(t)\|_{L^p} + \left\| \mathcal{F}\left(\begin{bmatrix} F \\ f \end{bmatrix} \right)(t) \right\|_{L^p\times L^p} \\
&\leqslant \int_{-\infty}^t \left\| e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v\otimes v) \\ v\eta \end{bmatrix}(\tau) \right\|_{L^p\times L^p} d\tau + \int_{-\infty}^t \left\| e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(h\eta)(\tau) \\ 0 \end{bmatrix} \right\|_{L^p\times L^p} d\tau \\
&\quad + \int_{-\infty}^t \left\| e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(F) \\ f \end{bmatrix}(\tau) \right\|_{L^p\times L^p} d\tau \\
&\leqslant \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{p}+\frac{1}{d}} e^{-\beta_2(t-\tau)} \left\| \begin{bmatrix} v\otimes v \\ v\eta \end{bmatrix}(\tau) \right\|_{L^{\frac{p}{2}}\times L^{\frac{p}{2}}} d\tau \\
&\quad + \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{p}} e^{-\beta_1(t-\tau)} \left\| \begin{bmatrix} (h\eta)(\tau) \\ 0 \end{bmatrix} \right\|_{L^{\frac{p}{2}}\times L^{\frac{p}{2}}} d\tau \\
&\quad + \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{r}-\frac{1}{p}} e^{-\beta_3(t-\tau)} \left\| \begin{bmatrix} F \\ f \end{bmatrix}(\tau) \right\|_{L^r\times L^r} d\tau \\
&\leqslant \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{p}+\frac{1}{d}} e^{-\beta_2(t-\tau)} \left\| \begin{bmatrix} v \\ \eta \end{bmatrix}(\tau) \right\|_{L^p\times L^p} \left\| \begin{bmatrix} v \\ \eta \end{bmatrix}(\tau) \right\|_{L^p\times L^p} d\tau \\
&\quad + \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{p}} e^{-\beta_1(t-\tau)} \|h(\tau)\|_{L^p} \|\eta(\tau)\|_{L^p} d\tau \\
&\quad + \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{r}-\frac{1}{p}} e^{-\beta_3(t-\tau)} \left\| \begin{bmatrix} F \\ f \end{bmatrix}(\tau) \right\|_{L^r\times L^r} d\tau \\
&\leqslant \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{p}} e^{-\beta_1(t-\tau)} d\tau \|h\|_{\mathscr{H}} \|\eta\|_{\mathscr{S}} + \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{p}+\frac{1}{d}} e^{-\beta_2(t-\tau)} d\tau \left\| \begin{bmatrix} v \\ \eta \end{bmatrix} \right\|^2_{\mathscr{X}\times\mathscr{S}} \\
&\quad + \int_{-\infty}^t [h_d(t-\tau)]^{\frac{1}{r}-\frac{1}{p}} e^{-\beta_3(t-\tau)} d\tau \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\mathscr{F}\times\mathscr{O}} \\
&\leqslant \int_0^{+\infty} C^{\frac{1}{p}} \left[\tau^{-\frac{d}{2p}} + 1 \right] e^{-\beta_1\tau} d\tau \|h\|_{\mathscr{H}} \|\eta\|_{\mathscr{S}} + \int_0^{+\infty} C^{\frac{1}{p}+\frac{1}{d}} \left[\tau^{-\frac{d}{2p}-\frac{1}{2}} + 1 \right] e^{-\beta_2\tau} d\tau \left\| \begin{bmatrix} v \\ \eta \end{bmatrix} \right\|^2_{\mathscr{X}\times\mathscr{S}} \\
&\quad + \int_0^{+\infty} C^{\frac{1}{r}-\frac{1}{p}} \left[\tau^{-\frac{d}{2r}+\frac{d}{2p}} + 1 \right] e^{-\beta_3\tau} d\tau \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\mathscr{F}\times\mathscr{O}} \\
&\leqslant C^{\frac{1}{p}} \left[\beta_1^{\frac{d}{2p}-1} \mathbf{\Gamma}\left(1 - \frac{d}{2p}\right) + \frac{1}{\beta_1} \right] \|h\|_{\mathscr{H}} \|\eta\|_{\mathscr{S}} + C^{\frac{1}{p}+\frac{1}{d}} \left[\beta_2^{\frac{d}{2p}-\frac{1}{2}} \mathbf{\Gamma}\left(\frac{1}{2} - \frac{d}{2p}\right) + \frac{1}{\beta_2} \right] \left\| \begin{bmatrix} v \\ \eta \end{bmatrix} \right\|^2_{\mathscr{X}\times\mathscr{S}} \\
&\quad + C^{\frac{1}{r}-\frac{1}{p}} \left[\beta_3^{\frac{d}{2r}-\frac{d}{2p}-1} \mathbf{\Gamma}\left(1 - \frac{d}{2r} + \frac{d}{2p}\right) + \frac{1}{\beta_3} \right] \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\mathscr{F}\times\mathscr{O}} \\
&\leqslant M_1 \left\| \begin{bmatrix} v \\ \eta \end{bmatrix} \right\|^2_{\mathscr{X}\times\mathscr{S}} + M_2 \|h\|_{\mathscr{H}} \left\| \begin{bmatrix} v \\ \eta \end{bmatrix} \right\|_{\mathscr{X}\times\mathscr{S}} + M_3 \left\| \begin{bmatrix} F \\ f \end{bmatrix} \right\|_{\mathscr{F}\times\mathscr{O}}. \qquad (13)
\end{aligned}$$

Here $\beta_1 := \gamma_{p/2,p}$, $\beta_2 := \frac{\gamma_{p,p}+\gamma_{p/2,p}}{2}$, $\beta_3 := \gamma_{r,p}$, and with the fact that $p > d \geqslant 2$ and $\frac{dp}{2p+d} < r \leqslant p$ the above gamma integrals are bounded by the following constants

$$\begin{aligned}
M_1 &= C^{\frac{1}{p}+\frac{1}{d}} \left[\beta_2^{\frac{d}{2p}-\frac{1}{2}} \mathbf{\Gamma}\left(\frac{1}{2} - \frac{d}{2p}\right) + \frac{1}{\beta_2} \right], \; M_2 = C^{\frac{1}{p}} \left[\beta_1^{\frac{d}{2p}-1} \mathbf{\Gamma}\left(1 - \frac{d}{2p}\right) + \frac{1}{\beta_1} \right], \\
M_3 &= C^{\frac{1}{r}-\frac{1}{p}} \left[\beta_3^{\frac{d}{2r}-\frac{d}{2p}-1} \mathbf{\Gamma}\left(1 - \frac{d}{2r} + \frac{d}{2p}\right) + \frac{1}{\beta_3} \right]. \qquad (14)
\end{aligned}$$

(*ii*) From the existence of the bounded mild solution of linear integral equation (9) on the whole line time-axis, we define a solution operator associating with equation (9) as follows

$$\begin{aligned} \mathcal{S} : \mathscr{X} \times \mathscr{S} \times \mathscr{H} \times \mathscr{F} \times \mathscr{O} &\longrightarrow \mathscr{X} \times \mathscr{S} \\ (v, \eta) &\longmapsto (u, \theta), \end{aligned}$$

where $(u, \theta)(t)$ is the bounded mild solution of equation (9). This means that

$$\mathcal{S}\,(v, \eta, h, F, f)\,(t) = (u, \theta)(t)$$

for all $t \in \mathbb{R}$.

Since $(v, \eta, h, F, f) \in \mathscr{X} \times \mathscr{S} \times \mathscr{H} \times \mathscr{F} \times \mathscr{O}$ is pseudo almost periodic, there exist the following functions

$$(v_1, \eta_1, h_1, F_1, f_1) \in AP\,(\mathbb{R}, L^p(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^p(\mathbf{M}; \mathbb{R}) \times L^p(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^r(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^r(\mathbf{M}; \mathbb{R}))$$

and

$$(v_2, \eta_2, h_2, F_2, f_2) \in PAP_0\,(\mathbb{R}_+, L^p(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^p(\mathbf{M}; \mathbb{R}) \times L^p(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^r(\mathbf{M}; \Gamma(T\mathbf{M})) \times L^r(\mathbf{M}; \mathbb{R}))$$

such that $v(t) = v_1(t) + v_2(t)$, $\eta(t) = \eta_1(t) + \eta_2(t)$, $h(t) = h_1(t) + h_2(t)$, $F(t) = F_1(t) + F_2(t)$ and $f(t) = f_1(t) + f_2(t)$ for all $t \in \mathbb{R}$.

Therefore, for all $t \in \mathbb{R}$, we have

$$\begin{aligned} & \mathcal{S}\,(v, \eta, h, F, f)\,(t) \\ = \; & \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v \otimes v) \\ v\eta \end{bmatrix} (\tau) d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(h\eta)(\tau) \\ 0 \end{bmatrix} d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(F) \\ f \end{bmatrix} (\tau) d\tau \\ = \; & \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v_1 \otimes v_1) \\ v_1\eta_1 \end{bmatrix} (\tau) d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(h_1\eta_1)(\tau) \\ 0 \end{bmatrix} d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(F_1) \\ f_1 \end{bmatrix} (\tau) d\tau \\ & + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v_1 \otimes v_2) \\ v_1\eta_2 \end{bmatrix} (\tau) d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v_2 \otimes v) \\ v_2\eta \end{bmatrix} (\tau) d\tau \\ & + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(h_1\eta_2)(\tau) \\ 0 \end{bmatrix} d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(h_2\eta)(\tau) \\ 0 \end{bmatrix} d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(F_2) \\ f_2 \end{bmatrix} (\tau) d\tau \\ = \; & \mathcal{S}(v_1, \eta_1, h_1, F_1, f_1)(t) + \mathcal{I}(t), \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}(t) = \; & \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v_1 \otimes v_2) \\ v_1\eta_2 \end{bmatrix} (\tau) d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v_2 \otimes v) \\ v_2\eta \end{bmatrix} (\tau) d\tau \\ & + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(h_1\eta_2)(\tau) \\ 0 \end{bmatrix} d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(h_2\eta)(\tau) \\ 0 \end{bmatrix} d\tau + \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}(F_2) \\ f_2 \end{bmatrix} (\tau) d\tau. \end{aligned}$$

In order to obtain that the function $\mathcal{S}\,(v, \eta, h, F, f)\,(\cdot)$ is pseudo almost periodic, we prove that $\mathcal{S}(v_1, \eta_1, h_1, F_1, f_1)(\cdot)$ is almost periodic and $\mathcal{I}(\cdot)$ is PAP_0-function.

Indeed, since $t \mapsto (v_1, \eta_1, h_1, F_1, f_1)(t)$ is almost periodic, we have: for each $\varepsilon > 0$, there is the constant $L_\varepsilon > 0$ such that every interval with length L_ε contains at least a number T in order to

$$\begin{aligned} \sup_{t \in \mathbb{R}} \big(& \|v_1(t+T) - v_1(t)\|_{L^p} + \|\eta_1(t+T) - \eta_1(t)\|_{L^p} + \|h_1(t+T) - h_1(t)\|_{L^p} \\ & + \|F_1(t+T) - F_1(t)\|_{L^r} + \|f_1(t+T) - f_1(t)\|_{L^r} \big) < \varepsilon. \end{aligned}$$

Therefore, we can estimate

$$\begin{aligned} & \left\| \mathcal{S}(v_1, \eta_1, h_1, F_1, f_1)(t+T) - \mathcal{S}(v_1, \eta_1, h_1, F_1, f_1)(t) \right\|_{L^p \times L^p} \\ \leqslant & \left\| \int_{-\infty}^{t+T} e^{-(t+T-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v_1 \otimes v_1) \\ v_1\eta_1 \end{bmatrix} (\tau) d\tau - \int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}(v_1 \otimes v_1) \\ v_1\eta_1 \end{bmatrix} (\tau) d\tau \right\|_{L^p \times L^p} \end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_{-\infty}^{t+T} e^{-(t+T-\tau)\mathcal{A}}\begin{bmatrix}\mathbb{P}(h_1\eta_1)(\tau)\\0\end{bmatrix}d\tau-\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}\begin{bmatrix}\mathbb{P}(h_1\eta_1)(\tau)\\0\end{bmatrix}d\tau\right\|_{L^p\times L^p}\\
&+\left\|\int_{-\infty}^{t+T} e^{-(t+T-\tau)\mathcal{A}}\begin{bmatrix}\mathbb{P}(F_1)\\f_1\end{bmatrix}(\tau)d\tau-\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}\begin{bmatrix}\mathbb{P}(F_1)\\f_1\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}\\
= &\left\|\int_{0}^{+\infty} e^{-z\mathcal{A}}\left(\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_1)\\v_1\eta_1\end{bmatrix}(t+T-z)-\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_1)\\v_1\eta_1\end{bmatrix}(t-z)\right)dz\right\|_{L^p\times L^p}\\
&+\left\|\int_{0}^{+\infty} e^{-z\mathcal{A}}\left(\begin{bmatrix}\mathbb{P}(h_1\eta_1)\\0\end{bmatrix}(t+T-z)-\begin{bmatrix}\mathbb{P}(h_1\eta_1)\\0\end{bmatrix}(t-z)\right)dz\right\|_{L^p\times L^p}\\
&+\left\|\int_{0}^{+\infty} e^{-z\mathcal{A}}\left(\begin{bmatrix}\mathbb{P}(F_1)\\f_1\end{bmatrix}(t+T-z)-\begin{bmatrix}\mathbb{P}(F_1)\\f_1\end{bmatrix}(t-z)\right)dz\right\|_{L^p\times L^p}\\
= &\left\|\int_{0}^{+\infty} e^{-z\mathcal{A}}\left(\mathrm{div}\begin{bmatrix}\mathbb{P}([v_1(t+T-z)-v_1(t-z)]\otimes v_1(t+T-z))\\ [v_1(t+T-z)-v_1(t-z)]\eta_1(t+T-z)\end{bmatrix}\right)dz\right\|_{L^p\times L^p}\\
&+\left\|\int_{0}^{+\infty} e^{-z\mathcal{A}}\left(\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1(t-z)\otimes[v_1(t+T-z)-v_1(t-z)])\\ v_1(t-z)[\eta_1(t+T-z)-\eta_1(t-z)]\end{bmatrix}\right)dz\right\|_{L^p\times L^p}\\
&+\left\|\int_{0}^{+\infty} e^{-z\mathcal{A}}\left(\begin{bmatrix}\mathbb{P}([h_1(t+T-z)-h_1(t-z)]\eta_1(t+T-z))\\0\end{bmatrix}\right)dz\right\|_{L^p\times L^p}\\
&+\left\|\int_{0}^{+\infty} e^{-z\mathcal{A}}\left(\begin{bmatrix}\mathbb{P}(h_1(t-z)[\eta_1(t+T-z)-\eta_1(t-z)])\\0\end{bmatrix}\right)dz\right\|_{L^p\times L^p}\\
&+\left\|\int_{0}^{+\infty} e^{-z\mathcal{A}}\left(\begin{bmatrix}\mathbb{P}(F_1)\\f_1\end{bmatrix}(t+T-z)-\begin{bmatrix}\mathbb{P}(F_1)\\f_1\end{bmatrix}(t-z)\right)dz\right\|_{L^p\times L^p}\\
\leqslant\ & 2M_1\left\|\begin{bmatrix}v_1\\\eta_1\end{bmatrix}\right\|_{\mathscr{X}\times\mathscr{S}}\sup_{z\in\mathbb{R}}\left\|\begin{bmatrix}v_1(z+T)-v_1(z)\\\eta_1(z+T)-\eta_1(z)\end{bmatrix}\right\|_{L^p\times L^p}\\
&+M_2\|\eta_1\|_{\mathscr{S}}\sup_{z\in\mathbb{R}}\|h_1(z+T)-h_1(z)\|_{L^p}+N\|h_1\|_{\mathscr{H}}\sup_{z\in\mathbb{R}}\|\eta_1(z+T)-\eta_1(z)\|_{L^p}\\
&+M_3\sup_{z\in\mathbb{R}}\left\|\begin{bmatrix}F_1\\f_1\end{bmatrix}(z+T)-\begin{bmatrix}F_1\\f_1\end{bmatrix}(z)\right\|_{L^r\times L^r}\\
<\ &\left((2M_1+M_2)\left\|\begin{bmatrix}v_1\\\eta_1\end{bmatrix}\right\|_{\mathscr{X}\times\mathscr{S}}+N\|h_1\|_{\mathscr{H}}+M_3\right)\varepsilon,\ \forall t\in\mathbb{R}.
\end{aligned}
$$

Here the constants M_1, M_2 and M_3 are determined in Assertion (i). This implies that $\mathcal{S}(v_1,\eta_1,h_1,F_1,f_1)(\cdot)$ is almost periodic.

We now prove that $\mathcal{I}(\cdot)$ is a PAP_0-function, i.e.,

$$\lim_{t\to+\infty}\frac{1}{2L}\int_{-L}^{L}\|\mathcal{I}(t)\|_{L^p\times L^p}dt=0 \tag{15}$$

provided that (v_2,η_2,h_2,F_2,f_2) is a PAP_0-function. We have

$$
\begin{aligned}
\|\mathcal{I}(t)\|_{L^p\times L^p}\ \leqslant\ &\left\|\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_2)\\v_1\eta_2\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}+\left\|\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}\mathrm{div}\begin{bmatrix}\mathbb{P}(v_2\otimes v)\\v_2\eta\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}\\
&+\left\|\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}\begin{bmatrix}\mathbb{P}(h_1\eta_2)(\tau)\\0\end{bmatrix}d\tau\right\|_{L^p\times L^p}+\left\|\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}\begin{bmatrix}\mathbb{P}(h_2\eta)(\tau)\\0\end{bmatrix}d\tau\right\|_{L^p\times L^p}\\
&+\left\|\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}\begin{bmatrix}\mathbb{P}(F_2)\\f_2\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}.
\end{aligned} \tag{16}
$$

Below, we prove that

$$\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L}\left\|\int_{-\infty}^{t} e^{-(t-\tau)\mathcal{A}}\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_2)\\v_1\eta_2\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}dt=0. \tag{17}$$

The limits of remain terms in right hand-side of (16) are zero by the same way and we obtain the desired limit (15). We have

$$\begin{aligned}
&\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L}\left\|\int_{-\infty}^{t}e^{-(t-\tau)\mathcal{A}}\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_2)\\ v_1\eta_2\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}dt\\
\leqslant\ &\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L}\left\|\int_{-\infty}^{-L}e^{-(t-\tau)\mathcal{A}}\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_2)\\ v_1\eta_2\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}dt\\
&+\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L}\left\|\int_{-L}^{t}e^{-(t-\tau)\mathcal{A}}\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_2)\\ v_1\eta_2\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}dt
\end{aligned}\tag{18}$$

By the boundedness of mild solutions obtained in Assertion (*i*) we have

$$\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L}\left\|\int_{-\infty}^{-L}e^{-(t-\tau)\mathcal{A}}\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_2)\\ v_1\eta_2\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}dt=\frac{1}{2L}\int_{-L}^{L}0dt=0.\tag{19}$$

Moreover, by the same estimations in Assertion (*i*), Fubini's theorem and changing variables we can estimate

$$\begin{aligned}
&\frac{1}{2L}\int_{-L}^{L}\left\|\int_{-L}^{t}e^{-(t-\tau)\mathcal{A}}\mathrm{div}\begin{bmatrix}\mathbb{P}(v_1\otimes v_2)\\ v_1\eta_2\end{bmatrix}(\tau)d\tau\right\|_{L^p\times L^p}dt\\
\leqslant\ &\frac{1}{2L}\int_{-L}^{L}\int_{-L}^{t}[h_d(t-\tau)]^{\frac{1}{p}+\frac{1}{d}}e^{-\beta_2(t-\tau)}\left\|\begin{bmatrix}v_1\otimes v_2\\ v_1\eta_2\end{bmatrix}(\tau)\right\|_{L^{\frac{p}{2}}\times L^{\frac{p}{2}}}d\tau dt\\
\leqslant\ &\frac{1}{2L}\int_{-L}^{L}\int_{0}^{t+L}[h_d(\tau)]^{\frac{1}{p}+\frac{1}{d}}e^{-\beta_2\tau}\left\|\begin{bmatrix}v_1\otimes v_2\\ v_1\eta_2\end{bmatrix}(t-\tau)\right\|_{L^{\frac{p}{2}}\times L^{\frac{p}{2}}}d\tau dt\\
\leqslant\ &\frac{1}{2L}\int_{0}^{2L}[h_d(\tau)]^{\frac{1}{p}+\frac{1}{d}}e^{-\beta_2\tau}\int_{\tau}^{L}\left\|\begin{bmatrix}v_1\otimes v_2\\ v_1\eta_2\end{bmatrix}(t-\tau)\right\|_{L^{\frac{p}{2}}\times L^{\frac{p}{2}}}dtd\tau\\
\leqslant\ &\int_{0}^{2L}[h_d(\tau)]^{\frac{1}{p}+\frac{1}{d}}e^{-\beta_2\tau}d\tau\frac{1}{2L}\int_{-L}^{L}\left(\|v_1(t)\|_{L^p}\|v_2(t)\|_{L^p}+\|v_1(t)\|_{L^p}\|\eta_2(t)\|_{L^p}\right)dtd\tau\\
\leqslant\ &\frac{M_1}{2L}\int_{-L}^{L}\left(\|v_1(t)\|_{L^p}\|v_2(t)\|_{L^p}+\|v_1(t)\|_{L^p}\|\eta_2(t)\|_{L^p}\right)dt\\
\leqslant\ &M_1\|v_1\|_{\infty,L^p}\left(\frac{1}{2L}\int_{-L}^{L}\|v_2(t)\|_{L^p}dt+\frac{1}{2L}\int_{-L}^{L}\|\eta_2(t)\|_{L^p}dt\right)\\
\longrightarrow\ &0
\end{aligned}\tag{20}$$

as $t\to+\infty$ since $(v_2,\eta_2)(\cdot)$ is a PAP_0-function.

Combining (18),(19) and (20) we obtain the desired limit (17). Our proof is complete. □

4. Wellposedness and exponential stability for semilinear systems

4.1. Existence and uniquness of PAP-mild solutions

We state and prove the existence and uniqueness of PAP-mild solution for the semilinear equation (5) in the following theorem.

Theorem 4.1. *(Global-in-time well-posedness). Let* $(\mathbf{M},g)$ *be a d-dimensional real hyperbolic manifold with* $d\geqslant 2$. *Suppose that the fields* $(h,F,f)\in PAP(\mathbb{R},L^p(\mathbf{M};\Gamma(T\mathbf{M}))\times L^r(\mathbf{M};\Gamma(T\mathbf{M}))\times L^r(\mathbf{M};\mathbb{R}))$ *for* $p>d$, $\frac{dp}{d+2p}<r\leqslant p$. *If the norms* $\|h\|_{\mathscr{H}}$ *and* $\left\|\begin{bmatrix}F\\ f\end{bmatrix}\right\|_{\mathscr{F}\times\mathscr{O}}$ *are small enough, then the semilinear equation* (5) *has a unique PAP-mild solution* $(\overline{u},\overline{\theta})$.

Proof. We set

$$\mathcal{B}_\rho^{PAP} := \{(v,\eta) \in PAP\left(\mathbb{R}, L^p(\mathbf{M};\Gamma(T\mathbf{M})) \times L^p(\mathbf{M};\mathbb{R})\right) \text{ such that } \|(v,\eta)\|_{\mathscr{X}\times\mathscr{S}} \leqslant \rho\}.$$

For a given $(v,\eta) \in \mathcal{B}_\rho^{PAP}$, we consider the linear equation

$$\begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} = \mathcal{B}\left(\begin{bmatrix} v \\ \eta \end{bmatrix}, \begin{bmatrix} v \\ \eta \end{bmatrix}\right)(t) + \mathcal{H}_h(\eta)(t) + \mathcal{F}\left(\begin{bmatrix} F \\ f \end{bmatrix}\right)(t). \tag{21}$$

By the boundedness (12) obtained in Theorem 3.2, we get that for a given $(v,\eta) \in \mathcal{B}_\rho^{PAP}$ there is a unique bounded mild solution (u,θ) to equation (21) satisfying

$$\begin{aligned} \left\|\begin{bmatrix} u \\ \theta \end{bmatrix}\right\|_{\mathscr{X}\times\mathscr{S}} &\leqslant M_1 \left\|\begin{bmatrix} v \\ \eta \end{bmatrix}\right\|^2_{\mathscr{X}\times\mathscr{S}} + M_2\|h\|_{\mathscr{H}}\|\eta\|_{\mathscr{S}} + M_3 \left\|\begin{bmatrix} F \\ f \end{bmatrix}\right\|_{\mathscr{F}\times\mathscr{O}} \\ &\leqslant M_1\rho^2 + M_2\rho\|h\|_{\mathscr{H}} + M_3 \left\|\begin{bmatrix} F \\ f \end{bmatrix}\right\|_{\mathscr{F}\times\mathscr{O}}. \end{aligned} \tag{22}$$

As a direct consequence, we can define a map $\Phi : \mathscr{X} \times \mathscr{S} \to \mathscr{X} \times \mathscr{S}$ as following:

$$\Phi \begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} u \\ \theta \end{bmatrix} \quad . \tag{23}$$

From inequality (22), we obtain that $\left\|\begin{bmatrix} u \\ \theta \end{bmatrix}\right\|_{\mathscr{X}\times\mathscr{S}} \leqslant \rho$ provided that ρ, $\|h\|_{\mathscr{H}}$ and $\left\|\begin{bmatrix} F \\ f \end{bmatrix}\right\|_{\mathscr{F}\times\mathscr{O}}$ are small enough. Hence, the operator Φ acts $\mathcal{B}_\rho^{PAP}$ into itself.

Moreover, it is clear from (23) that

$$\Phi\left(\begin{bmatrix} v \\ \eta \end{bmatrix}\right)(t) = \mathcal{B}\left(\begin{bmatrix} v \\ \eta \end{bmatrix}, \begin{bmatrix} v \\ \eta \end{bmatrix}\right)(t) + \mathcal{H}_h(\eta)(t) + \mathcal{F}\left(\begin{bmatrix} F \\ f \end{bmatrix}\right)(t). \tag{24}$$

Hence, for (v_1,η_1), $(v_2,\eta_2) \in \mathcal{B}_\rho^{PAP}$, by the same arguments for the bilinear estimates $\mathcal{B}(\cdot,\cdot)$ and the linear estimate for $\mathcal{H}(\cdot)$ as in the proof of Theorem 3.2, we see that

$$\begin{aligned} \left\|\Phi\left(\begin{bmatrix} v_1 \\ \eta_1 \end{bmatrix}\right) - \Phi\left(\begin{bmatrix} v_2 \\ \eta_2 \end{bmatrix}\right)\right\|_{\mathscr{X}\times\mathscr{S}} &= \left\|\mathcal{B}\left(\begin{bmatrix} v_1 \\ \eta_1 \end{bmatrix}, \begin{bmatrix} v_1 \\ \eta_1 \end{bmatrix}\right) - \mathcal{B}\left(\begin{bmatrix} v_2 \\ \eta_2 \end{bmatrix}, \begin{bmatrix} v_2 \\ \eta_2 \end{bmatrix}\right) + \mathcal{H}_h(\eta_1 - \eta_2)\right\|_{\mathscr{X}\times\mathscr{S}} \\ &\leqslant \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^t e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} v_2 \otimes (v_2 - v_1) + v_1 \otimes (v_2 - v_1) \\ v_2(\eta_1 - \eta_2) - \eta_1(v_2 - v_1) \end{bmatrix} (\tau) d\tau \right\|_{L^p\times L^p} \\ &\quad + \|\mathcal{H}_h(\eta_1 - \eta_2)\|_{\mathscr{X}\times\mathscr{S}} \\ &\leqslant \left\|\begin{bmatrix} v_1 - v_2 \\ \eta_1 - \eta_2 \end{bmatrix}\right\|_{\mathscr{X}\times\mathscr{S}} \left(M_1 \left\|\begin{bmatrix} v_1 \\ \eta_1 \end{bmatrix}\right\|_{\mathscr{X}\times\mathscr{S}} + M_1 \left\|\begin{bmatrix} v_2 \\ \eta_2 \end{bmatrix}\right\|_{\mathscr{X}\times\mathscr{S}} + M_2\|h\|_{\mathscr{H}}\right) \\ &\leqslant \left\|\begin{bmatrix} u_1 - u_2 \\ \eta_1 - \eta_2 \end{bmatrix}\right\|_{\mathscr{X}\times\mathscr{S}} (2M_1\rho + M_2\|h\|_{\mathscr{H}}). \end{aligned} \tag{25}$$

Therefore, if ρ and $\|h\|_{\mathscr{H}}$ are small enough such that $2M_1\rho + M_2\|h\|_{\mathscr{H}} < 1$, the operator Φ is a contraction on $\mathcal{B}_\rho^{PAP}$.

By using fixed point arguments there is a unique fixed point $(\overline{u},\overline{\theta})$ of Φ, and by the definition of Φ, this fixed point $(\overline{u},\overline{\theta})$ is a bounded mild solution to equation (5). The uniqueness of $(\overline{u},\overline{\theta})$ in the small ball $\mathcal{B}_\rho^{PAP}$ is clearly by using inequality (25). □

4.2. Exponential stability

This section we establish an exponential stability of the PAP-mild solutions obtained in Theorem 4.1.

Theorem 4.2. *(Exponential stability). Let* $(\mathbf{M}, g)$ *be a d-dimensional real hyperbolic manifold with* $d \geqslant 2$. *The PAP-mild solution* $(\overline{u}, \overline{\theta})$ *obtained in Theorem 4.1 is exponentially stable in the sense that for any other mild solution* $(\tilde{u}, \tilde{\theta}) \in \mathscr{X} \times \mathscr{S}$ *of the equation* (5)*, we have*

$$\left\| \begin{bmatrix} \overline{u} - \tilde{u} \\ \overline{\theta} - \tilde{\theta} \end{bmatrix} (t) \right\|_{L^p \times L^p} \leqslant Ce^{-\alpha t} \left\| \begin{bmatrix} \overline{u}(0) - \tilde{u}(0) \\ \overline{\theta}(0) - \tilde{\theta}(0) \end{bmatrix} \right\|_{L^p \times L^p} \text{ for all } t > 0, \tag{26}$$

where $\alpha < min\{\gamma_{p,p}, \beta_1, \beta_2\}$ *with* $\beta_1 = \gamma_{p/2,p},\ \beta_2 = \dfrac{\gamma_{p,p} + \gamma_{p/2,p}}{2}$.

Proof. Let $\tilde{\rho} > 0$ be satisfied that $\|(\tilde{u}, \tilde{\theta})\|_{\mathscr{X} \times \mathscr{S}} < \tilde{\rho}$. It is not hard to see that

$$\begin{aligned} \begin{bmatrix} \overline{u}(t) - \tilde{u}(t) \\ \overline{\theta}(t) - \tilde{\theta}(t) \end{bmatrix} &\leqslant e^{-t\mathcal{A}} \begin{bmatrix} \overline{u}(0) - \tilde{u}(0) \\ \overline{\theta}(0) - \tilde{\theta}(0) \end{bmatrix} + \int_0^t e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}[\tilde{u} \otimes (\tilde{u} - \overline{u}) + \overline{u} \otimes (\tilde{u} - \overline{u})] \\ \tilde{u}(\overline{\theta} - \tilde{\theta}) - \overline{\theta}(\tilde{u} - \overline{u}) \end{bmatrix} (\tau) d\tau \\ &\quad + \int_0^t e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}[h(\overline{\theta} - \tilde{\theta})] \\ 0 \end{bmatrix} (\tau) d\tau. \end{aligned} \tag{27}$$

By the same wave as in the proof of Theorem 3.2, we can obtain that

$$\begin{aligned} &\left\| \begin{bmatrix} \overline{u}(t) - \tilde{u}(t) \\ \overline{\theta}(t) - \tilde{\theta}(t) \end{bmatrix} \right\|_{L^p \times L^p} \\ \leqslant\ & \left\| e^{-t\mathcal{A}} \begin{bmatrix} \overline{u}(0) - \tilde{u}(0) \\ \overline{\theta}(0) - \tilde{\theta}(0) \end{bmatrix} \right\|_{L^p \times L^p} + \int_0^t \left\| e^{-(t-\tau)\mathcal{A}} \mathrm{div} \begin{bmatrix} \mathbb{P}[\tilde{u} \otimes (\tilde{u} - \overline{u}) + \overline{u} \otimes (\tilde{u} - \overline{u})] \\ \tilde{u}(\overline{\theta} - \tilde{\theta}) - \overline{\theta}(\tilde{u} - \overline{u}) \end{bmatrix} (\tau) \right\|_{L^p \times L^p} d\tau \\ & + \int_0^t \left\| e^{-(t-\tau)\mathcal{A}} \begin{bmatrix} \mathbb{P}[h(\overline{\theta} - \tilde{\theta})] \\ 0 \end{bmatrix} (\tau) \right\|_{L^p \times L^p} d\tau \\ \leqslant\ & e^{-\gamma_{p,p} t} \left\| \begin{bmatrix} \overline{u}(0) - \tilde{u}(0) \\ \overline{\theta}(0) - \tilde{\theta}(0) \end{bmatrix} \right\|_{L^p \times L^p} + \int_0^t [h_d(t-\tau)]^{\frac{1}{p} + \frac{1}{d}} e^{-\beta_2 (t-\tau)} \left\| \begin{bmatrix} \tilde{u} \otimes (\tilde{u} - \overline{u}) + \overline{u} \otimes (\tilde{u} - \overline{u}) \\ \tilde{u}(\overline{\theta} - \tilde{\theta}) - \overline{\theta}(\tilde{u} - \overline{u}) \end{bmatrix} (\tau) \right\|_{L^{\frac{p}{2}} \times L^{\frac{p}{2}}} d\tau \\ & + \int_0^t [h_d(t-\tau)]^{\frac{1}{p}} e^{-\beta_1 (t-\tau)} \left\| \begin{bmatrix} h(\overline{\theta} - \tilde{\theta}) \\ 0 \end{bmatrix} (\tau) \right\|_{L^{\frac{p}{2}} \times L^{\frac{p}{2}}} d\tau \\ \leqslant\ & e^{-\gamma_{p,p} t} \left\| \begin{bmatrix} \overline{u}(0) - \tilde{u}_0 \\ \overline{\theta}(0) - \tilde{\theta}_0 \end{bmatrix} \right\|_{L^p \times L^p} \\ & + \int_0^t [h_d(t-\tau)]^{\frac{1}{p} + \frac{1}{d}} e^{-\beta_2 (t-\tau)} \left\| \begin{bmatrix} \overline{u} - \tilde{u} \\ \overline{\theta} - \tilde{\theta} \end{bmatrix} (\tau) \right\|_{L^p \times L^p} \left(\left\| \begin{bmatrix} \overline{u} \\ \overline{\theta} \end{bmatrix} (\tau) \right\|_{L^p \times L^p} + \left\| \begin{bmatrix} \tilde{u} \\ \tilde{\theta} \end{bmatrix} (\tau) \right\|_{L^p \times L^p} \right) d\tau \\ & + \int_0^t [h_d(t-\tau)]^{\frac{1}{p}} e^{-\beta_1 (t-\tau)} \|h(\tau)\|_{L^p} \left\| \begin{bmatrix} 0 \\ \overline{\theta} - \tilde{\theta} \end{bmatrix} (\tau) \right\|_{L^p} d\tau \\ \leqslant\ & e^{-\gamma_{p,p} t} \left\| \begin{bmatrix} \overline{u}(0) - \tilde{u}(0) \\ \overline{\theta}(0) - \tilde{\theta}(0) \end{bmatrix} \right\|_{L^p \times L^p} + (\rho + \tilde{\rho}) \int_0^t [h_d(t-\tau)]^{\frac{1}{p} + \frac{1}{d}} e^{-\beta_2 (t-\tau)} \left\| \begin{bmatrix} \overline{u} - \tilde{u} \\ \overline{\theta} - \tilde{\theta} \end{bmatrix} (\tau) \right\|_{L^p \times L^p} d\tau \\ & + \|h\|_{\mathscr{H}} \int_0^t [h_d(t-\tau)]^{\frac{1}{p}} e^{-\beta_1 (t-\tau)} \left\| \begin{bmatrix} \overline{u} - \tilde{u} \\ \overline{\theta} - \tilde{\theta} \end{bmatrix} (\tau) \right\|_{L^p \times L^p} d\tau. \end{aligned} \tag{28}$$

Here $\beta_1 = \gamma_{p/2,p},\ \beta_2 = \dfrac{\gamma_{p,p} + \gamma_{p/2,p}}{2}$.

By setting $y(\tau) = e^{\alpha\tau} \left\| \begin{bmatrix} \overline{u}(\tau) - \tilde{u}(\tau) \\ \overline{\theta}(\tau) - \tilde{\theta}(\tau) \end{bmatrix} \right\|_{L^p}$ for $\alpha < \min\{\gamma_{p,p}, \beta_1, \beta_2\}$, we obtain from (28) that

$$y(t) \leqslant e^{-(\gamma_{p,p} - \alpha)t} \left\| \begin{bmatrix} \overline{u}(0) - \tilde{u}(0) \\ \overline{\theta}(0) - \tilde{\theta}(0) \end{bmatrix} \right\|_{L^p \times L^p} + (\rho + \tilde{\rho}) \int_0^t [h_d(t-\tau)]^{\frac{1}{p} + \frac{1}{d}} e^{-(\beta_2 - \alpha)(t-\tau)} y(\tau) d\tau$$

$$+\|h\|_{\mathscr{H}} \int_0^t [h_d(t-\tau)]^{\frac{1}{p}} e^{-(\beta_1-\alpha)(t-\tau)} y(\tau) d\tau. \tag{29}$$

By the following convergences

$$\int_0^t [h_d(t-\tau)]^{\frac{1}{p}+\frac{1}{d}} e^{-(\beta_2-\alpha)(t-\tau)} d\tau \leqslant Q_1 < +\infty \text{ and } \int_0^t [h_d(t-\tau)]^{\frac{1}{p}} e^{-(\beta_1-\alpha)(t-\tau)} d\tau \leqslant Q_2 < +\infty.$$

We can use Gronwall's inequality to obtain from (29) that

$$|y(t)| \leqslant e^{Q_1(\rho+\tilde{\rho})+Q_2\|h\|_{\mathscr{H}}} \left\| \begin{bmatrix} \overline{u}(0) - \tilde{u}(0) \\ \overline{\theta}(0) - \tilde{\theta}(0) \end{bmatrix} \right\|_{L^p \times L^p} \quad \text{for all } t > 0.$$

This implies the exponential stability (26). Our proof is complete. □

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