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Periodic solutions of impulsive differential systems with relativistic acceleration operator

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Abstract. This paper aims to investigate the second-order singular differential systems generated by instantaneous and non-instantaneous impulses. By proposing a new energy functional and solving the difficulties brought by impulsive effects, the existence of periodic solutions to the second-order singular differential systems is obtained via the variational method, which extends and enriches some previous results.

1. Introduction

In this paper, we study the following second-order singular differential systems with instantaneous and non-instantaneous impulses:

$$\begin{cases} -(P(t)u'(t))' + H(t)u(t) = \lambda_i K_i(t)\Psi(u(t)), \ a.e.\ t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ \Delta(P(t_i)u'(t_i)) = \nabla I_i(u(t_i)), \ i = 1, 2, ..., N, \\ P(t)u'(t) = P(t_i^+)u'(t_i^+), t \in (t_i, s_i], \ i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-), \ i = 1, 2, ..., N, \\ u(0) = u(T), \ u'(0) = u'(T), \end{cases} \tag{1}$$

where Ψ is a relativistic acceleration operator that comes from classical theory of relativity defined by

$$\Psi(u) = \frac{u}{\sqrt{1 - |u|^2}}, \ u \in B(1),$$

$T > 0$, $\lambda_i > 0$, $s_0 = 0 < t_1 < s_1 < t_2 < \cdots < s_N < t_{N+1} = T$, $t_{i+N+1} = t_i + T$, $s_{i+N} = s_i + T$, $P \in C([0, T]; \mathbb{R}^+)$ with $P(t + T) = P(t)$, $K_i \in L^\infty([0, T]; \mathbb{R}^+)$ with $K_i(t + T) = K_i(t)$, $\mathbb{R}^+ = (0, +\infty)$, $I_i \in C^1(\mathbb{R}^n; \mathbb{R})$, $I_i(0) = 0$, $\nabla I_{i+N+1}(u) = \nabla I_i(u)$, $\Delta(P(t_i)u'(t_i)) = P(t_i^+)u'(t_i^+) - P(t_i^-)u'(t_i^-)$, $P(t_i^\pm)u'(t_i^\pm) = \lim_{t \to t_i^\pm} P(t)u'(t)$, H is a $n \times n$ symmetric matrix, $H(t + T) = H(t)$ and $(H(t)u, u) \geq \mu|u|^2$ in which $H(t) = (h_{ij}(t))_{n\times n}$ with $h_{ij} \in L^\infty([0, T]; \mathbb{R})$ and $\mu > 0$.

As is known to all, the impulsive differential equation is an important theoretical tool to describe the discontinuous state of the development of things, which has some important applications in many fields

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such as biological model, control theory, economics, aerospace, etc. So, it is very meaningful to study the qualitative theory of impulsive differential equations. As its branches, boundary value problems (BVPs for short) of impulsive differential equations have been studied by many scholars (see [1-5] and references therein). In the recent years, the variational methods have been naturally applied to study the existence and multiplicity of solutions for BVPs of differential equations with instantaneous impulsive effects. In [6], Sun, Chen and Nieto considered the following second-order n-dimensional Hamiltonian system with instantaneous impulsive effects

$$\begin{cases} -u''(t) + A(t)u(t) = \nabla F(t, u(t)), \text{ a.e. } t \in J = [0, T] \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta(u'(t_j)) = I_{ij}(u^i(t_j)), i = 1, 2, ..., n, j = 1, 2, ..., m, \\ u(0) = u(T), u'(0) = u'(T) \end{cases} \tag{2}$$

and proved the existence of infinitely many periodic solutions by the variant fountain theorems. If $A(t)$ is equal to the unit matrix, it was investigated by Zhou and Li [7]. Moreover, for scalar case, it has been considered by Sun, Chen and Liu [8]. Furthermore, for the problems of periodic solutions to second-order Hamiltonian system without impulsive effects, one can read [9-13] and references therein.

On the other hand, in 2013, the non-instantaneous impulsive problem was firstly introduced by Hernández and O'Regan [14]. Its impulsive effects keep active on a finite time interval rather than some certain moments. From then on, it attracts more and more scholars' attention (see [15,16] and references therein). Recently, based on the Lax-Milgram Theorem, Bai and Nieto [17] got the existence of solutions to Dirichlet BVPs of second-order differential equations with non-instantaneous impulsive effects by establishing a new variational structure. Tian and Zhang [18] made further research on the existence of solutions for Dirichlet BVPs of second-order differential equations with non-instantaneous and instantaneous impulses via the Ekeland's variational principle as follows.

$$\begin{cases} -u''(t) = g_i(t, u(t)), t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N \\ \Delta(u'(t_i)) = I_i(u(t_i)), \ i = 1, 2, ..., N, \\ u'(t) = u(t_i^+), t \in (t_i, s_i], \ i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-), \ i = 1, 2, ..., N, \\ u(0) = u(T) = 0. \end{cases} \tag{3}$$

Moreover, Khaliq and ur Rehman [19] and Zhang and Liu [20] considered Dirichlet BVPs of fractional differential equations with non-instantaneous impulses via the variational methods. For further papers on this subject, please refer to [22-27].

Then a natural question is raised: Can we deal with the existence of the periodic solutions for second-order singular differential systems generated by instantaneous and non-instantaneous impulses (1.1)? In this paper, an affirmative answer will be given. Let us present the innovations of this paper: First, under the influence of non-instantaneous and instantaneous impulsive effects, a new energy functional is established for the periodic boundary conditions of the second-order singular differential systems via the truncation technique. Second, the relativistic acceleration operator Ψ is a singular operator, which make this problem become more interesting and difficult. Third, there are few papers dealing with periodic solutions of second-order differential systems with non-instantaneous impulsive effects.

Let $\lambda_{min} = \min\{\lambda_0, \lambda_1, ..., \lambda_N\}$. For stating our main results, the following conditions are given:

(I1) For any $u \in \mathbb{R}^n$, $(\nabla I_i(u), u) \geq 0$ and $I_i(u) \geq 0$, $i = 1, 2, ..., N$.

(I2) There exist constants $d > 0$, $\alpha_i > 0$ and $\gamma_i \in [2, +\infty)$, $i = 1, 2, ..., N$ such that

$$|I_i(u)| \leq \alpha_i |u|^{\gamma_i} \text{ for any } |u| \leq d.$$

Theorem 1.1. Assuming that the conditions (I1) and (I2) are satisfied, there exists a constant $\lambda_* > 0$ such that if $\lambda_{min} > \lambda_*$, the singular differential system (1) admits at least one nontrivial ground state weak solutions.

Remark 1.2. In the condition (I2), the impulsive terms can be local square growth or supersquare growth.

2. Preliminaries

Let $C := C([0,T];\mathbb{R}^n)$ with norm $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$, $L^2 := L^2([0,T];\mathbb{R}^n)$ with norm $\|u\|_{L^2} = (\int_0^T |u(t)|^2 dt)^{\frac{1}{2}}$, and $L^\infty := L^\infty([0,T];\mathbb{R}^n)$ with norm $\|u\|_{L^\infty} = \text{esssup}_{t\in[0,T]}|u(t)|$. Let $(\cdot,\cdot)$ represent the inner product in $\mathbb{R}^n$. Consider the following classical Sobolev space

$$W_T^{1,2} = \{u : [0,T] \to \mathbb{R}^n | u \text{ is absolutely continuous}, u' \in L^2,\ u(0) = u(T)\},$$

whose inner product is

$$< u, v >_0 = \int_0^T (u(t), v(t)) + (u'(t), v'(t))dt,\ \forall u,\ v \in W_T^{1,2}$$

and the norm is $\|u\|_0 =< u, u >^{\frac{1}{2}}$. In Sobolev space $W_T^{1,2}$, define the following inner

$$< u, v >= \int_0^T (P(t)u(t), v(t)) + (H(t)u'(t), v'(t))dt,\ \forall u,\ v \in W_T^{1,2}$$

with the norm $\|u\| =< u, u >^{\frac{1}{2}}$. For any $u \in W_T^{1,2}$, it follows that there exists a positive constant $\xi \le \sum_{i=1}^{n}\sum_{j=1}^{n} \|h_{ij}\|_{L^\infty}$ such that $(H(t)u, u) \le \xi|u|^2$, which together with $(H(t)u, u) \ge \mu|u|^2$ yield that $\beta_*\|u\|_0^2 \le \|u\|^2 \le \beta^*\|u\|_0^2$ where

$$\beta_* = \min\{P_{min}, \mu\},\ \beta^* = \max\{P_{max}, \xi\},$$
$$P_{max} = \max_{t\in[0,T]} P(t),\ P_{min} = \min_{t\in[0,T]} P(t).$$

Thus, the norm $\|u\|_0$ and $\|u\|$ are equivalent. From [12], it follows that the embedding $W_T^{1,2} \hookrightarrow C$ is compact. Thus, we can find a constant $c > 0$ such that $\|u\|_\infty \le c\,\|u\|$ for any $u \in W_T^{1,2}$. It should be mentioned that for each $u \in W_T^{1,2}$, u is absolutely continuous and $u' \in L^2$, which lead to the occurrence of impulsive effects.

Lemma 2.1. *If a function $u \in W_T^{1,2}$ with $\|u\|_\infty < 1$ is a solution of the problem (1), then the following identity*

$$\int_0^T (P(t)u'(t), v'(t))dt + \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} (H(t)u(t), v(t))dt + \sum_{i=1}^{N} (\nabla I_i(u(t_i)), v(t_i))$$
$$= \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} (\lambda_i K_i(t)\Psi(u(t)), v(t))dt \tag{4}$$

holds for any $v \in W_T^{1,2}$.

Proof. Since $P \in C([0,T];\mathbb{R}^+)$ with $P(t+T) = P(t)$, for any $v \in W_T^{1,2}$, it follows that

$$\int_0^T (P(t)u'(t), v'(t))dt$$
$$= \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} (P(t)u'(t), v'(t))dt + \sum_{i=1}^{N} \int_{t_i}^{s_i} (P(t)u'(t), v'(t))dt$$

$$= \sum_{i=0}^{N} (P(t)u'(t), v(t))\Big|_{s_i^+}^{t_{i+1}^-} - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} ((P(t)u'(t))', v(t))dt$$

$$+\sum_{i=1}^{N}(P(t)u'(t),v(t))\mid_{t_i^+}^{s_i^-}-\sum_{i=1}^{N}\int_{t_i}^{s_i}(\frac{d}{dt}(P(t)u'(t)),v(t))dt$$

$$=\sum_{i=1}^{N}(P(t_i^-)u'(t_i^-)-P(t_i^+)u'(t_i^+),v(t_i))$$

$$+\sum_{i=1}^{N}(P(s_i^-)u'(s_i^-)-P(s_i^-)u'(s_i^+),v(s_i))$$

$$+(P(T)u'(T),v(T))-(P(0)u'(0),v(0))$$

$$-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}((P(t)u'(t))',v(t))dt$$

$$=-\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}((P(t)u'(t))',v(t))dt$$

$$-\sum_{i=1}^{N}(\nabla I_i(u(t_i)),v(t_i)), \tag{5}$$

which together with

$$\int_0^T(H(t)u(t),v(t))dt = \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}(H(t)u(t),v(t))dt+\sum_{i=1}^{N}\int_{t_i}^{s_i}(H(t)u(t),v(t))dt \tag{6}$$

yield that

$$\int_0^T(P(t)u'(t),v'(t))dt+\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}(H(t)u(t),v(t))dt+\sum_{i=1}^{N}(\nabla I_i(u(t_i)),v(t_i))$$

$$=\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}(\lambda_iK_i(t)\Psi(u(t)),v(t))dt.$$

Definition 2.2. *A function $u \in W_T^{1,2}$ with $\|u\|_\infty < 1$ is called a weak solution of problem (1), if (4) is satisfied for any $v \in W_T^{1,2}$.*

Next, inspired by [9], we define the functional $\Phi : W_T^{1,2} \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_0^T P(t)\,|u'(t)|^2\,dt+\frac{1}{2}\int_0^T(H(t)u(t),u(t))dt$$

$$+\sum_{i=1}^{N}I_i(u(t_i))-\Gamma(\|u\|)(\frac{1}{2}\sum_{i=1}^{N}\int_{t_i}^{s_i}(H(t)u(t),u(t))dt$$

$$+\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}\lambda_iK_i(t)(1-\sqrt{1-|u(t)|^2})dt), \tag{7}$$

where $\Gamma \in C^1(\mathbb{R}^+,[0,1])$ and satisfies

$$\Gamma'(x) \le 0,\ \forall x \in [0,\frac{\eta}{c}],$$

$$\Gamma(x) = 0, \forall x \ge \frac{\eta}{c},$$

$$\Gamma(x) = 1, \forall x \leq \frac{\eta}{2c},$$

where $\eta = \min\{\frac{1}{2}, d\}$. Noting that I_i are continuously differentiable, standard arguments can prove that $\Phi \in C^1(W_T^{1,2}, \mathbb{R})$. Moreover, if $\|u\| \leq \frac{\eta}{2c}$ which implies $|u| \leq \|u\|_\infty \leq c\|u\| \leq \frac{\eta}{2}$, the critical points of Φ are the weak solutions of (1).

The following lemma will be used in our paper.

Lemma 2.3 ([21]). Assume that E is a real Banach space and $\Phi \in C^1(E, \mathbb{R})$ satisfy the (PS)-condition. If Φ is bounded from below, then $m = \inf_E \Phi$ is a critical value of Φ.

3. Main Result

Lemma 3.1. *If the assumptions of Theorem 1.1 are satisfied, there exists a constant $\lambda_* > 0$ such that if $\lambda_{min} > \lambda_*$, the critical point u_* of (7) which satisfies*

$$\Phi(u_*) = \inf_{u \in W_T^{1,2}} \Phi(u)$$

must be the nontrivial ground state weak solutions of (1).

Proof. In fact, we just need to show that $u_* \neq 0$ and $\|u_*\| \leq \frac{\eta}{2c}$ which implies that $\|u_*\|_\infty \leq c\|u_*\| \leq \frac{\eta}{2} < 1$. By choosing a function $u_0 \in W_T^{1,2} \setminus \{0\}$ with $\|u_0\| \leq \frac{\eta}{2c}$, for $0 < r < 1$ and $\gamma_i \in [2, +\infty)$, it is clear that

$$1 - \sqrt{1 - |ru_0|^2} \leq \frac{1}{\sqrt{1 - \frac{\eta^2}{4}}} |ru_0|^2 \text{ and} \tag{8}$$

$$\begin{aligned}
\Phi(ru_0) &= \frac{1}{2}\int_0^T P(t)\left|ru_0'(t)\right|^2 dt + \frac{1}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} (H(t)ru_0(t), ru_0(t))dt \\
&\quad + \sum_{i=1}^{N} I_i(ru_0(t_i)) - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda_i K_i(t)(1 - \sqrt{1 - |ru_0(t)|^2})dt) \\
&\leq \frac{r^2}{2}\int_0^T P(t)\left|u_0'(t)\right|^2 dt + \frac{r^2}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} (H(t)u_0(t), u_0(t))dt \\
&\quad + \sum_{i=1}^{N} r^{\gamma_i}\alpha_i c^{\gamma_i}\|u_0\|^{\gamma_i} - \frac{r^2}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda_i K_i(t)|u_0(t)|^2 dt \\
&\leq \frac{r^2\eta^2}{8c^2} + r^{\gamma_{min}}\sum_{i=1}^{N}\frac{\alpha_i\eta^{\gamma_i}}{2^{\gamma_i}} - \frac{r^2\lambda_{min}}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} K_i(t)|u_0(t)|^2 dt,
\end{aligned}$$

where $\gamma_{min} = \min\{\gamma_1, \gamma_2, ..., \gamma_N\}$. Thus, we can find a

$$\lambda_* = \begin{cases} \dfrac{\eta^2}{4c^2\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} K_i(t)|u_0(t)|^2 dt}, \gamma_{min} > 2, \\[2ex] \dfrac{\frac{\eta^2}{4c^2} + \sum_{i=1}^{N}\frac{\alpha_i\eta^{\gamma_i}}{2^{\gamma_i - 1}}}{\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} K_i(t)|u_0(t)|^2 dt}, \gamma_{min} = 2. \end{cases}$$

If $\lambda_{min} > \lambda_*$, for a small enough r, we can obtain that

$$\Phi(ru_0) < -\frac{\lambda_{max}K_{max}\eta^2 T}{\sqrt{1 - \eta^2}},$$

where $\lambda_{max} = \max\{\lambda_0, \lambda_1, ..., \lambda_N\}$ and $K_{max} = \max\{\|K_0\|_{L^\infty}, \|K_1\|_{L^\infty}, ..., \|K_N\|_{L^\infty}\}$. Furthermore, if $u \in W_T^{1,2}$ with $\frac{\eta}{2c} < \|u\| \le \frac{\eta}{c}$, we have

$$\begin{aligned}\Phi(u) &\ge \frac{1}{2}\int_0^T P(t)\,|u'(t)|^2\,dt + \frac{1}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} (H(t)u(t), u(t))dt \\ &\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \frac{\lambda_i K_i(t)}{\sqrt{1-\eta^2}}|u(t)|^2 dt \\ &\ge -\frac{\lambda_{max}K_{max}\eta^2 T}{\sqrt{1-\eta^2}},\end{aligned}$$

which together with $\Phi(0) = 0$ yield that $u_* \neq 0$ and $\|u_*\| \le \frac{\eta}{2c}$. Thus, the critical point u_* of Φ is a nontrivial ground state weak solutions (1).

Lemma 3.2. *If the assumptions of Theorem 1.1 are satisfied, then $\Phi(u)$ meets the (PS)-condition, i.e., for any $\{u_n\} \in W_T^{1,2}$, if*

$$\{\Phi(u_n)\} \text{ is bounded and } \Phi'(u_n) \to 0 \text{ as } n \to +\infty,$$

then $\{u_n\}$ has a convergent subsequence in $W_T^{1,2}$.

Proof. For any $u \in W_T^{1,2}$, from the definition of $\Phi(u)$ and (I1), if $\|u\| \ge \frac{\eta}{c}$, it follows that

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_0^T P(t)\,|u'(t)|^2\,dt + \frac{1}{2}\int_0^T (H(t)u(t), u(t))dt \\ &\quad + \sum_{i=1}^{N} I_i(u(t_i)) \ge \frac{1}{2}\|u\|^2,\end{aligned}$$

which implies that

$$\Phi(u) \to +\infty \text{ as } \|u\| \to +\infty. \tag{9}$$

It means that $\Phi(u)$ is coercive and bounded from below. Thus, for any $\{u_n\} \in W_T^{1,2}$, if $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \to 0$, it follows that $\{u_n\}$ is bounded in $W_T^{1,2}$ by (9). If $\|u_n\| \ge \frac{\eta}{c}$, one has

$$\begin{aligned}\Phi'(u_n)u_n &= \int_0^T P(t)|u_n'(t)|^2 dt + \int_0^T (H(t)u_n(t), u_n(t))dt \\ &\quad + \sum_{i=1}^{N} (\nabla I_i(u_n(t_i)), u_n(t_i)) \ge \|u_n\|^2,\end{aligned}$$

which yields that $\Phi'(u_n) \nrightarrow 0$ as $n \to +\infty$. Hence, we only need to investigate the case of $\|u_n\| \le \frac{\eta}{c}$. Since $W_T^{1,2}$ is a reflexive Banach space, $\{u_n\}$ has a convergent subsequence (denoted again $\{u_n\}$) such that $u_n \rightharpoonup u$ in $W_T^{1,2}$ and $u_n \to u$ uniformly in C. Noting that $|u_n| \le \|u_n\|_\infty \le c\|u_n\| \le \eta$, it follows that

$$\begin{aligned}&\Big|\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} (\lambda_i K_i(t)\Psi(u_n(t)), u_n(t) - u(t))dt\Big| \\ &\le \lambda_{max}K_{max}\int_0^T |\Psi(u_n(t))||u_n(t) - u(t)|dt \\ &\le \frac{\lambda_{max}K_{max}\|u_n\|_\infty}{\sqrt{1-\eta^2}}\int_0^T |u_n(t) - u(t)|dt \to 0\end{aligned} \tag{10}$$

as $n \to +\infty$, which together with $\Phi'(u_n) \to 0$ as $n \to +\infty$ imply that

$$\begin{aligned} o(1) &= \Phi'(u_n)(u_n - u) \\ &= < u_n, u_n - u > + \sum_{i=1}^{N} (\nabla I_i(u_n(t_i)), u_n(t_i) - u(t_i)) \\ &\quad -\Gamma'(\|u_n\|) < \frac{u_n(t)}{\|u_n\|}, u_n - u > (\frac{1}{2}\sum_{i=1}^{N}\int_{t_i}^{s_i}(H(t)u_n(t), u_n(t))dt \\ &\quad + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda_i K_i(t)(1 - \sqrt{1 - |u_n(t)|^2})dt) \\ &\quad -\Gamma(\|u_n\|)(\sum_{i=1}^{N}\int_{t_i}^{s_i}(H(t)u_n(t), u_n(t) - u(t))dt \\ &\quad + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}}(\lambda_i K_i(t)\Psi(u_n(t)), u_n(t) - u(t))dt) \\ &= < u_n, u_n - u > (1 - \Gamma'(\|u_n\|)(\frac{\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda_i K_i(t)(1 - \sqrt{1 - |u_n(t)|^2})dt}{\|u_n\|} \\ &\quad + \frac{\sum_{i=1}^{N}\int_{t_i}^{s_i}(H(t)u_n(t), u_n(t))dt}{\|u_n\|}) + o(1), \end{aligned}$$

which together with $\Gamma'(\|u_n\|) \leq 0$ yield that

$$< u_n, u_n - u > \to 0 \ \text{ as } n \to +\infty.$$

By $u_n \rightharpoonup u$ in $W_T^{1,2}$, it follows that $u_n \to u$ in $W_T^{1,2}$. Thus, $\Phi(u)$ satisfies the (PS)-condition.

Proof of Theorem 1.1. By Lemma 3.2, we know that $\Phi(u)$ satisfies the (PS)-condition and is bounded from below. So, from Lemma 2.2, it follows that there exists a critical point u_* for $\Phi(u)$ such that

$$\Phi(u_*) = \inf_{u \in W_T^{1,2}} \Phi(u).$$

Moreover, based on Lemma 3.1, we have $u_* \neq 0$ and $\|u_*\| \leq \frac{\eta}{2c}$. Thus, the critical point u_* of Φ a nontrivial ground state weak solutions (1).

An Example. The example provided is intended to confirm our primary results.

$$\begin{cases} -(P(t)u'(t))' + H(t)u(t) = \lambda_i K_i(t)\Psi(u(t)), \ a.e.\ t \in (s_i, t_{i+1}], i = 0, 1, \\ \Delta(P(t_1)u'(t_1)) = \nabla I_1(u(t_1)), \\ P(t)u'(t) = P(t_1^+)u'(t_1^+), t \in (t_1, s_1], \\ u'(s_1^+) = u'(s_1^-), \\ u(0) = u(1), \ u'(0) = u'(1), \end{cases} \tag{11}$$

where $p(t) = 1$, $H(t)$ is an identity matrix, $T = 1$, $K_i(t) = 1, i = 0, 1, I_1(u) = u^4$. Therefore, the conditions (I1) and (I2) are satisfied, there exists a constant $\lambda_* > 0$ such that if $\lambda_{min} > \lambda_*$, the singular differential systems (11) admits at least one nontrivial ground state weak solutions.

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Competing interests

The author declare no competing interest

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