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Boundedness of some operators in grand generalized weighted Morrey spaces on RD-spaces

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Abstract. This paper aims to investigate the boundedness of some operators on grand generalized weighted Morrey spaces $\mathcal{L}_{\varphi}^{p),\phi}(\omega)$ over RD-spaces. Under assumption that functions φ and ϕ satisfy certain conditions, we prove that Hardy-Littlewood maximal operator and θ-type Calderón-Zygmund operator are bounded on these spaces. Moreover, the boundedness of the commutator $[b, T_\theta]$ generated by the θ-type Calderon-Zygmund operator and locally integrable function b is also established. The results presented for grand generalized weighted Morrey spaces are also new even in the context of Euclidean domains.

1. Introduction

The space of homogeneous type, first introduced by Coifman and Weiss [2, 3], provides a general framework for the study of Calderón-Zygmund operators and function spaces. Around the 1970s, Coifman and Weiss began exploring various harmonic analysis problems on metric spaces, which are referred to as spaces of homogeneous type (X, d, μ). These spaces are equipped with a metric d and a regular Borel measure μ that satisfies the doubling condition, i.e., there exists a constant $C_0 > 1$ such that for any ball $B(x, r) := \{y \in X : d(x, y) < r\}$, where $x \in X$ and $r > 0$, the following inequality holds:

$$\mu(B(x, 2r)) \le C_0 \mu(B(x, r)).$$

Since then, many classical results have been extended to spaces of homogeneous type in the sense of Coifman and Weiss. However, certain results have so far been established only for RD-spaces. An RD-space is a space of homogeneous type (X, d, μ) that satisfies an additional condition: there exist positive constants $a > 1$ and $b > 0$ such that,

$$b\mu(B(x, r)) \le \mu(B(x, ar)) \tag{1}$$

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holds for all $x \in X$ and $r \in (0, \mathrm{diam}(X)/a)$. For developments and research on operators over RD-spaces, we refer readers to see [22, 30, 31].

Morrey spaces were first introduced by Morrey in 1938 [24] in connection with local regularity problems of solutions to second-order elliptic partial differential equations. In 2009, Komori and Shirai [20] extended this concept by introducing weighted Morrey spaces in the Euclidean setting. Since then, numerous studies have focused on Morrey spaces and weighted Morrey spaces in various contexts; see, for example, [1, 5, 6, 13, 25, 28, 32]. The concept of generalized weighted Morrey spaces over RD-spaces was introduced in [4], where the boundedness of the Hardy-Littlewood maximal operator and Calderón-Zygmund operator was established. More recently, Li *et al.* [21] investigated the boundedness of commutators generated by the θ-Calderón-Zygmund operator and BMO functions in generalized weighted Morrey spaces over RD-spaces.

The theory of grand Lebesgue spaces, introduced by Iwaniec and Sbordone [14], has become one of the rapidly developing areas in modern analysis. These spaces have proven useful in various applications, including partial differential equations, geometric function theory, and the theory of Sobolev spaces; see [8, 10, 11, 26, 27] for example. Since their introduction, numerous classical operators have been extensively studied on grand function spaces. For example, Kokilashvili [15] provided criteria for the boundedness of several well-known operators in generalized weighted grand Lebesgue spaces. In 2019, Kokilashvili *et al.* [18] established weighted extrapolation results for grand Morrey spaces and applied these findings to partial differential equations. More recently, the boundedness of certain operators on grand generalized Morrey spaces over non-homogeneous spaces was studied in [12]. For additional research on the boundedness of operators in grand spaces, we refer readers to see [16, 17, 19, 23] and references therein.

Inspired by the above studies, this paper aims to establish the boundedness of the Hardy-Littlewood maximal operator and θ-type Calderón-Zygmund operators on grand generalized weighted Morrey spaces over RD-spaces.

Let $1 < p < \infty$ and φ be a function on $(0, p-1]$ which is a positive bounded and satisfies $\lim_{x\to 0} \varphi(x) = 0$. The class of such functions will be simply denoted by Φ_p. Then the norm of functions f in weighted grand Lebesgue space $L^{p)}_{\varphi}(\omega)$ is defined by

$$\|f\|_{L^{p)}_{\varphi}(\omega)} = \sup_{0<\varepsilon<p-1} [\varphi(\varepsilon)]^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\omega)}, \tag{2}$$

where $L^r(\omega)$ is the classical Lebesgue space with respect to a measure μ, and defined by the norm:

$$\|f\|_{L^r(\omega)} := \left(\int_X |f(x)|^r \omega(x) \mathrm{d}\mu(x) \right)^{\frac{1}{r}}, \quad 1 \le r < \infty.$$

On the base of weighted grand Lebesgue space $L^{p)}_{\varphi}(\omega)$, we give the definition of grand generalized weighted Morrey spaces as follows.

Definition 1.1. *Let $1 < p < \infty$, ω be a weight and $\varphi \in \Phi_p$. Suppose that $\phi : (0, \infty) \to (0, \infty)$ is an increasing function. Then grand generalized weighted Morrey space $\mathcal{L}^{p),\phi}_{\varphi}(\omega)$ is defined by*

$$\|f\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} := \left\{ f \in L^1_{\mathrm{loc}}(\omega) : \|f\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \left(\int_B |f(x)|^{p-\varepsilon} \omega(x) \mathrm{d}\mu(x) \right)^{\frac{1}{p-\varepsilon}}.$$

Especially, if we take $\varphi(\varepsilon) = \varepsilon^{\theta}$ with $\theta > 0$ in (4), then we can denote $\|f\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} := \|f\|_{\mathcal{L}^{p),\phi}_{\theta}(\omega)}$.

Remark 1.2. (1) When $\phi(x) = 1$, $\mathcal{L}_{\varphi}^{p),\phi}(\omega) = L_{\varphi}^{p)}(\omega)$. Therefore, the grand generalized weighted Morrey space $\mathcal{L}_{\varphi}^{p),\phi}(\omega)$ is an extension of the grand weighted Lebesgue space.

(2) If $\omega \in A_p(\mu)$, the generalized weighted Morrey space $\mathcal{L}^{p,\phi}(\omega)$ (see [4]), which is defined with respect to the norm:

$$\|f\|_{\mathcal{L}^{p,\phi}(\omega)} := \sup_{B\subset X}\left(\frac{1}{\phi(\omega(B))}\int_B |f(x)|^p\omega(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p}}, \quad 1 \le p < \infty. \tag{3}$$

(3) If we take function $\phi(t) = t^{\frac{p}{q}-1}$ for $t > 0$ and $1 < p \le q < \infty$, then grand generalized weighted Morrey space $\mathcal{L}_{\varphi}^{p),\phi}(\omega)$ defined as in (3) is just the grand weighted Morrey space $\mathcal{L}_{\varphi}^{p),q}(\omega)$ which is sightly modified in [18], that is,

$$\|f\|_{\mathcal{L}_{\varphi}^{p),q}(\omega)} = \sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\sup_B[\omega(B)]^{\frac{1}{q}-\frac{1}{p-\varepsilon}}\|f\|_{L^{p-\varepsilon}(\omega)}. \tag{4}$$

We recall that a weight function ω belongs to the Muckenhoupt class $A_p(\mu)(1 < p < \infty)$ if

$$\|\omega\|_{A_p} := \sup_B\left(\frac{1}{\mu(B)}\int_B\omega(x)\mathrm{d}\mu(x)\right)\left(\frac{1}{\mu(B)}\int_B[\omega(x)]^{1-p'}\mathrm{d}\mu(x)\right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset X$.

Further, $\omega \in A_1(\mu)$ if there is a positive constant C such that, for any ball $B \subset X$,

$$\frac{1}{\mu(B)}\int_B\omega(x)\mathrm{d}\mu(x) \le C\operatorname*{ess\,inf}_{y\in B}\omega(y),$$

as in the classical setting, let $A_\infty(\mu) = \bigcup_{p=1}^{\infty}A_p(\mu)$.

We end this section by stating some conventions on notation. The constant C denotes a positive constant that is independent of the main parameters. The symbol p' represents the conjugate exponent, i.e., $\frac{1}{p}+\frac{1}{p'} = 1$. The ball $B(x,r)$ is defined as $\{y \in X : d(x,y) < r\}$. For any $x, y \in X$ and $\delta \in (0,\infty)$, let $V(x,y) := \mu(B(x,d(x,y)))$ and $V_\delta := \mu(B(x,\delta))$. From the doubling condition, it follows that $V(x,y) = V(y,x)$. We also assume that $\mu(X) < \infty$.

2. Hardy-Littlewood maximal operator on $\mathcal{L}_{\varphi}^{p),\phi}(\omega)$

2.1. Weighted boundedness of the maximal operator

In this subsection we study the one-weighted problem for the Hardy-Littlewood maximal operator M defined by setting

$$M(f)(x) := \sup_{r>0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f(y)|\mathrm{d}\mu(y), \quad \text{for all x} \in \text{X} \tag{5}$$

Lemma 2.1. *[29] Let $p \in (1,\infty)$ and $\omega \in A_p(\mu)$. There exist positive constants C_1 and C_2 such that for any ball $B \subset X$ and each measurable set $E \subseteq B$,*

$$\frac{\omega(E)}{\omega(B)} \le C_1\left[\frac{\mu(E)}{\mu(B)}\right]^{\frac{1}{p}} \quad \textit{and} \quad \frac{\omega(E)}{\omega(B)} \ge C_2\left[\frac{\mu(E)}{\mu(B)}\right]^{p}.$$

Lemma 2.2. *[4] Let (X,d,μ) be an RD-space, if $\omega \in A_p(\mu)$, $p \in (1,\infty)$, then there exist positive constants $C_3, C_4 > 1$ such that for any ball $B \subset X$,*

$$\omega(2B) \ge C_3\omega(B), \tag{6}$$

and

$$\omega(2B) \le C_4\omega(B). \tag{7}$$

Lemma 2.3. *[4] Let $1 < p < \infty$, $\omega \in A_p(\mu)$, $\varphi \in \Phi_p$ and $\phi : (0,\infty) \to (0,\infty)$ be an increasing function. Assume that the mapping $t \mapsto \frac{\phi(t)}{t}$ is almost decreasing. Then M be as in (5) is bounded on $\mathcal{L}^{p,\phi}(\omega)$.*

Theorem 2.4. *Let $1 < p < \infty$, $\omega \in A_p(\mu)$, $\varphi \in \Phi_p$ and $\phi : (0,\infty) \to (0,\infty)$ be an increasing function. Let M be as in (5). Assume that the mapping $t \mapsto \frac{\phi(t)}{t}$ is almost decreasing, namely, there exists a positive constant C such that*

$$\frac{\phi(t)}{t} \le C\frac{\phi(s)}{s}, \tag{8}$$

for $s \ge t$. Then there exists a positive constant C such that for any $f \in \mathcal{L}_{\varphi}^{p),\phi}(\omega)$,

$$\|M(f)\|_{\mathcal{L}_{\varphi}^{p),\phi}(\omega)} \le C\|f\|_{\mathcal{L}_{\varphi}^{p),\phi}(\omega)}.$$

Proof. Choosing a number δ such that $0 < \varepsilon \le \delta < p-1$, observe that

$$\begin{aligned}
\|M(f)\|_{\mathcal{L}_{\varphi}^{p),\phi}(\omega)} &= \sup_{0<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \|M(f)\|_{L^{p-\varepsilon}(\omega)} \\
&\le \sup_{0<\varepsilon\le\delta} \varphi(\varepsilon) \sup_{B}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \|M(f)\|_{L^{p-\varepsilon}(\omega)} \\
&\quad + \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \|M(f)\|_{L^{p-\varepsilon}(\omega)} \\
&=: \mathrm{E}_1 + \mathrm{E}_2.
\end{aligned}$$

The estimates for E_1 goes as follows. By applying the $\mathcal{L}^{p,\phi}(\omega)$-boundedness of M (see[4]) and Definition 1.1, we can deduce that

$$\begin{aligned}
&\sup_{0<\varepsilon\le\delta} \varphi(\varepsilon) \sup_{B}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \|M(f)\|_{L^{p-\varepsilon}(\omega)} \\
&\quad = \sup_{0<\varepsilon\le\delta} \varphi(\varepsilon) \|M(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \\
&\quad \le C\|f\|_{\mathcal{L}_{\varphi}^{p),\phi}(\omega)}.
\end{aligned}$$

Now let us estimate E_2. Since $\delta < \varepsilon < p-1$, then we have $\frac{p-\delta}{p-\varepsilon} > 1$. Further, by virtue of Hölder's inequality and $\mathcal{L}^{p,\phi}(\omega)$-boundedness of M, we get

$$\begin{aligned}
\mathrm{E}_2 &= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \|M(f)\|_{L^{p-\varepsilon}(\omega)} \\
&\le \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \|M(f)\|_{L^{p-\delta}(\omega)} (\omega(B))^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}} \\
&= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)[\varphi(\delta)]^{-1}\varphi(\delta) \sup_{B}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} [\phi(\omega(B))]^{\frac{1}{p-\delta}} \\
&\qquad \times [\phi(\omega(B))]^{-\frac{1}{p-\delta}} \|M(f)\|_{L^{p-\delta}(\omega)} (\omega(B))^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}} \\
&= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)[\varphi(\delta)]^{-1}\varphi(\delta) \sup_{B}[\phi(\omega(B))]^{\frac{1}{p-\delta}-\frac{1}{p-\varepsilon}} [\omega(B)]^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}} \\
&\qquad \times [\phi(\omega(B))]^{-\frac{1}{p-\delta}} \|M(f)\|_{L^{p-\delta}(\omega)} \\
&\le C\|f\|_{\mathcal{L}_{\varphi}^{p),\phi}(\omega)}.
\end{aligned}$$

Which, together with the estimate for E_1, the Theorem 2.4 is proved. □

With an argument similar to that used in the proof of Theorem 2.4, it is easy to obtain the following result on the maximal operator $\widetilde{M}_r$.

Corollary 2.2. Let $1 < p < \infty$, $\omega \in A_p(\mu)$, $\varphi \in \Phi_p$ and $\phi : (0, \infty) \to (0, \infty)$ be an increasing function. Assume that the mapping $t \mapsto \frac{\phi(t)}{t}$ is almost decreasing function satisfying (8). Then non-centered maximal operator $\widetilde{M}_r$ is bounded on $\mathcal{L}^{p),\phi}_{\varphi}(\omega)$, where $\widetilde{M}_r$ is defined by

$$\widetilde{M}_r(f)(x) := \sup_{x \in B} \Big(\frac{1}{\mu(B)} \int_B |f(y)|^r \mathrm{d}\mu(y) \Big)^{\frac{1}{r}}.$$

2.2. Vector-valued extension

To discuss the vector-valued extension of Theorem 2.4, we need the following assumption on ϕ: there exists a positive constant C such that

$$\int_r^\infty \frac{\phi(t)}{t} \frac{\mathrm{d}t}{t} \leq C \frac{\phi(r)}{r} \quad \text{for any r} \in (0, \infty). \tag{9}$$

Lemma 2.5. *[4] Let $p \in (1, \infty)$, $\omega \in A_p(\mu)$ and $\phi : (0, \infty) \to (0, \infty)$ be an increasing function which satisfies (9), assume that the mapping $t \mapsto \frac{\phi(t)}{t}$ satisfies (8). Then there exists a positive constant C such that for any ball $B \subset X$,*

$$\sum_{k=1}^{\infty} \Big[\frac{\phi(\omega(2^k B))}{\omega(2^k B)} \Big]^{\frac{1}{p}} \leq C \Big[\frac{\phi(\omega(B))}{\omega(B)} \Big]^{\frac{1}{p}}.$$

Lemma 2.6. *[9] Let $r \in (1, \infty), p \in (1, \infty)$ and $\omega \in A_p(\mu)$. Then there exists a positive constant C depending on p and r, such that, for any $\{f_i\}_{i=1}^{\infty} \subset L^p(\omega)$,*

$$\Big\| \Big\{ \sum_{j \in \mathbb{N}} \big[M(f_j) \big]^r \Big\}^{\frac{1}{r}} \Big\|_{\mathcal{L}^p(\omega)} \leq C \Big\| \Big\{ \sum_{j \in \mathbb{N}} |f_j|^r \Big\}^{\frac{1}{r}} \Big\|_{\mathcal{L}^p(\omega)}.$$

Theorem 2.7. *Let $1 < p, r < \infty$, $\omega \in A_p(\mu)$, $\varphi \in \Phi_p$ and $\phi : (0, \infty) \to (0, \infty)$ be an increasing function that satisfies (9). Assume that the mapping $t \mapsto \frac{\phi(t)}{t}$ satisfies (8). Then there exists a positive constant C depending on p and r, such that for any $\{f_j\}_{j=1}^{\infty} \subset \mathcal{L}^{p),\phi}_{\varphi}(\omega)$,*

$$\Big\| \Big\{ \sum_{j \in \mathbb{N}} \big[M(f_j) \big]^r \Big\}^{\frac{1}{r}} \Big\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} \leq C \Big\| \Big\{ \sum_{j \in \mathbb{N}} |f_j|^r \Big\}^{\frac{1}{r}} \Big\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)}.$$

Proof. Choosing a small δ such that $0 < \varepsilon \leq \delta < p - 1$, then, by Definition 1.1, observe that

$$\begin{aligned} \Big\| \Big\{ \sum_{j \in \mathbb{N}} \big[M(f_j) \big]^r \Big\}^{\frac{1}{r}} \Big\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} &= \sup_{0 < \varepsilon < p-1} \varphi(\varepsilon) \Big\| \Big\{ \sum_{j \in \mathbb{N}} \big[M(f_j) \big]^r \Big\}^{\frac{1}{r}} \Big\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \\ &\leq \sup_{0 < \varepsilon < \delta} \varphi(\varepsilon) \Big\| \Big\{ \sum_{j \in \mathbb{N}} \big[M(f_j) \big]^r \Big\}^{\frac{1}{r}} \Big\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \\ &\quad + \sup_{\delta < \varepsilon < p-1} \varphi(\varepsilon) \Big\| \Big\{ \sum_{j \in \mathbb{N}} \big[M(f_j) \big]^r \Big\}^{\frac{1}{r}} \Big\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \\ &=: \mathrm{F}_1 + \mathrm{F}_2. \end{aligned}$$

The estimates for F_1 is given as follows. By the $\mathcal{L}^{p,\phi}(\omega)-$boundedness of M (see[4]), it follows that

$$\sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\left\|\left\{\sum_{j\in\mathbb{N}}\left[M(f_j)\right]^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)}\leq C\sup_{0<\varepsilon<\delta}\varphi(\varepsilon)\left\|\left\{\sum_{j\in\mathbb{N}}|f_j|^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)}$$
$$\leq C\left\|\left\{\sum_{j\in\mathbb{N}}|f_j|^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)}.$$

Similar to the estimate of E_2 in the proof of Theorem 2.4, By virtue of Hölder's inequality and Lemma 2.6, we have

$$\begin{aligned}
&\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\sup_{B\subset X}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}}\left\|\left\{\sum_{j\in\mathbb{N}}\left[M(f_j)\right]^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p-\varepsilon}(\omega)}\\
&\quad\leq\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\sup_{B\subset X}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}}\left\|\left\{\sum_{j\in\mathbb{N}}\left[M(f_j)\right]^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p-\delta}(\omega)}(\omega(B))^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}}\\
&\quad\leq C\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)\sup_{B\subset X}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}}\left\|\left\{\sum_{j\in\mathbb{N}}|f_j|^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p-\delta}(\omega)}(\omega(B))^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}}\\
&\quad\leq C\sup_{\delta<\varepsilon<p-1}\varphi(\varepsilon)[\varphi(\delta)]^{-1}\varphi(\delta)\sup_{B\subset X}[\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}}[\phi(\omega(B))]^{\frac{1}{p-\delta}}\\
&\qquad\times[\phi(\omega(B))]^{-\frac{1}{p-\delta}}\left\|\left\{\sum_{j\in\mathbb{N}}|f_j|^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p-\delta}(\omega)}(\omega(B))^{\frac{\varepsilon-\delta}{(p-\varepsilon)(p-\delta)}}\\
&\quad\leq C\phi(p-1)[\phi(\delta)]^{-1}\left\|\left\{\sum_{j\in\mathbb{N}}|f_j|^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)}\\
&\quad\leq C\left\|\left\{\sum_{j\in\mathbb{N}}|f_j|^r\right\}^{\frac{1}{r}}\right\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)}.
\end{aligned}$$

Which, together with the estimate for F_1, is our desired result. □

3. θ-Type Calderón-Zygmund operators on $\mathcal{L}^{p),\phi}_{\varphi}(\omega)$

In this section, we shall deal with the boundedness of the θ-type Calderón-Zygmund operators and its commutator on grand generalized weighted Morrey space $\mathcal{L}^{p),\phi}_{\varphi}(\omega)$ over RD-spaces.

The following definition see, Duong *et. al.* [7].

Definition 3.1. Let θ be a non-negative and non-decreasing function on $[0,\infty)$ with satisfying

$$\int_0^1\frac{\theta(t)}{t}\mathrm{d}t<\infty. \tag{10}$$

And the measurable function $K(\cdot,\cdot)$ on $X\times X\backslash\{(x,y):x\in X\}$ is called a θ-type Calderón-Zygmund kernel, if for any $x\neq y$,

$$|K(x,y)|\leq\frac{C}{V(x,y)}, \tag{11}$$

and for $d(x,z) < \frac{d(x,y)}{2}$,

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq \frac{C}{V(x,y)}\theta\Big(\frac{d(x,z)}{d(x,y)}\Big). \tag{12}$$

Remark 3.2. If we take the function $\theta(t) = t^{\delta}$ with $t > 0$ and $\delta \in (0,1]$. Then $K(x,y)$ defined as in Definition 3.1 is just the standard Calderón-Zygmund kernel.

Definition 3.3. Let b a real valued $\mu-$measurable function on X, if $b \in L^1_{loc}(\mu)$ and its norm is

$$\|b\|_* := \sup_B \frac{1}{\mu(B)} \int_B |b(x) - b_B| \mathrm{d}\mu(x) < \infty,$$

then b is called a BMO(μ) function, where the supremum is taken over all $B \subset X$ and

$$b_B := \frac{1}{\mu(B)} \int_B b(y) \mathrm{d}\mu(y).$$

Let $L^{\infty}_b(\mu)$ be the space of all $L^{\infty}(\mu)$ functions with bounded support. Let $K(x,y)$ be a θ-type Calderón-Zygmund kernel. A linear operator T_θ is called a θ-type Calderón-Zygmund operator with kernel $K(x,y)$ if T_θ can be extended to a bounded linear operator on $L^2(X)$, and

$$T_\theta f(x) := \int_X K(x,y) f(y) \mathrm{d}\mu(y) \tag{13}$$

for all $f \in L^{\infty}_b(\mu)$ and $x \notin \mathrm{supp}(f)$.

Given a locally integrable function b on X, then the commutator $[b, T_\theta]$ of θ-type Calderón-Zygmund operator T_θ is defined as

$$[b, T_\theta] f(x) := b(x) T_\theta f(x) - T_\theta(bf)(x) = \int_X [b(x) - b(y)] K(x,y) f(y) \mathrm{d}\mu(y). \tag{14}$$

The main theorems of this section is stated as follows.

Theorem 3.4. *Let $p \in (1,\infty)$, $\omega \in A_p(\mu)$, $\varphi \in \Phi_p$. Let $\phi : (0,\infty) \to (0,\infty)$ be an increasing function, continuous function satisfying conditions (8) and (9). Then T_θ defined as in (13) is bounded on $\mathcal{L}^{p),\phi}_{\varphi}(\omega)$, that is, there exists a constant $C > 0$ such that, for all $f \in \mathcal{L}^{p),\phi}_{\varphi}(\omega)$,*

$$\|T_\theta(f)\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} \leq C \|f\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)}.$$

Theorem 3.5. *Let $p \in (1,\infty)$, $\omega \in A_p(\mu)$, $b \in \mathrm{BMO}(\mu)$, $\varphi \in \Phi_p$. Let $\phi : (0,\infty) \to (0,\infty)$ be an increasing function, continuous function satisfying conditions (8) and (9). Then $[b, T_\theta]$ defined as in (14) is bounded on $\mathcal{L}^{p),\phi}_{\varphi}(\omega)$, that is, there exists a constant $C > 0$ such that, for all $f \in \mathcal{L}^{p),\phi}_{\varphi}(\omega)$,*

$$\|[b, T_\theta] f\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} \leq C \|b\|_{BMO(\mu)} \|f\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)}.$$

To formulate the above theorems we also need the following lemma.

Lemma 3.6. Let $p \in (1,\infty)$, $\omega \in A_p(\mu)$. Let $\phi : (0,\infty) \to (0,\infty)$ be an increasing function, continuous function satisfying conditions (8) and (9) and θ be a non-negative, non-decreasing function on $(0,\infty)$ with satisfying (10). Then T_θ defined as in (13) is bounded on the generalized weighted Morrey space, that is, there exists a constant $C > 0$ such that, for all $f \in \mathcal{L}^{p,\phi}(\omega)$,

$$\|T_\theta(f)\|_{\mathcal{L}^{p,\phi}(\omega)} \leq C \|f\|_{\mathcal{L}^{p,\phi}(\omega)}.$$

Proof. Let $p \in (1,\infty)$, we only need to consider that for any fixed ball $B = B(x_0, r) \subset X$,

$$\left\{\frac{1}{\phi(\omega(B))}\int_B [T_\theta f(x)]^p \omega(x)\mathrm{d}\mu(x)\right\}^{\frac{1}{p}} \le C\|f\|_{\mathcal{L}^{p,\phi}(\omega)}. \tag{15}$$

To estimate (15), we decompose f as $f := f_1 + f_2$, where $f_1 := f\chi_{2B}$ and $2B = B(x_0, 2r)$, write

$$\begin{aligned}
&\left\{\frac{1}{\phi(\omega(B))}\int_B [T_\theta(f)(x)]^p \omega(x)\mathrm{d}\mu(x)\right\}^{\frac{1}{p}} \\
&\quad\le \left\{\frac{1}{\phi(\omega(B))}\int_B [T_\theta(f_1)(x)]^p \omega(x)\mathrm{d}\mu(x)\right\}^{\frac{1}{p}} + \left\{\frac{1}{\phi(\omega(B))}\int_B [T_\theta(f_2)(x)]^p \omega(x)\mathrm{d}\mu(x)\right\}^{\frac{1}{p}} \\
&\quad= G_1 + G_2.
\end{aligned}$$

The estimate for G_1 goes as follows. From [7, Theorem 1.3], slightly modified, we know that the T_θ is bounded on $L^p(\omega)$ for $p \in (1,\infty)$. By applying (7) and (8), implies that

$$\begin{aligned}
G_1 &\le \frac{1}{[\phi(\omega(B))]^{\frac{1}{p}}}\Big[\int_X |f_1(x)|^p \omega(x)\mathrm{d}\mu(x)\Big]^{\frac{1}{p}} \\
&\le C\Big[\frac{1}{\phi(\omega(2B))}\int_{2B} |f(x)|^p \omega(x)\mathrm{d}\mu(x)\Big]^{\frac{1}{p}}\Big[\frac{\phi(\omega(2B))}{\phi(\omega(B))}\Big]^{\frac{1}{p}} \\
&\le C\|f\|_{\mathcal{L}^{p,\phi}(\omega)}\Big[\frac{\omega(2B)}{\omega(B)}\Big]^{\frac{1}{p}} \\
&\le C\|f\|_{\mathcal{L}^{p,\phi}(\omega)}.
\end{aligned}$$

For term G_2, notice that, for any $x \in B$ and $y \in (2B)^c$, we obtain $d(x,y) \sim d(x_0,y)$ and $V(x,y) \sim V(x_0,y)$, by virtue of Hölder inequality and Lemma 2.5,

$$\begin{aligned}
|T_\theta(f_2)(x)| &\le \int_{d(y,x_0)\ge 2r} |K(x,y)f(y)|\mathrm{d}\mu(y) \\
&\le C\int_{d(y,x_0)\ge 2r} \frac{|f(y)|}{V(x,y)}\mathrm{d}\mu(y) \\
&\sim C\int_{d(y,x_0)\ge 2r} \frac{|f(y)|}{V(x_0,y)}\mathrm{d}\mu(y) \\
&\le C\sum_{k=1}^{\infty}\int_{2^k r\le d(y,x_0)\le 2^{k+1}r} \frac{|f(y)|}{V(x_0,y)}\mathrm{d}\mu(y) \\
&\le C\sum_{k=1}^{\infty}\frac{1}{V_{2^k r}(x_0)}\Big[\int_{B(x_0,2^{k+1}r)} |f(y)|^p \omega(y)\mathrm{d}\mu(y)\Big]^{\frac{1}{p}}\Big[\int_{B(x_0,2^{k+1}r)} \omega(y)^{1-p'}\mathrm{d}\mu(y)\Big]^{\frac{1}{p'}} \\
&\le C\sum_{k=1}^{\infty}\frac{[\phi(\omega(B(x_0,2^{k+1}r)))]^{\frac{1}{p}}}{V_{2^k r}(x_0)}\cdot\frac{V_{2^{k+1}r}(x_0)}{[\omega(B(x_0,2^{k+1}r))]^{\frac{1}{p}}}\|f\|_{\mathcal{L}^{p,\phi}(\omega)} \\
&\le C\Big[\frac{\phi(\omega(B))}{\omega(B)}\Big]^{\frac{1}{p}}\|f\|_{\mathcal{L}^{p,\phi}(\omega)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left\{\frac{1}{\phi(\omega(B))}\int_B [T_\theta(f_2)(x)]^p \omega(x)\mathrm{d}\mu(x)\right\}^{\frac{1}{p}} &\le C\Big[\frac{\phi(\omega(B))}{\omega(B)}\Big]^{\frac{1}{p}}\Big[\frac{\omega(B)}{\phi(\omega(B))}\Big]^{\frac{1}{p}}\|f\|_{\mathcal{L}^{p,\phi}(\omega)} \\
&\le C\|f\|_{\mathcal{L}^{p,\phi}(\omega)}.
\end{aligned}$$

Which, together with estimate of G_1, we obtain the desired result. □

Lemma 3.7. *[21] Let $p \in (1,\infty)$, $\omega \in A_p(\mu)$ and $b \in \mathrm{BMO}(\mu)$. Let $\phi : (0,\infty) \to (0,\infty)$ be an increasing function, continuous function satisfying conditions (8) and (9) and θ be a non-negative, non-decreasing function on $(0,\infty)$ with satisfying (10). Then the commutator $[b, T_\theta]$ defined as in (14) is bounded on $\mathcal{L}^{p,\phi}(\omega)$.*

Proof of Theorem 3.4. Let δ be a fixed constant satisfying $0 < \varepsilon < \delta < p-1$. By Definition 1.1, observe that

$$\begin{aligned}\|T_\theta(f)\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)} &= \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|T_\theta(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \\ &\le \sup_{0<\varepsilon<\delta} \varphi(\varepsilon)\|T_\theta(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} + \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon)\|T_\theta(f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \\ &= \mathrm{H}_1 + \mathrm{H}_2.\end{aligned}$$

The estimates for H_1 goes as follows. From Lemma 3.6, it follows that

$$\begin{aligned}\mathrm{H}_1 &\le C \sup_{0<\varepsilon<\delta} \varphi(\varepsilon)\|f\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \\ &\le C\|f\|_{\mathcal{L}^{p),\phi}_{\varphi}(\omega)}.\end{aligned}$$

Fix $\varepsilon \in (\delta, p-1)$ so that $\frac{p-\delta}{p-\varepsilon} > 1$. Using Hölder inequality with respect to the $\left(\frac{p-\delta}{p-\varepsilon}\right)' = \frac{p-\delta}{\varepsilon-\delta}$ and the boundedness of T_θ in $L^p(\omega)$ for $p \in (1,\infty)$, we can deduce that

$$\begin{aligned}\mathrm{H}_2 &= \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \left(\int_B |T_\theta(f)(x)|^{p-\varepsilon}\omega(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p-\varepsilon}} \\ &\le \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \left(\int_B |T_\theta(f)(x)|^{p-\delta}\omega(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p-\delta}} \omega(B)^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \\ &\le C \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \left(\int_B |f(x)|^{p-\delta}\omega(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p-\delta}} \omega(B)^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \\ &\le C \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} [\phi(\omega(B))]^{\frac{1}{p-\delta}} \omega(B)^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \\ &\times [\phi(\omega(B))]^{-\frac{1}{p-\delta}} \left(\int_B |f(x)|^{p-\delta}\omega(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p-\delta}}.\end{aligned}$$

Let

$$S = [\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} [\phi(\omega(B))]^{\frac{1}{p-\delta}} \omega(B)^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}}.$$

Since $\delta < p-1$ and $\varepsilon \in (\delta, p-1)$, imply that

$$0 < \frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)} < \frac{p-1-\delta}{p-\delta}.$$

By applying the monotonicity of ϕ, we can deduce that

$$\begin{aligned}S &\le [\phi(\omega(B))]^{-\frac{1}{p-\delta}} [\phi(\omega(B))]^{\frac{1}{p-\delta}} \omega(B)^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \\ &\le \omega(B)^{\frac{p-1-\delta}{(p-\delta)(p-\delta)}} \le C.\end{aligned}$$

Then combing the above estimates, we further obtain that

$$\begin{aligned}
\mathrm{H}_2 &\leq C \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\delta}} \Big(\int_B |f(x)|^{p-\delta} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p-\delta}} \\
&\leq C \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) [\phi(\delta)]^{-1} \phi(\delta) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\delta}} \Big(\int_B |f(x)|^{p-\delta} \omega(x) \mathrm{d}\mu(x) \Big)^{\frac{1}{p-\delta}} \\
&\leq C\varphi(p-1)[\phi(\delta)]^{-1} \|f\|_{\mathcal{L}_\varphi^{p),\phi}(\omega)} \\
&\leq \mathrm{C}\|f\|_{\mathcal{L}_\varphi^{p),\phi}(\omega)}.
\end{aligned}$$

Which, together with estimate of H_1, the proof of Theorem 3.4 is finished. □

Proof of Theorem 3.5. First observe that the $L^p(\omega)$-boundedness of $[b, T_\theta]$ (see [7]) and the interpolation theorem imply that there is a number δ, $\delta \in (0, p-1)$, such that

$$\|[b, T_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \leq \mathrm{C}\|b\|_{BMO(\mu)} \|f\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)}, \quad \varepsilon \in (0, \delta].$$

Fix $\varepsilon \in (\delta, p-1)$ so that $\frac{p-\delta}{p-\varepsilon} > 1$, by virtue of Hölder's inequality and Lemma 3.7, we can obtain that

$$\begin{aligned}
&\|[b, T_\theta](f)\|_{\mathcal{L}_\varphi^{p),\phi}(\omega)} = \max\Big\{ \sup_{0<\varepsilon<\delta} \varphi(\varepsilon) \|[b, T_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)}, \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \|[b, T_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \Big\} \\
&\leq \max\Big\{ \sup_{0<\varepsilon<\delta} \varphi(\varepsilon) \|[b, T_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)}, \\
&\qquad \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\varepsilon}} \|[b, T_\theta](f)\|_{L^{p-\delta}(\omega)} \omega(B)^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \Big\} \\
&\leq \max\Big\{ \sup_{0<\varepsilon<\delta} \varphi(\varepsilon) \|[b, T_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)}, \\
&\qquad \sup_{\delta<\varepsilon<p-1} \varphi(\varepsilon) \sup_{B\subset X} [\phi(\omega(B))]^{-\frac{1}{p-\delta}} \|[b, T_\theta](f)\|_{L^{p-\delta}(\omega)} \Big[\frac{\omega(B)}{\phi(\omega(B))} \Big]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \Big\} \\
&\leq \max\Big\{ \sup_{0<\varepsilon<\delta} \varphi(\varepsilon) \|[b, T_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)}, \\
&\qquad \sup_{\delta<\varepsilon<p-1} \Big[\frac{\omega(B)}{\phi(\omega(B))} \Big]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}} \sup_{0<\varepsilon<\delta} \varphi(\varepsilon) \|[b, T_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)} \Big\}.
\end{aligned}$$

Let $S := \sup_{0<\varepsilon<\delta} \varphi(\varepsilon) \|[b, T_\theta](f)\|_{\mathcal{L}^{p-\varepsilon,\phi}(\omega)}$ and $T := \sup_{\delta<\varepsilon<p-1} \Big[\frac{\omega(B)}{\phi(\omega(B))} \Big]^{\frac{\varepsilon-\delta}{(p-\delta)(p-\varepsilon)}}$. Then

$$\|[b, T_\theta](f)\|_{\mathcal{L}_\varphi^{p),\phi}(\omega)} \leq \max\{1, T\} \cdot S \leq \mathrm{C}\|b\|_{BMO(\mu)} \|f\|_{\mathcal{L}_\varphi^{p),\phi}(\omega)}.$$

This completes the proof of Theorem 3.5. □

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