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Characterization theorem for third degree symmetric semiclassical forms of class two

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Abstract. Relying on their third degree character, this paper offers a comprehensive description of a large family of symmetric semiclassical forms of class two. A characterization theorem for all third degree symmetric semiclassical forms of class two is stated and proved. In fact, by using the Stieltjes function and the moments of those forms, we give necessary and sufficient conditions for a regular form to be at the same time of strict third degree (resp. second degree), symmetric and semiclassical of class two under condition $\Phi(0) = 0$. Thus, we focus our attention on the link between these forms and the Jacobi forms $\mathcal{V}_q^{k,l} := \mathcal{J}(k + q/3, l - q/3), k + l \geq -1, k, l \in \mathbb{Z}, q \in \{1, 2\}$ (resp. $\mathcal{T}_{p,q} := \mathcal{J}(p - 1/2, q - 1/2), p + q \geq 0, p, q \in \mathbb{Z}$). All of them are rational transformations of the Jacobi form $\mathcal{V} := \mathcal{J}(-2/3, -1/3)$ (resp. the Tchebychev form of first kind $\mathcal{T} := \mathcal{J}(-1/2, -1/2)$).

1. Introduction

This work is grounded in the theory of semiclassical orthogonal polynomials. Following J. Shohat's pioneering paper on semiclassical orthogonal polynomials [33], many authors have explored this area. Specifically, an orthogonal polynomial sequence associated with a regular linear form (linear functional) w is considered semiclassical if there exists a monic polynomial Φ and a polynomial Ψ with $\deg \Psi \geq 1$, such that $(\Phi w)' + \Psi w = 0$. These polynomials naturally generalize the well-known classical orthogonal polynomials, including Hermite, Laguerre, Jacobi, and Bessel polynomials. Over the past four decades, this theory has been extensively developed and studied. P. Maroni has contributed significantly to its advancement, particularly from an algebraic and structural viewpoint (see [25] for a comprehensive review, as well as [15] for a discussion on the role of semiclassical linear forms in the analysis of polynomial sequences orthogonal with respect to Sobolev inner products).

The study of semiclassical forms with class greater than or equal to one presents a significant challenge. In [5], class-one semiclassical linear forms w are described through the Pearson equations they satisfy (also referenced in [13, 28, 29]). Examples of semiclassical forms of class two can be found in [17, 18, 31, 34], among others. Due to the complexities in solving the Laguerre-Freud equations, as highlighted in [6], it becomes increasingly essential to utilize alternative methods for constructing semiclassical forms (see

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[2, 3, 8, 30, 31], among others), either by employing algebraic equations satisfied by the associated Stieltjes function or through their integral representation.

On the other hand, studying regular forms whose associated formal Stieltjes function

$$S(w)(z) := -\sum_{n\geq 0} \langle w, x^n \rangle / z^{n+1}$$

satisfies an algebraic equation offers a compelling approach for analyzing their properties.

Second degree linear forms are defined via a quadratic equation with polynomial coefficients

$$M(z)S^2(w)(z) + N(z)S(w)(z) + R(z) = 0.$$

These forms were first introduced in [27]. Examples of second degree linear forms have been examined in various contexts, including the classical case (as seen in [4]), in the semiclassical case with class $s = 1$ (see [2, 3, 9, 11, 17, 30], among others), and in the semiclassical case of class $s = 2$ in [17], [31].

Third degree linear forms (TDRF, in short)) are characterized by the fact that their formal Stieltjes function satisfies a cubic equation with polynomial coefficients

$$A(z)S^3(w)(z) + B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0.$$

A regular form w is said to be a strict third degree form (STDRF, in short) if it is a TDRF and its Stieltjes function does not satisfy a quadratic equation with polynomial coefficients, i. e., it is not a second degree form. It is important to notice that a third degree form belongs to the Laguerre-Hahn class [12]. In [7] (resp. [4]) all classical forms which are STDRFs (resp. second degree forms) are determined. It is worthy to mention that the unique classical STDRFs (resp. second degree forms) are the Jacobi forms $\mathcal{J}(k + q/3, l - q/3)$, where $q \in \{1, 2\}$ and k, l are integer numbers with $k + l \geq -1$ (see [7]) (resp. $\mathcal{J}(p - 1/2, q - 1/2)$, where p, q are integer numbers with $p + q \geq 0$ (see [4])). Properties and examples of TDRF are given in [7]. On the other hand, an approach to construct certain families of TDRF is presented there, either through spectral perturbations of the linear form (see [10]) or by employing a cubic decomposition of the associated sequences of orthogonal polynomials (see [8], [18]).

Characterizations of some families of semiclassical linear forms of class $s \leq 2$ which are TDRF are given in (see [2–4, 7–9, 11, 17, 19, 20, 22, 30, 31], as well as the nice Table 1 in [20] which summarizes the Pearson equations for some useful examples of third degree semiclassical forms of class at most 2).

Our work focuses on the analysis of semiclassical forms of class $s = 2$ that are TDRF. Specifically, we aim to describe a large family of symmetric forms using their strict third degree and second-degree characteristics. These forms are such that their corresponding sequences of orthogonal polynomials $\{W_n\}_{n\geq 0}$ satisfy the three-term recurrence relation (TTRR) $W_{n+2}(x) = xW_{n+1}(x) - \gamma_{n+1}W_n(x)$, $n \geq 0$, with initial conditions $W_0(x) = 1$ and $W_1(x) = x$, where $\gamma_{n+1} \neq 0$ for all n. Symmetric forms have been the subject of numerous studies (see [1, 14, 28, 31], among others).

The goal of our contribution is to describe not only all symmetric semiclassical forms of class two under the condition $\Phi(0) = 0$ that are STDRFs (or second-degree forms) and symmetric, but also to go further by addressing the identification of this family of forms in relation to classical forms that are STDRFs (or second-degree forms). Our primary method relies on the explicit expression of their Stieltjes function.

We now provide a brief overview of the paper's structure. In Section 2, we review essential notations, definitions, and results that will be utilized in the following sections. In Section 3, we first revisit the definitions and key properties of third degree forms. Next, we present some results regarding strict third degree classical forms, denoted by $\mathcal{V}_q^{k,l} := \mathcal{J}(k + q/3, l - q/3)$, $k + l \geq -1$, $k, l \in \mathbb{Z}$, $q \in \{1, 2\}$ (resp. second degree classical forms, denoted by $\mathcal{T}_{p,q} := \mathcal{J}(p - 1/2, q - 1/2)$, $p + q \geq 0$, $p, q \in \mathbb{Z}$), which are needed in the sequel. In Section 4, we provide and characterize all forms that meet our objective. In Section 5, we present the main results of the paper, which establish the equivalence between conditions (a), (b), (c), (d), (e), and (f) as stated in Theorem 5.1 (and similarly, (i), (ii), (iii), (iv), (v), and (vi) in Theorem 5.2). Specifically, we derive necessary and sufficient conditions for a regular form to simultaneously be a symmetric, strict third degree (or second-degree) and semiclassical form of class two, under the condition $\Phi(0) = 0$. Furthermore, we establish the connection between these forms and the Jacobi forms $\mathcal{V}_q^{k,l}$, showing that all of these forms are rational transformations of the Jacobi form $\mathcal{V} = \mathcal{J}(-2/3, -1/3)$ (or, for the second degree case, the Chebyshev form of the first kind $\mathcal{T} := \mathcal{J}(-1/2, -1/2)$).

2. Notation and basic background

Let $\mathcal{P}$ be the vector space of polynomials in one variable with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. The elements of $\mathcal{P}'$ will be called linear forms (linear functionals). By $\langle \cdot, \cdot \rangle$, we denote the duality brackets between $\mathcal{P}$ and $\mathcal{P}'$. Given a linear form $w \in \mathcal{P}'$, we denote by $(w)_n =: \langle w, x^n \rangle$ the moment of order n of w. In the sequel $\{(w)_n\}_{n\geq 0}$ will denote the sequence of moments of w with respect to the monomial sequence $\{x^n\}_{n\geq 0}$.

Let us define the following operations on $\mathcal{P}'$. For any $a \in \mathbb{C} - \{0\}$, $b, c \in \mathbb{C}$, $f, g \in \mathcal{P}$, and $w \in \mathcal{P}'$ (see [25])

$$\langle gw, f \rangle = \langle w, gf \rangle, \quad \langle w', f \rangle = -\langle w, f' \rangle, \quad \langle (x-c)^{-1} w, f \rangle = \langle w, \theta_c f \rangle = \left\langle w, \frac{f(x) - f(c)}{x - c} \right\rangle,$$
$$\langle h_a w, f \rangle = \langle w, h_a f \rangle = \langle w, f(ax) \rangle, \quad \langle \tau_b w, f \rangle = \langle w, \tau_{-b} f \rangle = \langle w, f(x+b) \rangle.$$

For $f \in \mathcal{P}$ and $w \in \mathcal{P}'$, the right product wf is the polynomial $(wf)(x) := \left\langle w, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \right\rangle$. This allows us to define the Cauchy product of two forms

$$\langle vw, f \rangle := \langle v, wf \rangle, \quad v, w \in \mathcal{P}', \quad f \in \mathcal{P}.$$

The above product is commutative, associative and distributive with respect to the sum of forms.
In $\mathcal{P}'$, we have the well-known formula

$$\tau_b \circ h_a = h_a \circ \tau_{a^{-1}b}, \quad a \in \mathbb{C} - \{0\},\ b \in \mathbb{C}. \tag{1}$$

The linear form $w \in \mathcal{P}'$ is said to be a rational perturbation of $v \in \mathcal{P}'$ if there exist polynomials M and N, such that

$$M(x)w = N(x)v.$$

The even part of a form w is given by

$$\langle \sigma(w), f \rangle = \langle w, \sigma(f) \rangle, \ \ f \in \mathcal{P},$$

where the linear operator $\sigma : \mathcal{P} \to \mathcal{P}$ is defined by $\sigma(f)(x) := f(x^2)$ for every $f \in \mathcal{P}$.

We introduce the so-called anti-symmetrization operator $\boldsymbol{\alpha} : \mathcal{P}' \to \mathcal{P}'$, [25], such that for $\varpi \in \mathcal{P}'$

$$\big(\boldsymbol{\alpha}(\varpi)\big)_{2n} = 0, \quad \big(\boldsymbol{\alpha}(\varpi)\big)_{2n+1} = (\varpi)_n, \ \ n \geq 0. \tag{2}$$

We will also use the so-called formal Stieltjes function associated with $w \in \mathcal{P}'$ that is defined by [14, 25]

$$S(w)(z) = -\sum_{n\geq 0} \frac{(w)_n}{z^{n+1}}.$$

Remark 2.1. *For any $p \in \mathcal{P}$ and $w \in \mathcal{P}'$, $S(w)(z) = p(z)$ if and only if $w = 0$ and $f = 0$.*

For any $f \in \mathcal{P}$ and $w, v \in \mathcal{P}'$, the following properties hold [25]

$$S(wv)(z) = -zS(w)(z)S(v)(z), \tag{3}$$
$$S(fw)(z) = f(z)S(w)(z) + (w\theta_0 f)(z). \tag{4}$$

Let us recall that a form w is said to be regular, see [25], (quasi-definite according to [14]) if there exists a monic polynomial sequence $\{W_n\}_{n\geq 0}$ with $\deg W_n = n$ such that $\langle w, W_n W_m \rangle = r_n \delta_{n,m},\ \ n, m \geq 0$, where $\{r_n\}_{n\geq 0}$ is a sequence of nonzero complex numbers and $\delta_{n,m}$ is the Kronecker symbol.
$\{W_n\}_{n\geq 0}$ is called a monic orthogonal polynomial sequence (MOPS, in short) with respect to the form w. It is characterized by the following three-term recurrence relation

$$\begin{aligned} &W_0(x) = 1,\ \ W_1(x) = x - \beta_0, \\ &W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x),\ n \geq 0. \end{aligned} \tag{5}$$

Here $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ are sequences of complex numbers such that $\gamma_{n+1} \neq 0$ for all n. This is the so called Favard's theorem (see [14], [16]). The form w is said to be normalized if $(w)_0 = 1$. In the sequel, only normalized forms will be considered.

In this work, we will deal with a MOPS $\{W_n\}_{n\geq 0}$ with respect to the form w fulfilling a three term recurrence relation (5) with coefficients

$$\beta_n = 0,\ \ n \geq 0.$$

In this case, we will say that $\{W_n\}_{n\geq 0}$ is symmetric. The corresponding form w is symmetric.

In the sequel, the following results will be useful.

Lemma 2.2. *[1, 14] Let $\{W_n\}_{n\geq 0}$ be a MOPS with respect to the form w. The following statements are equivalent:*

(a) $\{W_n\}_{n\geq 0}$ *is symmetric.*
(b) $W_n(-x) = (-1)^n W_n(x),\ n \geq 0.$
(c) $(w)_{2n+1} = 0,\ n \geq 0.$
(d) The sequence $\{W_n\}_{n\geq 0}$ has the following quadratic decomposition

$$W_{2n}(x) = P_n\left(x^2\right), \quad W_{2n+1}(x) = xR_n\left(x^2\right), \quad n \geq 0,$$

where $\{P_n\}_{n\geq 0}$ is orthogonal with respect to the form $u = \sigma(w)$ and $\{R_n\}_{n\geq 0}$ is orthogonal with respect to the form $v = \gamma_1^{-1} x\sigma(w)$.

Thus,

$$\begin{aligned} &S\big(w\big)(z) = zS\big(u\big)(z^2), \\ &S\big(v\big)(z) = \gamma_1^{-1} zS\big(u\big)(z) + \gamma_1^{-1}. \end{aligned} \tag{6}$$

A form w is called semiclassical (see [25]) when it is regular and there exist two polynomials Φ and Ψ, Φ monic, $\deg \Phi \geq 0,\ \deg \Psi \geq 1$, such that w satisfies a Pearson's equation

$$(\Phi w)' + \Psi w = 0. \tag{7}$$

Equivalently, the formal Stieltjes function of w satisfies a nonhomogeneous first order linear differential equation with polynomial coefficients

$$A_0(z)S'(w)(z) = C_0(z)S(w)(z) + D_0(z),$$

where

$$A_0 = \Phi,\ \ C_0 = -\Phi' - \Psi,\ \ D_0 = -(w\theta_0\Phi)' - (w\theta_0\Psi). \tag{8}$$

Furthermore, if the polynomials A_0, C_0, and D_0 appearing in (8) are coprime, then the class of w is defined by

$$s = \max\{\deg C_0 - 1, \deg D_0\}.$$

If $\{W_n\}_{n\geq 0}$ is a MOPS with respect to a semiclassical form w of class s, then $\{W_n\}_{n\geq 0}$ is called a semiclassical OPS of class s. In particular, when $s = 0$, i. e., deg $\Phi \leq 2$ and deg $\Psi = 1$, one obtains, up to an affine change of variables, the four families of classical forms : Hermite, $\mathcal{H}$; Laguerre, $\mathcal{L}(\alpha)$; Jacobi, $\mathcal{J}(\alpha,\beta)$ and Bessel, $\mathcal{B}(\alpha)$ (see [26]). Taking into account Jacobi forms $\mathcal{J}(\alpha,\beta)$ will be used in the sequel, we point out that $\phi(x) = x^2 - 1, \psi(x) = -(\alpha+\beta+2)x + (\alpha-\beta)$.

To finish this section, we point out that the semiclassical character of a form is preserved by shifting (or by an affine transformation). Indeed, the shifted form $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w, a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C}$, is also semiclassical having the same class as that of w and satisfies

$$\left(a^{-\deg\phi}\phi(ax+b)\widehat{w}\right)' + a^{1-\deg\phi}\psi(ax+b)\widehat{w} = 0.$$

Hence, a displacement does not change neither the semiclassical character nor the class of a semiclassical form [25]. Therefore, we can take canonical functional equations, by re-situating the zeros of in Eq. (7). This will be put in evidence in the sequel.

The sequence $\{\widehat{W}_n\}_{n\geq 0}$, where $\widehat{W}_n(x) = a^{-n}W_n(ax+b)$, $n \geq 0$, is orthogonal with respect to $\widehat{w}$. The recurrence coefficients are given by $\widehat{\beta_n} = a^{-1}(\beta_n - b)$, $\widehat{\gamma_{n+1}} = a^{-2}\gamma_{n+1}$, $n \geq 0$ [25].
For any $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$, the moments of the shifted form $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$ are

$$(\widehat{w})_n = n!a^{-n} \sum_{\nu+\mu=n} \frac{(-b)^\nu}{\nu!\mu!}(w)_\mu, \quad n \geq 0. \tag{9}$$

The formal Stieltjes function of the shifted form $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w, a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C}$ satisfies [7]

$$S\left(\widehat{w}\right)(z) = aS(w)(az+b). \tag{10}$$

3. Third degree semiclassical forms

3.1. Third degree form

In this subsection, we briefly review the definitions and list some basic properties of the third degree regular forms. Afterwards, we will give some results concerning strict third degree classical forms (resp. second degree classical forms) which are needed later on in this paper.

Definition 3.1. *The form w is called a third degree regular form (TDRF, in short) if it is regular and if there exist four polynomials, A monic, B, C and D, such that*

$$A(z)S^3(w)(z) + B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0. \tag{11}$$

Notice that D depends on A, B, C and w. Indeed, $D(z) = \left(w^3\theta_0^3 A\right)(z) - \left(w^2\theta_0^2 B\right)(z) + \left(w\theta_0 C\right)(z)$.

Remark 3.2. 1. *A regular form w is called a second degree form if the corresponding Stieltjes function satisfies a quadratic equation with polynomial coefficients B, C, D (see [27]))*

$$B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0, \tag{12}$$

where $B \neq 0, C^2 - 4BD \neq 0, D \neq 0$ since the regularity of w.

2. *When the form w is a TDRF and it is not a second degree form, it will be said to be a strict third degree regular form (STDRF, in short) [7].*

Remark 3.3. *Among the most well-known STDRFs (resp. second degree forms) you get the Jacobi form* $\mathcal{V} := \mathcal{J}(-\frac{2}{3}, -\frac{1}{3})$ *[7] (resp. the Tchebychev form of first kind* $\mathcal{T} := \mathcal{J}\big(-1/2, -1/2\big)$ *[4]). Indeed, its formal Stieltjes function is* $S(\mathcal{V})(z) = -(z+1)^{-2/3}(z-1)^{-1/3}$ *(resp.* $S(\mathcal{T})(z) = -(z^2-1)^{-1/2}$*) and satisfies the cubic equation*

$$(z+1)^2(z-1)S^3(\mathcal{V})(z) + 1 = 0.$$

(resp. the quadratic equation

$$(z^2-1)S^2(\mathcal{T})(z) - 1 = 0.)$$

The third degree character is preserved by an affine transformation. Indeed,

Proposition 3.4. *[7] Let* w *be a TDRF and satisfying* (11). *Then the shifted form* $\widehat{w} = (h_{a^{-1}} \circ \tau_{-b})w$, $a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C}$, *fulfils:*

$$\widehat{A}(z)S^3(\widehat{w})(z) + \widehat{B}(z)S^2(\widehat{w})(z) + \widehat{C}(z)S(\widehat{w})(z) + \widehat{D}(z) = 0,$$

with

$$\widehat{A}(z) = a^{\deg A}A(az+b), \qquad \widehat{B}(z) = a^{1-\deg A}B(az+b),$$
$$\widehat{C}(z) = a^{2-\deg A}C(az+b), \quad \widehat{D}(z) = a^{3-\deg A}D(az+b).$$

Elementary transformations of linear forms as k-associated and k-anti-associated perturbations, shift, multiplication and division by a polynomial, inversion, among others preserve the family of linear forms of third degree [7, 10, 23]. In particular, if you deal with a the following rational spectral transformations (see [10], [35]), then,

Lemma 3.5. *[7] Let* w *and* v *be two regular forms satisfying* $M(x)w = N(x)v$, *where* $M(x)$ *and* $N(x)$ *are polynomials. If one of the two forms* w *and* v *is a third degree form, then it is the same for the other one. If* w *is a third degree form satisfying* (12), *then* v *is also a third degree form and*

$$A_vS^3(v) + B_vS^2(v) + C_vS(v) + D_v = 0,$$

with

$$A_v = AN^3, \quad B_v = N^2\Big\{BM + 3A\Big((v\theta_0N) - (u\theta_0M)\Big)\Big\},$$
$$C_v = N\Big\{CM^2 + 2BM\Big((v\theta_0N) - (u\theta_0M)\Big) + 3A\Big((v\theta_0N) - (u\theta_0M)\Big)^2\Big\},$$
$$D_v = DM^3 + CM^2\Big((v\theta_0N) - (u\theta_0M)\Big) + BM\Big((v\theta_0N) - (u\theta_0M)\Big)^2 + A\Big((v\theta_0N) - (u\theta_0M)\Big)^3.$$

3.2. Third degree classical forms

As mentioned in the introduction, the classical forms which are of STDRFs (resp. second degree forms) are determined in [7] (resp. in [4]). More precisely, only some Jacobi forms are TDRF. Indeed,

Theorem 3.6. *[4, 7] Among the classical forms, only the Jacobi forms* $\mathcal{J}(k + q/3, l - q/3)$, *where* $k + l \geq -1$, $k, l \in \mathbb{Z}$, $q \in \{1, 2\}$ *(resp.* $\mathcal{J}(p - 1/2, q - 1/2)$, *where* $p + q \geq 0$, $p, q \in \mathbb{Z}$*), are STDRFs (resp. are second degree forms).*

Remark 3.7. *Throughout this paper the following notation will be used:*

$$\mathcal{V}_q^{k,l} := \mathcal{J}(k + q/3, l - q/3), \text{ with } k + l \geq -1,\ k, l \in \mathbb{Z},\ q \in \{1, 2\}, \tag{13}$$
$$\mathcal{T}_{p,q} := \mathcal{J}(p - 1/2, q - 1/2), \text{ with } p + q \geq 0,\ p, q \in \mathbb{Z}. \tag{14}$$

The next lemma provides us with fundamental relations to be used in the sequel. More precisely, it emphasizes the fact that the strict third degree classical forms $\mathcal{V}_q^{k,l}$ (resp. second degree classical forms $\mathcal{T}_{p,q}$) are perturbations as (3.5) of $h_{(-1)^{q-1}}\mathcal{V}$ (resp. $\mathcal{T}$).

Lemma 3.8. *[9, 18] Let $q \in \{1, 2\}$ and $k, l \in \mathbb{Z}$ with $k + l \geq -1$ (resp. $p, q \in \mathbb{Z}$ with $p + q \geq 0$). The forms $\mathcal{V}_q^{k,l}$ and $\mathcal{V}$ (resp. $\mathcal{T}_{p,q}$ and $\mathcal{T}$) are related by*

$$f_q^{k,l}\mathcal{V}_q^{k,l} = g_q^{k,l} h_{(-1)^{q-1}}\mathcal{V}, \tag{15}$$

$$(resp. \quad L_{p,q}\mathcal{T}_{p,q} = R_{p,q}\mathcal{T},) \tag{16}$$

where $f_q^{k,l}$ and $g_q^{k,l}$ (resp. $L_{p,q}$ and $R_{p,q}$) are polynomials with

$$f_q^{k,l}(x) := \Big\langle h_{(-1)^{q-1}}\mathcal{V}, (x+1)^{\frac{|k+1|+k+1}{2}}(x-1)^{\frac{|l|+l}{2}} \Big\rangle (x+1)^{\frac{|k+1|-(k+1)}{2}}(x-1)^{\frac{|l|-l}{2}}, \tag{17}$$

$$g_q^{k,l}(x) := \Big\langle \mathcal{V}_q^{k,l}, (x+1)^{\frac{|k+1|-(k+1)}{2}}(x-1)^{\frac{|l|-l}{2}} \Big\rangle (x+1)^{\frac{|k+1|+k+1}{2}}(x-1)^{\frac{|l|+l}{2}}. \tag{18}$$

$\Big($*resp.*

$$L_{p,q}(x) := \Big\langle \mathcal{T}, (x+1)^{\frac{|p|+p}{2}}(x-1)^{\frac{|q|+q}{2}} \Big\rangle (x+1)^{\frac{|p|-p}{2}}(x-1)^{\frac{|q|-q}{2}}, \tag{19}$$

$$R_{p,q}(x) := \Big\langle \mathcal{T}_{p,q}, (x+1)^{\frac{|p|-p}{2}}(x-1)^{\frac{|q|-q}{2}} \Big\rangle (x+1)^{\frac{|p|+p}{2}}(x-1)^{\frac{|q|+q}{2}}.\Big) \tag{20}$$

In the following remark, we summarize some results concerning the forms $\mathcal{V}_q^{k,l}$ and $\mathcal{T}_{p,q}$ defined above.

Remark 3.9. *Let us recall that the moments of the Jacobi form $\mathcal{V}_q^{k,l}$ with $k + l \geq -1,\ k, l \in \mathbb{Z}, q \in \{1, 2\}$ (resp. $\mathcal{T}_{p,q}$ with $p + q \geq 0,\ p, q \in \mathbb{Z}$), are*

$$\left(\mathcal{V}_q^{k,l}\right)_n = \sum_{\nu=0}^{n} \binom{n}{\nu} 2^{\nu-1} \frac{\Gamma(k+l+2)}{\Gamma(\nu+k+l+2)} F_{n,\nu}\left(k+\tfrac{q}{3}, l-\tfrac{q}{3}\right),\ \ n \geq 0, \tag{21}$$

where

$$F_{n,\nu}\left(k+\tfrac{q}{3}, l-\tfrac{q}{3}\right) = (-1)^{n-\nu}\frac{\Gamma(\nu+k+\frac{q}{3}+1)}{\Gamma(k+\frac{q}{3}+1)} + (-1)^{\nu}\frac{\Gamma(\nu+l-\frac{q}{3}+1)}{\Gamma(l-\frac{q}{3}+1)}, \tag{22}$$

$\Big($*resp.*

$$\left(\mathcal{T}_{p,q}\right)_n = \sum_{\nu=0}^{n} \binom{n}{\nu} 2^{\nu-1} \frac{\Gamma(p+q+1)}{\Gamma(\nu+p+q+1)} F_{n,\nu}\left(p-\tfrac{1}{2}, q-\tfrac{1}{2}\right),\ \ n \geq 0, \tag{23}$$

where

$$F_{n,\nu}\left(p-\tfrac{1}{2}, q-\tfrac{1}{2}\right) = (-1)^{n-\nu}\frac{\Gamma(\nu+p+\frac{1}{2})}{\Gamma(p+\frac{1}{2})} + (-1)^{\nu}\frac{\Gamma(\nu+q+\frac{1}{2})}{\Gamma(q+\frac{1}{2})},\Big) \tag{24}$$

and Γ is the gamma function [26].

Remark 3.10. *Let $q \in \{1, 2\}$ and $k, l \in \mathbb{Z}$ with $k + l \geq -1$. Notations that will be used In the sequel are introduced*

$$\widehat{\mathcal{V}}_q^{k,l} := (h_{-1/2} \circ \tau_{-1})\mathcal{V}_q^{k,l}, \tag{25}$$

$$\widehat{\mathcal{V}}^{\pm} := (h_{\pm 1/2} \circ \tau_1)\mathcal{V}, \tag{26}$$

$$\widehat{\mathcal{V}}^q := \begin{cases} \mathcal{V}\widehat{\mathcal{V}}^+ & if\ q = 1, \\ \widehat{\mathcal{V}}^-\widehat{\mathcal{V}}^+ & if\ q = 2, \end{cases} \tag{27}$$

where $\mathcal{V}$ and $\mathcal{V}_q^{k,l}$ denote the forms given in Remark 3.3 and Remark 3.7, respectively.

On the other hand,

Lemma 3.11. *Let $q \in \{1, 2\}$, then*

$$-2S\Big(h_{(-1)^{q-1}}\mathcal{V}\Big)(-2z^2+1) = z^{-1}S\Big(\widehat{\mathcal{V}^q}\Big)(z). \tag{28}$$

Remark 3.12. *Let $p, q \in \mathbb{Z}$ with $p+q \geq 0$. Notations that will be used In the sequel are introduced*

$$\widehat{\mathcal{T}}_{p,q} := (h_{-1/2} \circ \tau_{-1})\mathcal{T}_{p,q}, \tag{29}$$

$$\widehat{\mathcal{T}} := (h_{-1/2} \circ \tau_{-1})\mathcal{T}, \tag{30}$$

where $\mathcal{T}$ and $\mathcal{T}_{p,q}$ denote the forms given in Remark 3.3 and Remark 3.7, respectively.

Lemma 3.13. *[11] One has*

$$S(\widehat{\mathcal{T}})(z^2) = z^{-1}S\Big(\mathcal{T}\Big)(z). \tag{31}$$

The following proposition plays an important role in order to prove our main results.

Proposition 3.14. *[9, 18] Let w be a regular symmetric form. The following statements are equivalent*

(a) w is a second degree regular form (resp. STDRF).
(b) $u = \sigma(w)$ is a second degree regular form (resp. STDRF).
(c) $v = \gamma_1^{-1}x\sigma(w)$ is a second degree form (resp. STDRF).

4. Third degree symmetric semiclassical forms of class $s = 2$ with $\Phi(0) = 0$

In this section, we obtain several characterizations for ymmetric semiclassical forms of class two under condition $\Phi(0) = 0$ which are STDRFs , by pointing out the connection with the forms $\mathcal{T}_{p,q}$, their corresponding Stieltjes function and their moments.

Recently, in [32], the description of symmetric semiclassical linear forms of class two with $\Phi(0) = 0$ is presented. In this particular case, the authors state that it is possible to characterize these semiclassical forms of class $s = 2$, as they describe in the following result.

Theorem 4.1. *[32] The following statements are equivalent:*

(a) u is a symmetric semiclassical normalized form of class $s = 2$ satisfying (7) with $\Phi(0) = 0$.

(b) There exist a symmetric semiclassical normalized linear form v of class $\tilde{s} \leq 1$, and $(\tilde{a}_0, \tilde{a}_2, \tilde{c}_1, \tilde{c}_3) \in \mathbb{C}^4$ such that :

$$u = -\lambda x^{-2}v + \delta_0, \quad \lambda = -(u)_2,$$
$$\begin{cases} \Big(\Big(\tilde{c}_3x^3 + \tilde{c}_1x\Big)v\Big)' + \Big(\tilde{a}_2x^2 + \tilde{a}_0\Big)v = 0, \\ |\tilde{c}_3| + |\tilde{c}_1| \neq 0, \quad \tilde{a}_2 \neq 0, \quad \tilde{a}_0 \neq 0 \end{cases}$$
$$-(2\tilde{c}_3 + \tilde{a}_2)\lambda + (2\tilde{c}_1 + \tilde{a}_0) \neq 0.$$

There are three cases in terms of the canonical choice of the polynomial Φ. Precisely,

- **First case:** $\Phi(x) = x^2$. In this case, $\Psi(x) = 2x^3 - (2\alpha+1)x$ and $\sigma v = \mathcal{L}(\alpha)$ where $\mathcal{L}(\alpha)$ is the classical Laguerre form.
- **Second case:** $\Phi(x) = x^4$. In this case, $\Psi(x) = (-4\alpha+1)x^3 - 4x$ and $\sigma v = \mathcal{B}(\alpha)$ where $\mathcal{B}(\alpha)$ is the classical Bessel form.

- **Third case:** $\Phi(x) = x^2(x^2-1)$. In this case, $\Psi(x) = -(2\alpha+2\beta+3)x^3+(2\beta+1)x$ and $\sigma v = \left(h_{\frac{1}{2}} \circ \tau_1\right)\mathcal{J}(\alpha,\beta)$, where $\mathcal{J}(\alpha,\beta)$ is the classical Jacobi form.

Keeping the same notation in [31], let $\mathcal{L}(\alpha,\beta,\lambda)$ be the form satisfying the following distributional equation:

$$\left(x^2\left(x^2-1\right)\mathcal{L}(\alpha,\beta,\lambda)\right)' + \left((-2\alpha-2\beta-3)x^3+(2\beta+1)x\right)\mathcal{L}(\alpha,\beta,\lambda) = 0, \quad \lambda(\alpha+\beta+1) \neq \beta.$$

In addition, we have

$$\begin{cases} \gamma_1 = \lambda, \quad \gamma_{2n+2} = \frac{(n+\beta+1)(n+\alpha+\beta+1)d_{n+1}(\lambda)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)d_n(\lambda)}, n \geq 0, \\ \gamma_{2n+3} = \frac{(n+1)(n+\alpha+1)d_n(\lambda)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)d_{n+1}(\lambda)}, n \geq 0. \end{cases}$$

with

$$d_n(\lambda) = \begin{cases} \lambda\frac{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)} + \frac{\beta}{\alpha+\beta+1} - \lambda, \beta(\alpha+\beta+1) \neq 0, n \geq 0, \\ 1 - \lambda\sum_{k=0}^{n-1}\frac{(2k+1)\Gamma(\alpha+k+1)\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+k+2)}, \alpha+\beta = -1, n \geq 0, \\ \frac{1}{\alpha+1} - \lambda\sum_{k=0}^{n-1}\frac{2k+\alpha+2}{(k+1)(k+\alpha+1)}, \beta = 0, n \geq 0, \left(\sum_0^{-1} = 0\right). \end{cases}$$

The regularity condition is

$$\alpha \neq -n-1, \quad \beta \neq -n-1, \quad \alpha+\beta \neq -n-1, \ \lambda \neq 0, \quad d_n(\lambda) \neq 0, \quad n \in \mathbb{N}.$$

Moreover,

$$\left(h_{(-1/2)^{-1}} \circ \tau_{-1/2}\right)\left(x\sigma\mathcal{L}(\alpha,\beta,\lambda)\right) = \lambda\mathcal{J}(\alpha,\beta). \tag{32}$$

Now we are in a position to state and to prove the main result of this subsection. Only one solution appears, up to affine transformation. Indeed,

Theorem 4.2. *Among the symmetric semiclassical forms of class $s = 2$ satisfying (7) with $\Phi(0) = 0$, only:*

- *the forms $\mathcal{L}\left(k+\frac{q}{3}, l-\frac{q}{3}, \lambda\right)$ are STDRF, provided $k+l \geq -1$, $\lambda^{-1} \neq \frac{3(k+l+1)}{3l-q}$, $k,l \in \mathbb{Z}, q \in \{1,2\}$;*
- *the forms $\mathcal{L}\left(p-\frac{1}{2}, q-\frac{1}{2}, \lambda\right)$ are second degree forms, provided $p+q \geq 0$, $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$, $p,q \in \mathbb{Z}$.*

Proof. The forms that emerge in the first and second cases (families) are not STDRFs (resp. are not second degree forms). Indeed, for the first and second cases, since $\mathcal{L}(\alpha)$ and $\mathcal{B}(\alpha)$ are not STDRFs [7] (resp. are not second degree forms [4]), according to Proposition 3.14 and Lemma 3.5, we conclude that these forms are not STDRFs (resp. are not second degree forms).

Some of the forms that emerge in the third case (or family) are TDRFs. Indeed, Using (32) and according to Theorem 3.6, Proposition 3.14 and Lemma 3.5, we get that w is a STDRF (resp. second degree form) symmetric semiclassical form of class three if and only if $\alpha = k+q/3$ and $\beta = l-q/3$, with $k+l \geq -1$, $k,l \in \mathbb{Z}$, $q \in \{1,2\}$ (resp. $\alpha = p-1/2$ and $\beta = q-1/2$, with $p+q \geq 0$, $p,q \in \mathbb{Z}$). □

Remark 4.3. *The result of the previous theorem for second degree forms was established in [9] and [31].*

5. A description of third degree symmetric semiclassical forms of class $s = 2$ with $\Phi(0) = 0$

The main object of interest of this section is to establish several characterizations of the symmetric semiclassical linear forms of class three that satisfy functional equation (7) with the condition $\Phi(0) = 0$, and which are TDRF. So, we only need to analyze separately the STDRF and the second degree forms, respectively.

5.1. Strict third degree symmetric semiclassical forms of class $s = 2$ with $\Phi(0) = 0$

The main result of this section provides a characterization of the strict third degree symmetric semiclassical forms of class two, that satisfy (7) with the condition $\Phi(0) = 0$, in terms of their formal Stieltjes function (that is explicitly given) and, as consequence, the moments are deduced.

Theorem 5.1. *Let w be a regular form. The following statements are equivalent.*

(a) The form w is a second degree symmetric semiclassical form of class two satisfying (7) *with $\Phi(0) = 0$.*

(b) (The connection between the regular forms via the operator σ). There exist $(k, l, \lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ and $q \in \{1, 2\}$, with $k + l \geq -1$ and $\lambda^{-1} \neq \frac{3(k+l+1)}{3l-q}$, such that

$$x\sigma w = \lambda \widehat{\mathcal{V}}_q^{k,l}.$$

(c) (The connection between the regular forms). There exist $(k, l, \lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ and $q \in \{1, 2\}$, with $k + l \geq -1$ and $\lambda^{-1} \neq \frac{3(k+l+1)}{3l-q}$, such that

$$x^2 f_q^{k,l}(-2x^2 + 1)w = \lambda g_q^{k,l}(-2x^2 + 1)\widehat{\mathcal{V}}^q, \tag{33}$$

and

$$\begin{aligned}\Big(w(x f_q^{k,l}(-2x^2 + 1))\Big)(z) =& 2\lambda z\Big(\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0 g_q^{k,l}\Big) - \Big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\Big)\Big)(-2z^2 + 1)\\ &+ \lambda\Big(\widehat{\mathcal{V}}^q\theta_0(g_q^{k,l}(-2x^2 + 1))\Big)(z) + z f_q^{k,l}(-2z^2 + 1),\end{aligned} \tag{34}$$

where $f_q^{k,l}$ and $g_q^{k,l}$ are polynomials defined by (19) *and* (20)*, respectively.*

(d) (The connection between the Stieltjes functions). There exist $(k, l, \lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ and $q \in \{1, 2\}$, with $k + l \geq -1$ and $\lambda^{-1} \neq \frac{3(k+l+1)}{3l-q}$, such that

$$zS(w)(z) = \lambda S\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)(z^2) - 1. \tag{35}$$

(e) (The connection between the regular forms via the operator $\boldsymbol{\alpha}$). There exist $(k, l, \lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ and $q \in \{1, 2\}$, with $k + l \geq -1$ and $\lambda^{-1} \neq \frac{3(k+l+1)}{3l-q}$, such that

$$xw = \lambda \boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big). \tag{36}$$

(f) (The moments). There exist $(k, l, \lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ and $q \in \{1, 2\}$, with $k + l \geq -1$ and $\lambda^{-1} \neq \frac{3(k+l+1)}{3l-q}$, such that

$$(w)_0 = 1, \quad (w)_{2n+1} = 0, \quad n \geq 0, \tag{37}$$

$$(w)_{2n+2} = \lambda n!(-2)^{-n} \sum_{\nu+\mu=n} \frac{1}{\nu!\mu!} \sum_{i=0}^{\mu} \binom{\mu}{i} 2^{i-1} \frac{\Gamma(k+l+2)}{\Gamma(i+k+l+2)} F_{\mu,i}\Big(k + \tfrac{q}{3}, l - \tfrac{q}{3}\Big), \quad n \geq 0, \tag{38}$$

where $F_{\mu,i}\Big(k + \frac{q}{3}, l - \frac{q}{3}\Big)$ is defined by (24).

Proof. $(a) \Rightarrow (b)$ Let w be a strict third degree symmetric semiclassical form of class two satisfying (7) with $\Phi(0) = 0$. Taking into account Theorem 4.2, there exist $(k, l, \lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ and $q \in \{1, 2\}$, with $k + l \geq -1$ and $\lambda^{-1} \neq \frac{3(k+l+1)}{3l-q}$, such that

$$w = \mathcal{L}\Big(k + \frac{q}{3}, l - \frac{q}{3}, \lambda\Big).$$

From (1), (32) becomes

$$x\sigma w = \lambda(h_{(-2)^{-1}} \circ \tau_{-1})\mathcal{V}_q^{k,l}. \tag{39}$$

$(b) \Rightarrow (a)$ According to (6), (10) and (39), we get

$$zS(w)(z) = -2\lambda S(\mathcal{V}_q^{k,l})(-2z^2+1) - 1.$$

Multiplying both sides of last equation by $zf_q^{k,l}(-2z^2+1)$, from (4) we deduce

$$\begin{aligned}
&z^2 f_q^{k,l}(-2z^2+1)S(w)(z)\\
&= -2\lambda z S\Big(f_q^{k,l}\mathcal{V}_q^{k,l}\Big)(-2z^2+1) + 2\lambda z\Big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\Big)(-2z^2+1) - zf_q^{k,l}(-2z^2+1)\\
&\overset{by\ (16)}{=} -2\lambda z S\Big(g_q^{k,l}h_{(-1)^{q-1}}\mathcal{V}\Big)(-2z^2+1) + 2\lambda z\Big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\Big)(-2z^2+1) - zf_q^{k,l}(-2z^2+1)\\
&\overset{by\ (4)}{=} -2\lambda z g_q^{k,l}(-2z^2+1)S(h_{(-1)^{q-1}}\mathcal{V})(-2z^2+1) - 2\lambda z\Big(\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0 g_q^{k,l}\Big) - \Big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\Big)\Big)(-2z^2+1)\\
&\quad - zf_q^{k,l}(-2z^2+1)\\
&\overset{by\ (28)}{=} \lambda g_q^{k,l}(-2z^2+1)S\Big(\widehat{\mathcal{V}}^q\Big)(z) - 2\lambda z\Big(\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0 g_q^{k,l}\Big) - \Big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\Big)\Big)(-2z^2+1) - zf_q^{k,l}(-2z^2+1).
\end{aligned}$$

Using (4), the above relation reads as

$$S\Big(z^2 f_q^{k,l}(-2x^2+1)w\Big)(z) = \lambda S\Big(g_q^{k,l}(-2x^2+1)h_{(-1)^{q-1}}\mathcal{V}\Big)(z) + P(z),$$

with

$$\begin{aligned}
P(z) =& -2\lambda z\Big(\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0 g_q^{k,l}\Big) - \Big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\Big)\Big)(-2z^2+1)\\
&+ \Big(w(xf_q^{k,l}(-2x^2+1))\Big)(z) - \lambda\Big(\widehat{\mathcal{V}}^q\theta_0(g_q^{k,l}(-2x^2+1))\Big)(z) - zf_q^{k,l}(-2z^2+1),
\end{aligned}$$

or equivalently,

$$S\Big(x^2 f_q^{k,l}(-2x^2+1)w - \lambda g_q^{k,l}(-2x^2+1)\widehat{\mathcal{V}}^q\Big)(z) = P(z) \in \mathcal{P}.$$

Thus, taking into Remark 2.1 we get

$$x^2 f_q^{k,l}(-2x^2+1)w - \lambda g_q^{k,l}(-2x^2+1)\widehat{\mathcal{V}}^q = 0 \quad \text{in } \mathcal{P}',$$

and

$$P(z) = 0.$$

Thus the result follows.
$(c) \Rightarrow (d)$ Applying the operator S to (33) and taking into account (4) we get

$$\begin{aligned}
z^2 f_q^{k,l}(-2z^2+1)S(w)(z) =& \lambda g_q^{k,l}(-2z^2+1)S\Big(h_{(-1)^{q-1}}\mathcal{V}\Big)(z) - \Big(w(xf_q^{k,l}(-2x^2+1))\Big)(z)\\
&+ \lambda\widehat{\mathcal{V}}^q\theta_0(g_q^{k,l}(-2x^2+1))\Big)(z).
\end{aligned}$$

Thus, from (28)

$$\begin{aligned}z^2 f_q^{k,l}(-2z^2+1)S(w)(z) &= -2\lambda z g_q^{k,l}(-2z^2+1)S\Big(h_{(-1)^{q-1}}\mathcal{V}\Big)(-2z^2+1) - \Big(w(xf_q^{k,l}(-2x^2+1))\Big)(z)\\ &\quad + \lambda\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0(g_q^{k,l}(-2x^2+1))\Big)(z)\\ &\overset{by\ (4)-(16)}{=} -2\lambda z S\Big(f_q^{k,l}\mathcal{V}_q^{k,l}\Big)(-2z^2+1) - \Big(w(xf_q^{k,l}(-2x^2+1))\Big)(z)\\ &\quad + \lambda\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0(g_q^{k,l}(-2x^2+1))\Big)(z) + 2\lambda z\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0 g_q^{k,l}\Big)(-2z^2+1)\\ &\overset{by\ (4)}{=} -2\lambda z f_q^{k,l}(-2z^2+1)S\Big(\mathcal{V}_q^{k,l}\Big)(-2z^2+1) - \Big(w(xf_q^{k,l}(-2x^2+1))\Big)(z)\\ &\quad + \lambda\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0(g_q^{k,l}(-2x^2+1))\Big)(z) + 2\lambda z\Big(h_{(-1)^{q-1}}\mathcal{V}\theta_0 g_q^{k,l}\Big)(-2z^2+1)\\ &\quad - 2\lambda z\Big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\Big)(-2z^2+1).\end{aligned}$$

Therefore, by using (34), the last equation becomes

$$z^2 f_q^{k,l}(-2z^2+1)S(w)(z) = -2\lambda z f_q^{k,l}(-2z^2+1)S\Big(\mathcal{V}_q^{k,l}\Big)(-2z^2+1) - z f_q^{k,l}(-2z^2+1).$$

As a consequence,

$$zS(w)(z) = -2\lambda S\Big(\mathcal{V}_q^{k,l}\Big)(-2z^2+1) - 1.$$

The statement (d) holds.

$(d) \Rightarrow (e)$ First, observe that

$$S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\right)(z) = -\sum_{n\geq 0}\frac{\Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\Big)_n}{z^{n+1}} \overset{by\ (2)}{=} -\sum_{n\geq 0}\frac{\Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\Big)_{2n+1}}{z^{2n+2}} \overset{by\ (2)}{=} -\sum_{n\geq 0}\frac{\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)_n}{z^{2(n+1)}} = S\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)(z^2).$$

Together with (35) we have

$$zS(w)(z) = \lambda S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\right)(z) - 1.$$

Therefore, using (4), the last equation becomes

$$S(xw)(z) = \lambda S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\right)(z).$$

By using Remark 2.1 the desired relation holds.

$(e) \Rightarrow (f)$

$$(w)_{2n+1} = (xw)_{2n} \overset{by\ (36)}{=} \left(\lambda\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\right)_{2n} = \lambda\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\right)_{2n} \overset{by\ (2)}{=} 0, \quad n \geq 0. \tag{40}$$

$$(w)_{2n+2} = (xw)_{2n+1} \overset{by\ (36)}{=} \left(\lambda\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\right)_{2n+1} = \lambda\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\right)_{2n+1} \overset{by\ (2)}{=} \lambda\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)_n, \quad n \geq 0. \tag{41}$$

Using (9) and taking into account (21)-(22), (37)-(38) follow in a straightforward way.

$(f) \Rightarrow (a)$ By hypothesis we have

$$(w)_0 = 1, \quad (w)_{2n+1} = 0, \quad (w)_{2n+2} = \lambda\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)_n, \quad n \geq 0.$$

It remains to show that

$$\lambda S\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)\Big(z^2\Big) = -\sum_{n\geq 0}\frac{\lambda\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)_n}{z^{2n+2}} = -\sum_{n\geq 0}\frac{(w)_{2n+2}}{z^{2n+2}} = -z\sum_{n\geq 1}\frac{(w)_{2n}}{z^{2n+1}} = z\Big(S(w)(z) + \frac{1}{z}\Big) = zS(w)(z) + 1. \tag{42}$$

Moreover, the fact that the affine transformation of a strict third degree form is also a second degree form yields $\widehat{\mathcal{V}}_q^{k,l}$ is a strict third degree form such that

$$\widehat{A}_q^{k,l}(z)S^3\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)(z) + \widehat{B}_q^{k,l}(z)S^2\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)(z) + \widehat{C}_q^{k,l}(z)S\Big(\widehat{\mathcal{V}}_q^{k,l}\Big)(z) + \widehat{D}_q^{k,l}(z) = 0, \tag{43}$$

with

$$\widehat{A}_q^{k,l}(z) = (-2)^{-t}A_q^{k,l}(-2z+1), \quad \widehat{B}_q^{k,l}(z) = (-2)^{1-t}B_q^{k,l}(-2z+1),$$
$$\widehat{C}_q^{k,l}(z) = (-2)^{2-t}C_q^{k,l}(-2z+1), \quad \widehat{D}_q^{k,l}(z) = (-2)^{2-t}D_q^{k,l}(-2z+1),$$

where $t = \frac{3(|k+1|+k+1+|l|+l)}{2} + 3$, and $A_q^{k,l}$, $B_q^{k,l}$, $C_q^{k,l}$ and $D_q^{k,l}$ are polynomials

$$\begin{aligned}
A_q^{k,l}(z) &= (z^2-1)\big(z+(-1)^{q-1}\big)(f_q^{k,l})^3(z),\\
B_q^{k,l}(z) &= 3(z^2-1)\big(z+(-1)^{q-1}\big)\Big(\big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\Big)(f_q^{k,l})^2(z),\\
C_q^{k,l}(z) &= 3(z^2-1)\big(z+(-1)^{q-1}\big)\Big(\big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\Big)^2 f_q^{k,l}(z),\\
D_q^{k,l}(z) &= (g_q^{k,l})^3(z) + (z^2-1)\big(z+(-1)^{q-1}\big)\Big(\big(\mathcal{V}_q^{k,l}\theta_0 f_q^{k,l}\big)(z) - \big((h_{(-1)^{q-1}}\mathcal{V})\theta_0 g_q^{k,l}\big)(z)\Big)^3(z).
\end{aligned} \tag{44}$$

Making $z \leftarrow z^2$ in (43), multiplying this equation by λ^3 and taking into account (42) we get

$$A_w(z)S^3(w)(z) + B_w(z)S^2(w)(z) + C_w(z)S(w)(z) + D_w(z) = 0,$$

where

$$\begin{aligned}
A_w(z) &= z^3\widehat{A}_q^{k,l}(z^2),\\
B_w(z) &= 3z^2\widehat{A}_q^{k,l}(z^2) + \lambda z^2\widehat{B}_q^{k,l}(z^2),\\
C_w(z) &= 3z\widehat{A}_q^{k,l}(z^2) + 2\lambda z\widehat{B}_q^{k,l}(z^2) + \lambda^2 z\widehat{C}_q^{k,l}(z^2),\\
D_w(z) &= \widehat{A}_q^{k,l}(z^2) + \lambda\widehat{B}_q^{k,l}(z^2) + \lambda^2\widehat{C}_q^{k,l}(z^2) + \lambda^3\widehat{D}_q^{k,l}(z^2).
\end{aligned}$$

As a consequence, w is a strict third degree form.
To finish the proof, it remains to prove that the class of w is two. Based on (10), the relation (42) becomes

$$zS(w)(z) = -2\lambda S(\mathcal{V}_q^{k,l})(-2z^2+1) - 1. \tag{45}$$

Taking formal derivatives in the last equation we get

$$S(w)(z) + zS'(w)(z) = 8\lambda z S'(\mathcal{V}_q^{k,l})(-2z^2+1).$$

From the above expression we obtain

$$S'(\mathcal{V}_q^{k,l})\left(-2z^2+1\right) = \frac{zS'(w)(z) + S(w)(z)}{8\lambda z}. \tag{46}$$

Using the first order linear differential equation satisfied by the Stieltjes function of the Jacobi form [25], it is a straightforward exercise to prove that $S\big(\mathcal{V}_q^{k,l}\big)(z)$ satisfies

$$\phi(z)S'\big(\mathcal{V}_q^{k,l}\big)(z) = C_{0,q}^{k,l}(z)S\big(\mathcal{V}_q^{k,l}\big)(z) + D_{0,q}^{k,l}(z), \tag{47}$$

with Φ, $C_{0,q}^{k,l}$, and $D_{0,q}^{k,l}$ given by

$$\begin{aligned}
\phi(z) &= z^2 - 1,\\
C_{0,q}^{k,l}(z) &= (k+l)z + l - k - \frac{2q}{3},\\
D_{0,q}^{k,l}(z) &= k + l + 1.
\end{aligned} \tag{48}$$

In (47) the change of variable $z \leftarrow -2z^2+1$ yields

$$\phi(-2z^2+1)S'(\mathcal{V}_q^{k,l})(-2z^2+1) = C_0^{p,q}(-2z^2+1)S(\mathcal{V}_q^{k,l})(-2z^2+1) + D_0^{p,q}(-2z^2+1). \tag{49}$$

Replacing (45) and (46) in (49), and multiplying both sides of the resulting equation by $8\lambda z$, one obtains

$$\Phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z), \tag{50}$$

where the polynomials Φ_w, C_w and D_w are

$$\Phi_w(z) = z\phi(-2z^2+1),$$
$$C_w(z) = -\phi(-2z^2+1) - 4z^2 C_{0,q}^{k,l}(-2z^2+1),$$
$$D_w(z) = -4zC_{0,q}^{k,l}(-2z^2+1) + 8\lambda z D_{0,q}^{k,l}(-2z^2+1).$$

Therefore, from (48) $S(w)(z)$ fulfils (50) with

$$\begin{aligned} \Phi_w(z) &= 4z^3(z^2-1),\\ C_w(z) &= -4z^2(z^2-1) - 4z^2\Big[(k+l)(-2z^2+1) + l - k - \frac{2q}{3}\Big],\\ D_w(z) &= -4z\Big[(k+l)(-2z^2+1) + l - k - \frac{2q}{3}\Big] + 8\lambda(k+l+1)z. \end{aligned} \tag{51}$$

Therefore, the polynomials Φ_w, C_w, and D_w given by (51) have $4z^2$ as a common factor, and so dividing these polynomials by $4z^2$ we obtain

$$\Phi_w(z) = z^2(z^2-1),$$
$$C_w(z) = (2k+2l-1)z^3 + \Big(-2l+1+\tfrac{2q}{3}\Big)z,$$
$$D_w(z) = 2(k+l)z^2 + 2\lambda(k+l+1) - 2\Big(l - \tfrac{q}{3}\Big).$$

Now, taking into account that $2k+2l-1 \neq 0$ and $\lambda(k+l+1) - \Big(l - \frac{q}{3}\Big) \neq 0$ hold, then Φ_w, C_w, and D_w are coprime. As a consequence, since $\deg D_w \leq 2$ and $\deg C_w = 3$, the class of w is two. □

5.2. *Second degree symmetric semiclassical forms of class $s = 2$ with $\Phi(0) = 0$*

The main result of this section provides a characterization of the second degree symmetric semiclassical forms of class two with $\Phi(0) = 0$ in terms of their formal Stieltjes function (that is explicitly given) and, as consequence, the moments are deduced.

Theorem 5.2. *Let w be a regular form. The following statements are equivalent.*

(i) The form w is a second degree symmetric semiclassical form of class two satisfying (7) with $\Phi(0) = 0$.

(ii) (The connection between the regular forms via the operator σ). There exists $(p,q,\lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ with $p+q \geq 0$ and $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$ such that

$$x\sigma w = \lambda \widehat{\mathcal{T}}_{p,q}.$$

(iii) (The connection between the regular forms). There exists $(p,q,\lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ with $p+q \geq 0$ and $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$ such that

$$x^2 L_{p,q}(-2x^2+1)w = \lambda R_{p,q}(-2x^2+1)\mathcal{T}, \tag{52}$$

and

$$\begin{aligned} \Big(w(xL_{p,q}(-2x^2+1))\Big)(z) =& 2\lambda z\Big(\Big(\mathcal{T}\theta_0 R_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 L_{p,q}\Big)\Big)(-2z^2+1)\\ &+ \lambda\Big(\mathcal{T}\theta_0(R_{p,q}(-2x^2+1))\Big)(z) + zL_{p,q}(-2z^2+1), \end{aligned} \tag{53}$$

where $L_{p,q}$ and $R_{p,q}$ are polynomials defined by (19) and (20), respectively.

(iv) *(The connection between the Stieltjes functions). There exists* $(p,q,\lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ *with* $p+q \geq 0$ *and* $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$ *such that*

$$zS(w)(z) = \lambda S\Big(\widehat{\mathcal{T}}_{p,q}\Big)(z^2) - 1. \tag{54}$$

(v) *(The connection between the regular forms via the operator* $\boldsymbol{\alpha}$*). There exists* $(p,q,\lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ *with* $p+q \geq 0$ *and* $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$ *such that*

$$xw = \lambda \boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big). \tag{55}$$

(vi) *(The moments). There exists* $(p,q,\lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ *with* $p+q \geq 0$ *and* $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$ *such that*

$$(w)_0 = 1, \quad (w)_{2n+1} = 0, \quad n \geq 0, \tag{56}$$

$$(w)_{2n+2} = \lambda n!(-2)^{-n} \sum_{\nu+\mu=n} \frac{1}{\nu!\mu!} \sum_{i=0}^{\mu} \binom{\mu}{i} 2^{i-1} \frac{\Gamma(p+q+1)}{\Gamma(i+p+q+1)} F_{\mu,i}\Big(p-\tfrac{1}{2}, q-\tfrac{1}{2}\Big), \quad n \geq 0, \tag{57}$$

where $F_{\mu,i}\Big(p-\frac{1}{2}, q-\frac{1}{2}\Big)$ *is defined by* (24).

Proof. $(i) \Rightarrow (ii)$ Let w be a second degree symmetric semiclassical form of class two satisfying (7) with $\Phi(0) = 0$. Taking into account Theorem 4.2, there exists $(p,q,\lambda) \in \mathbb{Z}^2 \times (\mathbb{C} - \{0\})$ with $p+q \geq 0$ and $\lambda^{-1} \neq \frac{2(p+q)}{2q-1}$ such that

$$w = \mathcal{L}\Big(p - \frac{1}{2}, q - \frac{1}{2}, \lambda\Big).$$

From (1), (32) becomes

$$x\sigma w = \lambda (h_{(-2)^{-1}} \circ \tau_{-1}) \mathcal{T}_{p,q}. \tag{58}$$

$(ii) \Rightarrow (iii)$ According to (6), (10) and (58), we get

$$zS(w)(z) = -2\lambda S(\mathcal{T}_{p,q})(-2z^2+1) - 1.$$

Multiplying both sides of last equation by $zL_{p,q}(-2z^2+1)$, from (4) we deduce

$$\begin{aligned}
&z^2 L_{p,q}(-2z^2+1) S(w)(z) \\
&= -2\lambda z S\Big(L_{p,q}\mathcal{T}_{p,q}\Big)(-2z^2+1) + 2\lambda z\Big(\mathcal{T}_{p,q}\theta_0 L_{p,q}\Big)(-2z^2+1) - zL_{p,q}(-2z^2+1) \\
&\overset{by\ (16)}{=} -2\lambda z S\Big(R_{p,q}\mathcal{T}\Big)(-2z^2+1) + 2\lambda z\Big(\mathcal{T}_{p,q}\theta_0 L_{p,q}\Big)(-2z^2+1) - zL_{p,q}(-2z^2+1) \\
&\overset{by\ (4)}{=} -2\lambda z R_{p,q}(-2z^2+1) S(\mathcal{T})(-2z^2+1) - 2\lambda z\Big(\Big(\mathcal{T}\theta_0 R_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 L_{p,q}\Big)\Big)(-2z^2+1) - zL_{p,q}(-2z^2+1) \\
&\overset{by\ (31)}{=} \lambda R_{p,q}(-2z^2+1) S\Big(\mathcal{T}\Big)(z) - 2\lambda z\Big(\Big(\mathcal{T}\theta_0 R_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 L_{p,q}\Big)\Big)(-2z^2+1) - zL_{p,q}(-2z^2+1).
\end{aligned}$$

Using (4), the above relation reads as

$$S\Big(z^2 L_{p,q}(-2x^2+1)w\Big)(z) = \lambda S\Big(R_{p,q}(-2x^2+1)\mathcal{T}\Big)(z) + P(z),$$

with

$$\begin{aligned}
P(z) = &-2\lambda z\Big(\Big(\mathcal{T}\theta_0 R_{p,q}\Big) - \Big(\mathcal{T}_{p,q}\theta_0 L_{p,q}\Big)\Big)(-2z^2+1) \\
&+ \Big(w(xL_{p,q}(-2x^2+1))\Big)(z) - \lambda\Big(\mathcal{T}\theta_0(R_{p,q}(-2x^2+1))\Big)(z) - zL_{p,q}(-2z^2+1),
\end{aligned}$$

or equivalently,

$$S\Big(x^2L_{p,q}(-2x^2+1)w-\lambda R_{p,q}(-2x^2+1)\mathcal{T}\Big)(z)=P(z)\in\mathcal{P}.$$

Thus, taking into Remark 2.1 we get

$$x^2L_{p,q}(-2x^2+1)w-\lambda R_{p,q}(-2x^2+1)\mathcal{T}=0\quad\text{in }\mathcal{P}',$$

and

$$P(z)=0.$$

Thus the result follows.
$(iii)\Rightarrow(iv)$ Applying the operator S to (52) and taking into account (4) we get

$$\begin{aligned}z^2L_{p,q}(-2z^2+1)S(w)(z)=&\lambda R_{p,q}(-2z^2+1)S\Big(\mathcal{T}\Big)(z)-\Big(w(xL_{p,q}(-2x^2+1))\Big)(z)\\&+\lambda\Big(\mathcal{T}\theta_0(R_{p,q}(-2x^2+1))\Big)(z).\end{aligned}$$

Thus, from (31)

$$\begin{aligned}z^2L_{p,q}(-2z^2+1)S(w)(z)=&-2\lambda zR_{p,q}(-2z^2+1)S\Big(\mathcal{T}\Big)(-2z^2+1)-\Big(w(xL_{p,q}(-2x^2+1))\Big)(z)\\&+\lambda\Big(\mathcal{T}\theta_0(R_{p,q}(-2x^2+1))\Big)(z)\\\overset{by\ (4)-(16)}{=}&-2\lambda zS\Big(L_{p,q}\mathcal{T}_{p,q}\Big)(-2z^2+1)-\Big(w(xL_{p,q}(-2x^2+1))\Big)(z)\\&+\lambda\Big(\mathcal{T}\theta_0(R_{p,q}(-2x^2+1))\Big)(z)+2\lambda z\Big(\mathcal{T}\theta_0R_{p,q}\Big)(-2z^2+1)\\\overset{by\ (4)}{=}&-2\lambda zL_{p,q}(-2z^2+1)S\Big(\mathcal{T}_{p,q}\Big)(-2z^2+1)-\Big(w(xL_{p,q}(-2x^2+1))\Big)(z)\\&+\lambda\Big(\mathcal{T}\theta_0(R_{p,q}(-2x^2+1))\Big)(z)+2\lambda z\Big(\mathcal{T}\theta_0R_{p,q}\Big)(-2z^2+1)-2\lambda z\Big(\mathcal{T}_{p,q}\theta_0L_{p,q}\Big)(-2z^2+1).\end{aligned}$$

Therefore, by using (53), the last equation becomes

$$z^2L_{p,q}(-2z^2+1)S(w)(z)=-2\lambda zL_{p,q}(-2z^2+1)S\Big(\mathcal{T}_{p,q}\Big)(-2z^2+1)-zL_{p,q}(-2z^2+1).$$

As a consequence,

$$zS(w)(z)=-2\lambda S\Big(\mathcal{T}_{p,q}\Big)(-2z^2+1)-1.$$

The statement (iv) holds.
$(iv)\Rightarrow(v)$ First, observe that

$$S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z)=-\sum_{n\geq0}\frac{\Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_n}{z^{n+1}}\overset{by\ (2)}{=}-\sum_{n\geq0}\frac{\Big(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1}}{z^{2n+2}}\overset{by\ (2)}{=}-\sum_{n\geq0}\frac{\Big(\widehat{\mathcal{T}}_{p,q}\Big)_n}{z^{2(n+1)}}=S\Big(\widehat{\mathcal{T}}_{p,q}\Big)(z^2).\tag{59}$$

Together with (54) we have

$$zS(w)(z)=\lambda S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z)-1.$$

Therefore, using (4), the last equation becomes

$$S(xw)(z)=\lambda S\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)(z).$$

By using Remark 2.1 the desired relation holds.
$(v)\Rightarrow(vi)$

$$(w)_{2n+1}=(xw)_{2n}\overset{by\ (55)}{=}\Big(\lambda\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n}=\lambda\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)_{2n}\overset{by\ (2)}{=}0,\quad n\geq0.\tag{60}$$

$$(w)_{2n+2}=(xw)_{2n+1}\overset{by\ (55)}{=}\Big(\lambda\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\Big)_{2n+1}=\lambda\left(\boldsymbol{\alpha}\Big(\widehat{\mathcal{T}}_{p,q}\Big)\right)_{2n+1}\overset{by\ (2)}{=}\lambda\Big(\widehat{\mathcal{T}}_{p,q}\Big)_n,\quad n\geq0.\tag{61}$$

Using (9) and taking into account (23)-(24), (56)-(57) follow in a straightforward way.
$(vi) \Rightarrow (i)$ By hypothesis we have

$$(w)_0 = 1, \quad (w)_{2n+1} = 0, \quad (w)_{2n+2} = \lambda\left(\widehat{\mathcal{T}}_{p,q}\right)_n, \quad n \geq 0.$$

It remains to show that

$$\lambda S\left(\widehat{\mathcal{T}}_{p,q}\right)\left(z^2\right) = -\sum_{n\geq 0} \frac{\lambda\left(\widehat{\mathcal{T}}_{p,q}\right)_n}{z^{2n+2}} = -\sum_{n\geq 0} \frac{(w)_{2n+2}}{z^{2n+2}} = -z\sum_{n\geq 1} \frac{(w)_{2n}}{z^{2n+1}} = z\Big(S(w)(z) + \frac{1}{z}\Big) = zS(w)(z) + 1. \tag{62}$$

Moreover, the fact that the affine transformation of a second degree form is also a second degree form yields $\widehat{\mathcal{T}}_{p,q}$ is a second degree form such that

$$\widehat{B}_{p,q}(z)S^2\Big(\widehat{\mathcal{T}}_{p,q}\Big)(z) + \widehat{C}_{p,q}(z)S\Big(\widehat{\mathcal{T}}_{p,q}\Big)(z) + \widehat{D}_{p,q}(z) = 0, \tag{63}$$

with

$$\widehat{B}_{p,q}(z) = (-2)^{-t}B_{p,q}(-2z+1), \;\; \widehat{C}_{p,q}(z) = (-2)^{1-t}C_{p,q}(-2z+1), \;\; \widehat{D}_{p,q}(z) = (-2)^{2-t}D_{p,q}(-2z+1),$$

where $t = \frac{|p|-p+|q|-q}{2}$ and $B_{p,q}$, $C_{p,q}$ and $D_{p,q}$ are polynomials

$$\begin{aligned}
B_{p,q}(z) &= \left(z^2-1\right)L^2_{p,q}(z)\\
C_{p,q}(z) &= 2\left(z^2-1\right)L_{p,q}(z)\left(\left(\mathcal{T}_{p,q}\theta_0 L_{p,q}\right)(z) - \left(\mathcal{T}\theta_0 R_{p,q}\right)(z)\right)\\
D_{p,q}(z) &= \left(z^2-1\right)\left(\left(\mathcal{T}_{p,q}\theta_0 L_{p,q}\right)(z) - \left(\mathcal{T}\theta_0 R_{p,q}\right)(z)\right)^2 - R^2_{p,q}(z).
\end{aligned}$$

Making $z \leftarrow z^2$ in (63), multiplying this equation by λ^2 and taking into account (62) we get

$$B_w(z)S^2(w)(z) + C_w(z)S(w)(z) + D_w(z) = 0,$$

where

$$\begin{aligned}
B_w(z) &= z^2\widehat{B}_{p,q}(z^2),\\
C_w(z) &= 2z\widehat{B}_{p,q}(z^2) + \lambda z\widehat{C}_{p,q}(z^2),\\
C_w(z) &= \widehat{B}_{p,q}(z^2) + \lambda\widehat{C}_{p,q}(z^2) + \lambda^2\widehat{D}_{p,q}(z^2).
\end{aligned}$$

As a consequence, w is a second degree form.
To finish the proof, it remains to prove that the class of w is two. Based on (10), the relation (62) becomes

$$zS(w)(z) = -2\lambda S(\mathcal{T}_{p,q})(-2z^2+1) - 1. \tag{64}$$

Taking formal derivatives in the last equation we get

$$S(w)(z) + zS'(w)(z) = 8\lambda z S'(\mathcal{T}_{p,q})(-2z^2+1).$$

From the above expression we obtain

$$S'(\mathcal{T}_{p,q})\left(-2z^2+1\right) = \frac{zS'(w)(z) + S(w)(z)}{8\lambda z}. \tag{65}$$

Using the first order linear differential equation satisfied by the Stieltjes function of the Jacobi form, it is a straightforward exercise to prove that $S\left(\mathcal{T}_{p,q}\right)(z)$ satisfies

$$\Phi(z)S'\left(\mathcal{T}_{p,q}\right)(z) = C_0^{p,q}(z)S\left(\mathcal{T}_{p,q}\right)(z) + D_0^{p,q}(z), \tag{66}$$

where $\Phi(z), C_0^{p,q}(z)$, and $D_0^{p,q}(z)$ are polynomials given by

$$\Phi(z) = z^2 - 1, \quad C_0^{p,q}(z) = (p+q-1)z + q - p, \quad D_0^{p,q}(z) = p+q. \tag{67}$$

In (66) the change of variable $z \leftarrow -2z^2 + 1$ yields

$$\Phi(-2z^2+1)S'(\mathcal{T}_{p,q})(-2z^2+1) = C_0^{p,q}(-2z^2+1)S(\mathcal{T}_{p,q})(-2z^2+1) + D_0^{p,q}(-2z^2+1). \tag{68}$$

Replacing (64) and (65) in (68), and multiplying both sides of the resulting equation by $8\lambda z$, one obtains

$$\Phi_w(z)S'(w)(z) = C_w(z)S(w)(z) + D_w(z), \tag{69}$$

where the polynomials Φ_w, C_w and D_w are

$$\begin{aligned}
\Phi_w(z) &= z\Phi(-2z^2+1),\\
C_w(z) &= -\Phi(-2z^2+1) - 4z^2 C_0^{p,q}(-2z^2+1),\\
D_w(z) &= -4zC_0^{p,q}(-2z^2+1) + 8\lambda z D_0^{p,q}(-2z^2+1).
\end{aligned}$$

Therefore, from (67) $S(w)(z)$ fulfils (69) with

$$\begin{aligned}
\Phi_w(z) &= 4z^3(z^2-1),\\
C_w(z) &= -4z^2(z^2-1) - 4z^2\Big[(p+q-1)(-2z^2+1) + q - p\Big],\\
D_w(z) &= -4z\Big[(p+q-1)(-2z^2+1) + q - p\Big] + 8\lambda(p+q)z.
\end{aligned} \tag{70}$$

Therefore, the polynomials Φ_w, C_w, and D_w given by (70) have $4z^2$ as a common factor, and so dividing these polynomials by $4z^2$ we obtain

$$\begin{aligned}
\Phi_w(z) &= z^2(z^2-1),\\
C_w(z) &= (2p+2q+1)z^3 - 2qz,\\
D_w(z) &= 2(p+q-1)z^2 + 2\lambda(p+q) - 2q + 1.
\end{aligned}$$

Now, taking into account that $2p + 2q + 1 \neq 0$ and $2\lambda(p+q) - 2q + 1 \neq 0$ hold, then Φ_w, C_w, and D_w are coprime. As a consequence, since $\deg D_w \leq 2$ and $\deg C_w = 3$, the class of w is two. □

6. Concluding remarks.

Very few examples of TDRFs that are semiclassical of class $s = 2$ are found in the literature. In this contribution, we have demonstrated several examples of TDRFs that are semiclassical of class two, generated through symmetrization processes. An intriguing question that arises is how to characterize all third-degree semiclassical forms of class $s = 2$.

The study of the analytic properties of MOPS associated with the linear forms described in Section 5 remains an open problem. More specifically, since the linear form w belongs to the semiclassical class with $s = 3$, the associated MOPS satisfies a second-order linear differential equation, derived from the raising and lowering operators tied to this linear form. Therefore, an electrostatic interpretation of the zeros of these MOPS would be highly valuable.

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