Filomat 39:7 (2025), 2331–2346
https://doi.org/10.2298/FIL2507331B

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and blow up of solutions for a m-biharmonic viscoelastic equation with variable exponents

Gülistan Butakın[a,*], Erhan Pişkin[b]

[a] *Dicle University, Institute of Natural and Applied Sciences, Diyarbakır, Turkey*
[b] *Dicle University, Department of Mathematics, Diyarbakır, Turkey*

Abstract. In this study investigate the m-biharmonic viscoelastic equation with variable exponents. Firstly we demonstrate the blow up of solutions for positive initial energy. Later we prove that nonnegative solutions of finite time blow up under negative initial energy.

1. Introduction

We deal with the following m-biharmonic viscoelastic equation with variable exponents

$$\begin{cases} z_t + \Delta^2 z - \int\limits_0^t k(t-s)\,\Delta_m^2 z(x,s)\,ds = |z|^{p(x)-2} z, & x \in \Omega,\ t > 0, \\ z(x,t) = \frac{\partial z}{\partial v}(x,t) = 0, & x \in \partial\Omega, t \geq 0, \\ z(x,0) = z_0(x), & x \in \Omega, \end{cases} \tag{1}$$

here $\Delta_m^2 z = \Delta\left(|\Delta z|^{m-2}\,\Delta z\right)$, $m \geq 2$ is a constant, Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $(n \geq 1)$, and the initial value $z_0 \in W_0^{2,p(\cdot)}(\Omega)$, the exponent $p(\cdot)$ is given measurable function and $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function.

Butakın and Pişkin [4] study the subsequent $m(x)$-biharmonic parabolic equation, with variable exponent

$$z_t - \Delta z + \Delta^2 z + \Delta_{m(x)}^2 z = |z|^{p(x)-2} z.$$

They proved the global existence and blow up of solutions with negative initial energy.

Butakın and Pişkin [5] discuss a viscoelastic $m(x)$-biharmonic equation with logarithmic source term

$$z_t + \Delta^2 z + \Delta_{m(x)}^2 z - \int_0^t g(t-s)\,\Delta^2 z ds = |z|^{p-2} z \ln|z|.$$

2020 *Mathematics Subject Classification*. Primary 35B44; Secondary 35K41, 35K61.
Keywords. Blow up, existence, m-biharmonic viscoelastic equation, variable exponent.
Received: 01 January 2025; Accepted: 03 January 2025
Communicated by Maria Alessandra Ragusa
* Corresponding author: Gülistan Butakın
Email addresses: gulistanbutakin@gmail.com (Gülistan Butakın), episkin@dicle.edu.tr (Erhan Pişkin)
ORCID iDs: https://orcid.org/0000-0003-1140-9672 (Gülistan Butakın), https://orcid.org/0000-0001-6587-4479 (Erhan Pişkin)

They proved the local existence of solutions by using the Faedo-Galerkin method. Later, we proved the blow up of solutions by using the concavity method.

Messaoudi [13] investigated the semilinear parabolic equation

$$z_t - \Delta z + \int_0^t g(t-s)\Delta z(x,s)\,ds = |z|^{p-2} z. \tag{2}$$

The author has demonstrated that the solutions to (2) blow up in finite time when the initial energy $E(0) \leq 0$. Later, Messaoudi [14] also proved that the solutions to (2) blow up in finite time when the initial energy is positive. Tian [25] obtain the lower bound for the problem (2).

Wu et. al. [26] studied the subsequent semilinear equations with p-Laplacian viscoelastic term

$$z_t - \Delta z + \int_0^t g(t-s)\Delta_p z(x,s)\,ds = |z|^{q(x)-2} z. \tag{3}$$

They proved that the weak solutions of the above problems blow up in finite time.

Problem (1) emerges from various mathematical models in engineering and the physical sciences. Over the recent decades, significant focus has been directed towards the examination of equations featuring viscoelastic components, leading to numerous findings concerning the existence, uniqueness, and properties of both weak and classical solutions. For further exploration, we direct interested readers to [1–7, 9, 15–21, 23, 24].

Inspired by the preceding research, we show that solutions to problem (1) blow up when given any positive initial energy and sufficiently large initial values, assuming $p(\cdot) \geq 2$ and $n \geq 1$. Also, we establish that nonnegative solutions of problem (1) finite time blow up under negative initial energy.

This work is organized as follows: In part 2, we will demonstrate the local existence of solutions. In part 3, we will establish that solutions finite time blow up.

2. Existence

In this part, we will prove the local existence of solutions.

Definition 2.1. *A function $z(x,t)$ is considered a weak solution to problem (1) if it belongs to $L^\infty\left(0,T;H_0^2(\Omega)\right) \cap L^m\left(0,T;W_0^{2,m}(\Omega)\right)$ and if $z_t \in L^2\left(0,T;L^2(\Omega)\right)$ subject to the given equality.*

$$\begin{aligned} &\int_\Omega z_t\theta dx + \int_\Omega \Delta z\Delta\theta dx - \int_\Omega\int_0^t k(t-s)\,|\Delta z(s)|^{m-2}\,\Delta z(s)\,\Delta\theta ds dx \\ = &\int_\Omega |z|^{p(\cdot)-2}\,z\theta dx, \end{aligned}$$

applies to all $\theta \in W_0^{2,m}(\Omega)$.

Lemma 2.2. *Suppose that $z(t)$ is a weak solution to problem (1), we introduce the energy functional $J(t)$ as follows:*

$$\begin{aligned} J(t) &= \frac{1}{2}\left(1-\int_0^t k(s)\,ds\right)\|\Delta z\|^2 \\ &\quad -\int_\Omega \frac{1}{p(x)}\,|z|^{p(x)}\,dx + \frac{1}{2}(k\circ\Delta z)(t), \end{aligned} \tag{4}$$

here

$$(k \circ \Delta z)(t) = \int_0^t k(t-s) \left\| \Delta z(t) - |\Delta z(s)|^{m-2} .\Delta z(s) \right\|_2^2 ds.$$

Thus $J(t)$ is a non-increasing function with respect to t.

Proof. Differentiating equation (4) and utilizing the initial equation of (1), it follows that

$$\begin{aligned} \frac{dJ(t)}{dt} &= -\int_\Omega (z_t)^2 dx + \frac{1}{2} \int_0^t k'(t-s) \left\| \Delta z(t) - |\Delta z|^{m-2} .\Delta z(s) \right\|_2^2 ds \\ &\quad -\frac{1}{2} k(t) \|\Delta z\|_2^2 \\ &\leq -\int_\Omega (z_t)^2 dx \leq 0. \end{aligned}$$

By integrating the above identity over the interval $(0, t)$, we derive

$$J(t) - J(0) \leq -\int_0^t \|z_\tau(\tau)\|_2^2 d\tau. \tag{5}$$

Therefore, the proof is concluded. □

We assume the exponent $p(x)$ and the relaxation function k, to be such that

$$1 < p^- = ess \inf_{x \in \Omega} p(x) \leq p(x) \leq p^+ = ess \sup_{x \in \Omega} p(x) < \infty, \tag{6}$$

$$\forall z \in \Omega, \epsilon \in \Omega, |z - \epsilon| < 1, \quad \left| p(z) - p(\epsilon) \right| \leq \omega(|z - \epsilon|), \tag{7}$$

here

$$\lim_{\tau \to 0^+} \omega(\tau) \ln \frac{1}{\tau} = C < \infty.$$

$$k(s) \geq 0, \ k'(s) \leq 0, \ 0 < \int_0^\infty k(s) ds < \frac{2p^- - 4}{2p^- - 3}, \ k = \frac{1 - \frac{3}{4} \int_0^\infty k(s) ds}{1 - \int_0^\infty k(s) ds}. \tag{8}$$

Where, we present our main result.

Theorem 2.3. *Suppose that (6), (7), (8) hold, and letting $p^+ \leq m < \frac{2n}{n-4}$, where $z_0 \in W_0^{2,m}(\Omega)$ then problem (1) has a local weak solution $z(x,t)$ satisfying: $z(x,t) \in L^\infty\left((0, T_0); H_0^2(\Omega)\right) \cap L^m\left(0, T_0; W_0^{2,m}(\Omega)\right)$ and $z_t \in L^2\left(0, T_0; L^2(\Omega)\right)$ for $T_0 > 0$.*

Proof. We will utilize the Faedo Galerkin method. Interested readers are referred to [10, 22] for similar proof of the existence of weak solutions.

Suppose $\{\omega_i\}_{i=1}^{\infty}$ forms an orthogonal basis in $W_0^{2,m}(\Omega)$. We define finite dimensional subspace V_r =span$\{\omega_1, \omega_2, ..., \omega_r\}$. Assuming

$$z_{0r}(x) = \sum_{i=1}^{r} \left(\int_\Omega z_0 \omega_i dx \right) \omega_i \to \widetilde{z_0} \text{ strongly in } W_0^{2,m}(\Omega) \text{ as } r \to +\infty.$$

For each r we aim to find r functions $a_1^r, a_2^r, a_3^r, ..., a_r^r$ so that

$$z_r(x.t) = \sum_{i=1}^{r} a_i^r(t)\, \omega_i(x) \quad r = 1, 2, ...,$$

solves the given problem

$$\begin{cases} \int_\Omega z_{rt}(x,t)\, \mu dx + \int_\Omega \Delta z_r(x,t)\, \Delta\mu dx - \int_\Omega \int_0^t k(t-s) \left|\Delta z_r(s)\right|^{m-2} \Delta z_r(s)\, \Delta\mu ds dx \\ = \int_\Omega \left|z_r(x,t)\right|^{p(x)-2} z_r(x,t)\, \mu dx, \\ z_r(0) = z_0^r, \end{cases} \tag{9}$$

for all $\mu \in W_0^{2,m}(\Omega)$ and $t \geq 0$.

For $i = 1, ..., r$ substituting $\mu = \omega_i$ into equation (9) results in the following Cauchy problem for a linear ordinary differential equation with unknown a_i^r :

$$\begin{cases} a_i^r(t) = T_i\left(t, a_1^r(t), a_2^r(t), ..., a_r^r(t)\right), \\ a_i^r(0) = \int_\Omega z_0 \omega_i dx, \end{cases} \tag{10}$$

here

$$\begin{aligned} T_i &= \int_\Omega \left| \sum_{i=1}^{r} a_i^r(t)\, \omega_i(x) \right|^{p(x)-2} \sum_{i=1}^{r} a_i^r(t)\, \omega_i(x)\, \omega_i(x)\, dx \\ &\quad - \int_\Omega \sum_{i=1}^{r} a_i^r(t)\, \Delta\omega_i(x)\, \Delta\omega_i(x)\, dx \\ &\quad + \int_\Omega \int_0^t k(t-s) \left| \sum_{i=1}^{r} a_i^r(t)\, \Delta\omega_i(x) \right|^{m-2} \sum_{i=1}^{r} a_i^r(t)\, \Delta\omega_i(x)\, \Delta\omega_i(x)\, dx ds. \end{aligned}$$

According to Peano's Theorem, for every i, the aforementioned Cauchy problem yields a unique local solution $a_i^r \in C^2[0, T]$.

Now, substituting $\mu = \omega_i$ into the second equation in (9) and then multiplying it by $a_i^r(t)$, and summing over s from 1 to r, we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|z_r(t)\right\|_2^2 + \left\|\Delta z_r(t)\right\|_2^2 \\ &- \int_\Omega \int_0^t k(t-s) \left|\Delta z_r(s)\right|^{m-2} \Delta z_r(s)\, \Delta z_r(t)\, dx ds \\ = &\int_\Omega \left|z_r(t)\right|^{p(x)} dx, \end{aligned} \tag{11}$$

with

$$
\begin{aligned}
& -\int_{\Omega}\int_{0}^{t} k(t-s)\left|\Delta z_r(s)\right|^{m-2}\Delta z_r(s)\,\Delta z_r(t)\,dxds \\
\geq\ & -\int_{\Omega}\int_{0}^{t} k(t-s)\left|\left|\Delta z_r(s)\right|^{m-2}\Delta z_r(s)\right|\left|\Delta z_r(t)\right| dxds \\
=\ & \int_{0}^{t} k(t-s)\int_{\Omega}\left[\left|\Delta z_r(t)\right|-\left|\left|\Delta z_r(s)\right|^{m-2}\Delta z_r(s)\right|\right]\left|\Delta z_r(t)\right| dxds \\
& -\int_{0}^{t} k(t-s)\int_{\Omega}\left|\Delta z_r(t)\right|\left|\Delta z_r(t)\right| dxds \\
\geq\ & \int_{0}^{t} k(t-s)\left\|\Delta z_r(t)-\left|\Delta z_r(s)\right|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\
& -\int_{0}^{t} k(t-s)\left\|\Delta z_r(t)\right\|_2^2 ds \\
=\ & \int_{0}^{t} k(t-s)\left\|\Delta z_r(t)-\left|\Delta z_r(s)\right|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\
& -\int_{0}^{t} k(s)\left\|\Delta z_r(t)\right\|_2^2 ds,
\end{aligned}
$$

through a direct calculation, this implies that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left\|z_r(t)\right\|_2^2+\left(1-\int_{0}^{t} k(s)\,ds\right)\left\|\Delta z_r(t)\right\|_2^2 \\
& +\int_{0}^{t} k(t-s)\left\|\Delta z_r(t)-\left|\Delta z_r(s)\right|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\
\leq\ & \int_{\Omega}\left|z_r(t)\right|^{p(x)} dx \leq \max\left\{\left\|z_r(t)\right\|_{p(x)}^{p^-},\left\|z_r(t)\right\|_{p(x)}^{p^+}\right\}. \qquad (12)
\end{aligned}
$$

Thus, we calculation the last term in the right-hand side $\max\left\{\left\|z_r(t)\right\|_{p(x)}^{p^-},\left\|z_r(t)\right\|_{p(x)}^{p^+}\right\}$.

Later, from equation (12), utilizing interpolation and Young's inequality, we derive

$$
\begin{aligned}
\max\left\{\left\|z_r(t)\right\|_{p(x)}^{p^-},\left\|z_r(t)\right\|_{p(x)}^{p^+}\right\} &\leq F_1\max\left\{\left\|z_r(t)\right\|_{m}^{p^-},\left\|z_r(t)\right\|_{m}^{p^+}\right\} \\
&\leq F_1\max\left\{\begin{array}{c} C_G\left\|\Delta z_r(t)\right\|_m^{\varphi p^-}\left\|z_r(t)\right\|_m^{(1-\varphi)p^-}, \\ C_G\left\|\Delta z_r(t)\right\|_m^{\varphi p^+}\left\|z_r(t)\right\|_m^{(1-\varphi)p^+} \end{array}\right\}
\end{aligned}
$$

$$\begin{aligned} &\leq C_G F_1 \max \left\{ \begin{array}{c} \varepsilon \|\Delta z_r(t)\|_2^{2^-} + C(\varepsilon) \|z_r(t)\|_2^{\frac{2(1-\varphi)p^-}{2-\varphi p^-}}, \\ \varepsilon \|\Delta z_r(t)\|_2^{2} + C(\varepsilon) \|z_r(t)\|_2^{\frac{2(1-\varphi)p^+}{2-\varphi p^+}} \end{array} \right\} \\ &= C_G F_1 \varepsilon \|\Delta z_r(t)\|_2^{2^-} \\ &\quad + C_G F_1 C(\varepsilon) \max \left\{ \|z_r(t)\|_2^{2\beta_1}, \|z_r(t)\|_2^{2\beta_2} \right\}, \end{aligned} \tag{13}$$

here $\varphi = \left(\frac{1}{2} - \frac{1}{m}\right)\left(\frac{1}{n} - \frac{1}{2} + \frac{1}{2}\right)^{-1} = \frac{n(m-2)}{2p}$. $\beta_1 = \frac{(1-\varphi)p^-}{2-\varphi p^-} > 1$, $\beta_2 = \frac{(1-\varphi)p^+}{2-\varphi p^+} > 1$. Where we select $p^+ \leq m < \frac{2n}{n-4}$.

By combining equations (12) and (13), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|z_r(t)\|_2^2 + \left(1 - \int_0^t k(s)\,ds\right)\|\Delta z_r(t)\|_2^2 \\ &+ \int_0^t k(t-s)\left\|\Delta z_r(t) - |\Delta z_r(s)|^{m-2}.\Delta z_r(s)\right\|_2^2 ds \\ \leq\ & C_G F_1 \varepsilon \|\Delta z_r(t)\|_2^2 + C_G F_1 C(\varepsilon) \max\left\{\|z_r(t)\|_2^{2\beta_1}, \|z_r(t)\|_2^{2\beta_2}\right\}. \end{aligned} \tag{14}$$

Due to the integral inequality of Gronwall-Bellman-Bihari type, there exists a positive constant T (independent of r) so that

$$\|z_r(t)\|_2^2 \leq C_T, \tag{15}$$

here

$$C_T = \max\left\{ \begin{array}{c} \left[2(1-\beta_2) C_G F_1 C(\varepsilon) T + \|z_{0m}\|_2^{2(1-\beta_1)}\right]^{\frac{1}{1-\beta_1}}, \\ \left[2(1-\beta_2) C_G F_1 C(\varepsilon) T + \|z_{0m}\|_2^{2(1-\beta_2)}\right]^{\frac{1}{1-\beta_2}} \end{array} \right\}.$$

Now, substituting $\mu = \omega_i$ into the second equation in (9) and multiplying it by $\left[a_i^r(t)\right]'$, then summing over s from 1 to m, we get

$$\begin{aligned} &\|z_{rt}(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\Delta z_r(t)\|_2^2 \\ &+ \frac{d}{dt}\left[\frac{1}{2}\int_0^t k(t-s)\left\|\Delta z_r(t) - |\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 ds\right] \\ &- \frac{d}{dt}\left[\frac{1}{2}\int_0^t k(s)\|\Delta z_r(t)\|_2^2 ds\right] \\ &- \frac{1}{2}\int_0^t k'(t-s)\left\|\Delta z_r(t) - |\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\ &+ \frac{1}{2}k(t)\|\Delta z_r(t)\|_2^2 \\ =\ & \frac{d}{dt}\int_\Omega \frac{1}{p(x)}|z_r(t)|^{p(x)} dx \\ \leq\ & \frac{d}{dt}\left(\frac{1}{p^-}\max\left\{\|z_r(t)\|_{p(x)}^{p^-}, \|z_r(t)\|_{p(x)}^{p^+}\right\}\right). \end{aligned} \tag{16}$$

By integrating equation (16) over the interval $[0,t]$, we derive

$$\begin{aligned}
&\int_0^t \|z_{rt}(t)\|_2^2 dt + \left(\frac{1}{2} - \frac{1}{2}\int_0^t k(s)\,ds\right)\|\Delta z_r(t)\|_2^2 \\
&+\frac{1}{2}\int_0^t k(t-s)\left\|\Delta z_r(t) - |\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\
&+\frac{1}{2}\int_0^t k(t)\|\Delta z_r(t)\|_2^2 dt \\
\leq\ & \frac{1}{p^-}\max\left\{\|z_r(t)\|_{p(x)}^{p^-}, \|z_r(t)\|_{p(x)}^{p^+}\right\} \\
&+\left(\frac{1}{2} - \frac{1}{2}\int_0^t k(s)\,ds\right)\|\Delta z_{0r}(t)\|_2^2 - \frac{1}{p^-}\max\left\{\|z_{0r}(t)\|_{p(x)}^{p^-}, \|z_{0r}(t)\|_{p(x)}^{p^+}\right\}. \qquad (17)
\end{aligned}$$

By combining equations (13) and (17), we obtain

$$\begin{aligned}
&\int_0^t \|z_{rt}(t)\|_2^2 dt + \left(\frac{1}{2} - \frac{1}{2}\int_0^t k(s)\,ds - \frac{1}{p^-}C_G F_1 \varepsilon\right)\|\Delta z_r(t)\|_2^2 \\
&+\frac{1}{2}\int_0^t k(t-s)\left\|\Delta z_r(t) - |\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\
&+\frac{1}{2}\int_0^t k(t)\|\Delta z_r(t)\|_2^2 dt \\
\leq\ & \frac{1}{p^-}\max\left\{\|z_{0r}(t)\|_{p(x)}^{p^-}, \|z_{0r}(t)\|_{p(x)}^{p^+}\right\} = c_T. \qquad (18)
\end{aligned}$$

From equation (18), Hölder's inequality, and Young's inequality, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^t k(t-s)\left\|\Delta z_r(t) - |\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\
\geq\ & \frac{1}{2}\int_0^t k(t-s)\left|\Delta z_r(t) - |\Delta z_r(s)|^{m-2}\Delta z_r(s)\right|^2 dxds \\
\geq\ & \frac{1}{2}\int_0^t k(t-s)\left[\begin{array}{c} \left(1-\frac{1}{\varepsilon}\right)\int_\Omega |\Delta z_r(t)|^2 dx \\ +(1-\varepsilon)\int_\Omega \left||\Delta z_r(s)|^{m-2}\Delta z_r(s)\right|^2 dx \end{array}\right] ds \\
\geq\ & \frac{1}{2}\int_0^t k(t-s)\left\||\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\
&-\frac{1}{2}\left(\frac{1}{\varepsilon}-1\right)\int_0^t k(s)\,ds\,\|\Delta z_r(t)\|_2^2,
\end{aligned}$$

here $0 < \varepsilon < 1$. From aforementioned inequality and equation (18), we conclude that

$$\int_0^t k(t-s)\left\|\Delta z_r(t) - |\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 ds \leq c_T.$$

Utilizing Hölder inequality, Young inequality and above inequality, we get

$$\begin{aligned}
&\int_0^t \|\Delta z_r(s)\|_m^m ds \\
\leq\ & \int_0^t \left[\int_\Omega \left(|\Delta z_r(s)|^m\right)^{\frac{2m-2}{m}} dx\right]^{\frac{m}{2m-2}} \left(\int_\Omega 1dx\right)^{\frac{m-2}{2m-2}} ds \\
\leq\ & \int_0^t \frac{m}{2m-2} k(t-s)\left\||\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 \\
&+\frac{m-2}{2m-2}[k(t-s)]^{-\frac{m}{m-2}}|\Omega|\, ds \\
\leq\ & \frac{m}{2m-2}\int_0^t k(t-s)\left\||\Delta z_r(s)|^{m-2}\Delta z_r(s)\right\|_2^2 ds \\
&+\frac{m}{2m-2}|\Omega|\int_0^t [k(s)]^{-\frac{m}{m-2}} ds \\
\leq\ & \frac{m}{2m-2}c_T + \frac{m-2}{2m-2}|\Omega|[k(T)]^{-\frac{m}{m-2}}T. \qquad (19)
\end{aligned}$$

It follows from equations (18) and (19) that

$$\begin{aligned}
z_r &\rightarrow z \text{ weakly star in } L^\infty\left([0,T];H_0^2(\Omega)\right) \\
z_{rt} &\rightarrow z_t \text{ weakly in } L^2\left([0,T];L^2(\Omega)\right) \\
z_r &\rightarrow z \text{ weakly in } L^p\left([0,T];W_0^{2,r}(\Omega)\right) \\
|\Delta z_r(s)|^{m-2}\Delta z_r(s) &\rightarrow \chi \text{ weakly in } L^2\left([0,T];L^2(\Omega)\right). \qquad (20)
\end{aligned}$$

Subsequently, using the method of Browder and Minty in the theory of monotone operators, we derive $\chi = |\Delta z|^{m-2}\Delta z$. By equation (20) and Aubin-Lions-Simon lemma, we get

$$z_r \rightarrow z \text{ weakly star in } C\left([0,T];L^2(\Omega)\right). \qquad (21)$$

Thus, $z_r(x,0)$ makes sense and $z_r(x,0) \rightarrow z(x,0) = z_0$ in $L^2(\Omega)$, that means $z_0 = \widetilde{z_0}$.

As $r \rightarrow +\infty$ taking the limit in equation (9), using equations (20) and (21), we can demonstrate that $z(t)$ satisfies the initial condition with $z(0) = z_0$ and

$$\begin{aligned}
&\int_\Omega z_t(x,t)\mu dx + \int_\Omega \Delta z(x,t)\Delta\mu dx \\
&+\int_\Omega\int_0^t k(t-s)|\Delta z_r(s)|^{m-2}\Delta z_r(s)\Delta\mu ds dx
\end{aligned}$$

$$= \int_\Omega |z(x,t)|^{p(x)-2} z(x,t)\,\mu dx,$$

for every $\mu \in W_0^{2,m}(\Omega)$ and for almost every $t \in [0,T]$. This concludes the proof of the theorem. □

3. Blow up

In this part, we will establish the occurrence of finite time blow up in solutions.

Lemma 3.1. *[11, 12] (Concavity methods) Assuming $\beta > 0$, suppose $\psi(t) \geq 0$ is weakly twice-differentiable on $(0,\infty)$ with $\psi(0) > 0$, $\psi'(0) > 0$ and*

$$\psi''(t)\psi(t) - (1+\beta)(\psi'(t))^2 \geq 0,$$

for every $t \in (0,\infty)$. Thus there exists a $T > 0$ so that

$$\lim_{t \to T^-} \psi(t) = \infty,$$

and

$$T \leq \frac{\psi(0)}{\beta\psi'(0)}.$$

Theorem 3.2. *Suppose that $z(x,t)$ be a weak solution of problem (1) in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$. Later for every $2k < p^- < p^+ < m < \frac{2n}{n-4}$ and $J(z_0) > 0$, $z_0 \in W_0^{2,m}(\Omega)$ the solution $z(x,t)$ blows up in finite time assuming that*

$$\|z_0\|_2^2 \geq \max\left\{C_1 J^{\frac{2}{p^-}}(z_0), C_2 J^{\frac{2}{p^-}}(z_0)\right\},$$

here

$$C_1 = \left(\frac{2kp^-}{p^- - 2k}\right)^{\frac{2}{p^-}} \left[\frac{p^- - 2}{(p^+ - 2)|\Omega|}\right]^{\frac{2-p^-}{p^-}}, C_2 = \left(\frac{2kp^+}{p^+ - 2k}\right)^{\frac{2}{p^+}} \left[\frac{p^-}{p^+|\Omega|}\right]^{\frac{2-p^+}{p^+}}.$$

Proof. To establish the finite time blow up of the solution under the theorem condition, we assume the contrary, that the solution $z(x,t)$ is global. By multiplying the first equation of problem (1) by z and integrating over Ω, we derive

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega z^2 dx &= -\int_\Omega |\Delta z|^2 dx \\ &+ \int_\Omega \Delta z(t) \int_0^t k(t-s)|\Delta z(s)|^{m-2} \Delta z(s)\,ds dx \\ &+ \int_\Omega |z|^{p(x)} dx. \end{aligned} \tag{22}$$

In the subsequent discussion, we will address two cases:

Case 1: $E(t) \geq 0$, for every $t > 0$. From equation (5), this implies $J(z_0) \geq J(t) \geq 0$. By combining the first equation of problems (1) and (4), applying the Schwartz inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega z^2 dx = \int zz_t dx$$

$$
\begin{aligned}
&\geq \int_{\Omega} |z|^{p(x)}\,dx - \int_{\Omega} |\Delta z|^2\,dx \\
&\quad - \int_0^t k(t-s) \int_{\Omega} \left| \Delta z(t) - |\Delta z(t)|^{m-2} \Delta z(s) \right|^2 dxds \\
&\quad + \frac{3}{4} \int_0^t k(t-s) \int_{\Omega} |\Delta z(t)|^2\,dxds \\
&\geq -\frac{1-\frac{3}{4}\int_0^t k(s)\,ds}{1-\int_0^t k(s)\,ds} 2J(t) + \left(1 - \frac{1-\frac{3}{4}\int_0^t k(s)\,ds}{1-\int_0^t k(s)\,ds} \frac{2}{p^-} \right) \int_{\Omega} |z|^{p(x)}\,dx \\
&\geq 2(\beta-1) gJ(t) - 2\beta gJ(t) + \left(1 - \frac{2g}{p^-}\right) \int_{\Omega} |z|^{p(x)}\,dx. \qquad (23)
\end{aligned}
$$

We select β complying with

$$
1 < \beta < \frac{p^- - 2k}{2gJ(0)} C_3, \qquad (24)
$$

here

$$
C_3 = \min\left\{ \frac{1}{p^-} \left[\frac{p^- - 2}{(p^+ - 2)|\Omega|} \right]^{\frac{p^- - 2}{p^-}} \|z_0\|_2^{p^-}, \frac{1}{p^+} \left[\frac{p^- - 2}{p^+ |\Omega|} \right]^{\frac{p^+ - 2}{p^+}} \|z_0\|_2^{p^+} \right\}. \qquad (25)
$$

By combining the fact that $J(t) \geq 0$ with the formula (23), it follows that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_{\Omega} z^2 dx &\geq -2\beta gJ(0) + 2g\beta \int_0^t \|z_\tau(\tau)\|_2^2\,d\tau \\
&\quad + \left(1 - \frac{2g}{p^-}\right) \int_{\Omega} |z|^{p(x)}\,dx. \qquad (26)
\end{aligned}
$$

Utilizing Hölder's inequality and Young's inequality results in

$$
\int_{\Omega} |z|^{p(x)}\,dx \geq \int_{\Omega} \frac{\eta p(x)}{2} z^2 dx - \int_{\Omega} \frac{p(x)-2}{2} \eta^{\frac{p(x)}{p(x)-2}}\,dx, \qquad (27)
$$

where η is a positive constant to be determined in (38). Substituting equation (27) into equation (26), we obtain

$$
\begin{aligned}
\frac{d}{dt} \int_{\Omega} z^2 dx &\geq 4g\beta \int_0^t \|z_\tau\|_2^2\,d\tau - 4g\beta J(0) + (p^- - 2g)\,\eta\,\|z\|_2^2 \\
&\quad - \frac{(p^- - 2g)(p^+ - 2)|\Omega|}{p^-} \max\left\{ \eta^{\frac{p^-}{p^- - 2}}, \eta^{\frac{p^+}{p^+ - 2}} \right\}. \qquad (28)
\end{aligned}
$$

It is clear

$$4g\beta\int_0^t \|z_\tau(\tau)\|_2^2\,d\tau \geq 0.$$

By solving nonhomogeneous ordinary differential equation, we can acquire

$$\begin{aligned}\int_\Omega z^2dx &\geq \|z_0\|_2^2\,e^{(p^- -2g)\eta t}\\ &\quad+\left[4g\beta J(0)+\frac{(p^- -2g)(p^+ -2)|\Omega|}{p^-}\max\left\{\eta^{\frac{p^-}{p^- -2}},\eta^{\frac{p^+}{p^+ -2}}\right\}\right]\\ &\quad\times\frac{\left(1-e^{\eta(p^- -2g)t}\right)}{\eta(p^- -2g)}.\end{aligned} \tag{29}$$

Let $L(t)=\int_0^t \|z_\tau(\tau)\|_2^2\,d\tau$. Given our assumption that the solution $z(x,t)$ is global, later $L(t)$ is a bounded function for every $t>0$.

$$L'(t)=\int_\Omega z^2dx,\quad L''(t)=\frac{d}{dt}\int_\Omega z^2dx.$$

By substituting equation (29) into equation (28), we conclude that

$$\begin{aligned}L''(t) &\geq 4g\beta\int_0^t \|z_\tau(\tau)\|_2^2\,d\tau\\ &\quad+e^{(p^- -2g)\eta t}\left[\begin{array}{c}(p^- -2g)\,\eta\,\|z_0\|_2^2-4g\beta J(0)\\ -\frac{(p^- -2g)(p^+ -2)|\Omega|}{p^-}\max\left\{\eta^{\frac{p^-}{p^- -2}},\eta^{\frac{p^+}{p^+ -2}}\right\}\end{array}\right].\end{aligned} \tag{30}$$

We can select ε to be sufficiently small so that

$$0<\varepsilon<\frac{(p^- -2g)\,C_3-2g\beta J(0)}{g\beta\,\|z_0\|_2^2}, \tag{31}$$

here C_3 is defined in equation (25).

We establish the auxiliary function ϕ as follows:

$$\phi(t)=L^2(t)+\varepsilon^{-1}\|z_0\|_2^2\,L(t)+C,$$

here C is large enough to satisfy $C>\frac{1}{4}\varepsilon^{-2}\|z_0\|_2^4$. Assume that $\delta=4C-\varepsilon^{-2}\|z_0\|_2^4$, so $\delta>0$. Thus

$$\phi'(t) = 2L(t)L'(t)+\varepsilon^{-1}\|z_0\|_2^2\,L'(t)=\left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)L'(t), \tag{32}$$

$$\phi''(t) = \left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)L''(t)+2\left[L'(t)\right]^2. \tag{33}$$

Utilizing identify (33), we can derive

$$\begin{aligned}\left[\phi'(t)\right]^2 &= \left[2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right]^2\left[L'(t)\right]^2\\ &= \left[4\phi(t)-\left(4C-\varepsilon^{-2}\|z_0\|_2^4\right)\right]\left[L'(t)\right]^2\\ &= \left(4\phi(t)-\delta\right)\left(L'(t)\right)^2,\end{aligned} \tag{34}$$

with

$$\begin{aligned} 2\phi''(t)\phi(t) &= 2\left[\left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)L''(t)+2\left(L'(t)\right)^2\right]\phi(t) \\ &= 2\left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)L''(t)\phi(t)+4\left[L'(t)\right]^2\phi(t). \end{aligned} \tag{35}$$

Observing that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega z^2 dx = \int_\Omega z(t) z_t(t)\, dx,$$

by integrating the aforementioned equality from 0 to t, we obtain

$$\|z(t)\|_2^2 - \|z_0\|_2^2 = 2\int_0^t \int_\Omega z(\tau) z_\tau(\tau)\, dx d\tau,$$

this implies

$$\left[L'(t)\right]^2 = \left[z(t)\right]_2^4 = \left(\|z_0\|_2^2 + 2\int_0^t \int_\Omega z(\tau) z_\tau(\tau)\, dx d\tau\right)^2.$$

By utilizing Hölder's and Young's inequalities, we can derive that

$$\begin{aligned} \left[L'(t)\right]^2 &= \left\{\|z_0\|_2^2 + 2\left[\int_0^t \left(\int_\Omega z^2(\tau)\, dx\right)^{\frac{1}{2}}\left(\int_\Omega \left(z_\tau(\tau)\right)^2 dx\right)^{\frac{1}{2}} d\tau\right]\right\}^2 \\ &\le \|z_0\|_2^4 + 4L(t)\int_0^t \|z_\tau(\tau)\|_2^2\, d\tau + 2\varepsilon\|z_0\|_2^2 L(t) \\ &\quad + 2\varepsilon^{-1}\|z_0\|_2^2 \int_0^t \|z_\tau(\tau)\|_2^2\, d\tau. \end{aligned} \tag{36}$$

From (34) and (35), we have

$$\begin{aligned} & 2\phi''(t)\phi(t) - (1+\alpha)\left[\phi'(t)\right]^2 \\ = \; & 2\left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)L''(t)\phi(t) - 4\alpha\phi(t)\left[L'(t)\right]^2 + \delta(1+\alpha)\left[L'(t)\right]^2. \end{aligned}$$

From (30) and (36), we derive

$$\begin{aligned} & 2\phi''(t)\phi(t) - (1+\alpha)\left[\phi'(t)\right]^2 \\ = \; & 2\left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)L''(t)\phi(t) - 4\alpha\phi(t)\left[L'(t)\right]^2 + \delta(1+\alpha)\left[L'(t)\right]^2 \\ \ge \; & 2\left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)L''(t)\phi(t) - 4\alpha\phi(t)\left[L'(t)\right]^2 \\ \ge \; & 2\phi(t)\left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)\left\{4g\beta\int_0^t \|z_\tau\|_2^2\, d\tau + e^{\eta\left(p^- - 2g\right)t}\right\}B(\eta) \end{aligned}$$

$$-4\alpha\phi(t)\left[\begin{array}{c}\|z_0\|_2^4+4L(t)\int\limits_0^t\|z_\tau\|_2^2\,d\tau+2\varepsilon\|z_0\|_2^2L(t)\\+2\varepsilon^{-1}\|z_0\|_2^2\int\limits_0^t\|z_\tau\|_2^2\,d\tau\end{array}\right], \tag{37}$$

here

$$\begin{aligned}B(\eta) &= (p^- - 2g)\,\eta\,\|z_0\|_2^2-4g\beta J(0)\\&-\frac{(p^- - 2g)(p^+ - 2)|\Omega|}{p^-}\max\left\{\eta^{\frac{p^-}{p^- - 2}},\eta^{\frac{p^+}{p^+ - 2}}\right\}.\end{aligned}$$

Thus

$$B'(\eta)=(p^- - 2g)\|z_0\|_2^2-\frac{(p^- - 2g)(p^+ - 2)|\Omega|}{p^-}\max\left\{\frac{p^-}{p^- - 2}\eta^{\frac{p^-}{p^- - 2}},\frac{p^+}{p^+ - 2}\eta^{\frac{p^+}{p^+ - 2}}\right\}.$$

Utilizing solving the equation $B'(\eta)=0$, we may find the maximum point

$$\eta_{\max 1}=\left[\frac{p^- - 2}{(p^+ - 2)|\Omega|}\right]^{\frac{p^- - 2}{2}}\|z_0\|_2^{p^- - 2}\ \text{or}\ \eta_{\max 2}=\left[\frac{p^-}{p^+|\Omega|}\right]^{\frac{p^+ - 2}{2}}\|z_0\|_2^{p^+ - 2}.$$

It is evident that

$$B(\eta_{\max 1})=\frac{2(p^- - 2g)}{p^-}\left[\frac{p^- - 2}{(p^+ - 2)|\Omega|}\right]^{\frac{p^- - 2}{2}}\|z_0\|_2^{p^-}-4g\beta J(0),$$

with

$$B(\eta_{\max 2})=\frac{2(p^- - 2g)}{p^+}\left[\frac{p^-}{p^+|\Omega|}\right]^{\frac{p^+ - 2}{2}}\|z_0\|_2^{p^+}-4g\beta J(0).$$

By considering the value of β in (24), we get $\min\{B(\eta_{\max 1}),B(\eta_{\max 2})\}>0$. We select η in (27) satisfying

$$B(\eta)=\min\{B(\eta_{\max 1}),B(\eta_{\max 2})\}. \tag{38}$$

Taking into consideration the value ε, we have

$$B(\eta)\geq 2\varepsilon g\beta\|z_0\|_2^2.$$

Given the facts that $\phi>0$ and $e^{\eta(p^- - 2g)t}>1$, we can obtain using (37)

$$\begin{aligned}&2\phi''(t)\phi(t)-(1+\alpha)\left[\phi'(t)\right]^2\\\geq\ &2\phi(t)\left(2L(t)+\varepsilon^{-1}\|z_0\|_2^2\right)\left\{4g\beta\int\limits_0^t\|z_\tau\|_2^2\,d\tau+2\varepsilon g\beta\|z_0\|_2^2\right\}\\&-4\alpha\phi(t)\left[\begin{array}{c}\|z_0\|_2^4+4L(t)\int\limits_0^t\|z_\tau(\tau)\|_2^2\,d\tau\\2\varepsilon\|z_0\|_2^2L(t)+2\varepsilon^{-1}\|z_0\|_2^2\int\limits_0^t\|z_\tau(\tau)\|_2^2\,d\tau\end{array}\right]\\\geq\ &\phi(t)\left[\begin{array}{c}16gL(t)\int\limits_0^t\|z_\tau\|_2^2\,d\tau+4g\beta\|z_0\|_2^4+8g\beta L(t)\|z_0\|_2^2\\+8g\beta\varepsilon^{-1}\|z_0\|_2^2\int\limits_0^t\|z_\tau\|_2^2\,d\tau\end{array}\right]\end{aligned}$$

$$
\begin{aligned}
&-\phi(t)\left[\begin{array}{c} 4\alpha\|z_0\|_2^4 + 16\alpha L(t)\int\limits_0^t \|z_\tau(\tau)\|_2^2\,d\tau + 8\varepsilon\alpha\|z_0\|_2^2 L(t) \\ +8\varepsilon^{-1}\alpha\|z_0\|_2^2\int\limits_0^t \|z_\tau(\tau)\|_2^2\,d\tau \end{array}\right] \\
&= 0,
\end{aligned} \tag{39}
$$

here $\alpha = g\beta$. Therefore,

$$
\phi''(t)\phi(t) - \left(1 + \frac{g\beta-1}{2}\right)\left(\phi'(t)\right)^2 \geq 0,
$$

given that $\phi(t) > 0$ and $\phi'(t) > 0$, we apply Lemma 3 to deduce that $\phi(t) \to +\infty$ as $t \to t^* \leq \frac{2\phi(0)}{(g\beta-1)\phi'(0)}$. Due to the continuity of ϕ with respect to L, it follows that $L(t)$ tends to infinity at some finite time, which is a opposition.

Case 2: We assume that there exists $t_0 > 0$ so that $J(z(t_0)) < 0$. We determine $v(x,t) = z(x, t+t_0)$, so $J(v(0)) = J(z(t_0)) < 0$. Given that $J(t)$ is decreasing in t, we have

$$
J(v(t)) \leq J(v(0)) \leq 0. \tag{40}
$$

Let $H(t) = \int\limits_\Omega v^2(x,t)\,dx$, later we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int\limits_\Omega v^2(x,t)\,dx &= \int\limits_\Omega v\left[-\Delta^2 v + \int\limits_0^t k(t-z)\Delta_m^2 v(x,z)\,dz + |v|^{p(x)-2}v\right]dx \\
&\geq -\frac{1-\frac{3}{4}\int\limits_0^t k(t-z)\,dz}{1-\int\limits_0^t k(z)\,dz}2J(v(t)) \\
&\quad + \left(\int\limits_\Omega |v|^{p(x)}\,dx - \frac{1-\frac{3}{4}\int\limits_0^t k(t-z)\,dz}{1-\int\limits_0^t k(z)\,dz}\int\limits_\Omega \frac{2|v|^{p(x)}}{p^-}dx\right) \\
&\geq \left(1-\frac{2g}{p^-}\right)\int\limits_\Omega |v|^{p(x)}\,dx.
\end{aligned} \tag{41}
$$

From (41), accordingly

$$
H'(t) \geq \frac{2(p^- - 2g)}{p^-}\int\limits_\Omega |v|^{p(x)}\,dx. \tag{42}
$$

Using the embedding theorem $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega)$, we deduce that

$$
\|v\|_2 \leq C_4\|v\|_{p(x)}. \tag{43}
$$

Referring to the definition in [8], we can obtain that

$$
\min\left\{\|v\|_{p(\cdot),\Omega}^{p^-}, \|v\|_{p(\cdot),\Omega}^{p^+}\right\} \leq \int\limits_\Omega |v|^{p(x)}\,dx \leq \max\left\{\|v\|_{p(\cdot),\Omega}^{p^-}, \|v\|_{p(\cdot),\Omega}^{p^+}\right\}. \tag{44}
$$

By inserting equations (43) and (44) into (42), we obtain

$$\begin{aligned} H'(t) &\geq \frac{2(p^- - 2g)}{p^-} \min\left\{\left(\frac{1}{C_4}\right)^{p^-} \|v\|_2^{p^-}, \left(\frac{1}{C_4}\right)^{p^+} \|v\|_2^{p+}\right\} \\ &\geq C_5 \min\left\{H^{\frac{p^-}{2}}(t), H^{\frac{p^+}{2}}(t)\right\}, \end{aligned} \tag{45}$$

here C_4 is a best embedding constant and $C_5 = \frac{2(p^- - 2g)}{p^-} \min\left\{\left(\frac{1}{C_4}\right)^{p^-}, \left(\frac{1}{C_4}\right)^{p^+}\right\}$.

Using $H'(t) > 0$, so that $H(t) \geq H(0)$. This allows us to conclude that

$$\left[\frac{H(t)}{H(0)}\right]^{\frac{p^+}{2}} \geq \left[\frac{H(t)}{H(0)}\right]^{\frac{p^-}{2}},$$

so that

$$[H(t)]^{\frac{p^+}{2}} \geq H(0)^{\frac{p^+ - p^-}{2}} [H(t)]^{\frac{p^-}{2}}. \tag{46}$$

Utilizing (45) and (46), we get

$$H'(t) \geq C_5 \min\left\{H^{\frac{p^-}{2}}(t), H(0)^{\frac{p^+ - p^-}{2}} [H(t)]^{\frac{p^-}{2}}\right\} \geq C_6 H^{\frac{p^-}{2}}(t), \tag{47}$$

here $C_6 = C_5 \min\left\{1, H(0)^{\frac{p^+ - p^-}{2}}\right\}$. Later, inequality (47) and Gronwall's inequality lead to

$$H^{\frac{p^- - 2}{2}}(t) \geq \frac{1}{H^{\frac{2-p^-}{2}}(0) - \frac{p^- - 2}{2} C_6 t}.$$

The inequality above indicates that $H(t)$ blow up at finite time $T^* \leq \frac{2H^{\frac{2-p^-}{2}}(0)}{(p^- - 2)C_6}$.

Therefore, by examining the aforementioned two cases, we can derive that $z(x,t)$ blows up in finite time. □

References

[1] S. N. Antontsev, J. Ferreira, E. Pişkin, Existence and blow up of solutions for a strongly damped Petrovsky equation with variable-exponent nonlinearities, EJDE, 2021(2021) 1-18.

[2] S. N. Antontsev, J. Ferreira, E. Pişkin, S. M. S. Cordeiro, Existence and non-existence of solutions for Timoshenko-type equations with variable exponents, Nonlinear Anal. Real World Appl., 61 (2021) 103341, 1-13.

[3] I. Ben Omrane, M. Ben Slimane, S. Gala, M.A. Ragusa, A new regularity criterion for the 3D nematic liquid crystal flows, Anal. Appl., 23(02) (2025) 287-306.

[4] G. Butakın, E. Pişkin, Existence and Blow up of Solutions for $m(x)$-Biharmonic equation with Variable Exponent Sources, Filomat, 38(22) (2024), 7871-7893.

[5] G. Butakın, E. Pişkin, Existence and Blow up of Solutions of a Viscoelastic $m(x)$-Biharmonic Equation with Logarithmic Source Term, Miskolc Math. Notes, 25(2) (2024), 629-643 .

[6] G. Butakın, E. Pişkin, Blow up of solutions for a fourth-order reaction-diffusion equation in variable-exponent Sobolev spaces, Filomat, 38(23) (2024), 8225–8242.

[7] N. Chem Eddine, M. A. Ragusa, D. D. Repovš, On the concentration-compactness principle for anisotropic variable exponent Sobolev spaces and its applications, Fract. Calc. Appl. Anal., 27 (2024) 725-756.

[8] L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, vol. 2017. Springer, Heidelberg (2011).

[9] J. Ferreira, W. S. Panni, S. A. Messaoudi, E. Pişkin, M. Shahrouzi, Existence and Asymptotic Behavior of Beam-Equation Solutions with Strong Damping and $p(x)$-Biharmonic Operator, J. Math. Phys. Anal. Geo., 18(4) (2022) 488-513.

[10] Y. Gao, W. Gao, Existence of weak solutions for viscoelastic hyperbolic equations with variable exponents, J. Bound. Value Probl., 2013(1), 1–8 (2013).

[11] V. Kalantarov, O. A. Ladyzhenskaya, The occurence of collapse for quasilinear equation of parabolic and hyperbolic types, J. Sov. Math., 10 (1978), 53–70.

[12] H. A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + \mathcal{F}(u)$, Arch. Ration. Mech. Anal., 51 (1973) 371-386.
[13] S. A. Messaodi, Blow up of solutions of a semilinear heat equation with a viscoelastic term, Prog. Nonlinear Differ. Equ. Appl. 2005(64), 351–356 (2005).
[14] S. A. Messaoudi, Blow-up of solutions of a semilinear heat equation with a memory term, Abstr. Appl. Anal. 2005(2), (2005) 87-94.
[15] Q.T. Ou, On the partial boundary value condition basing on the diffusion coefficient, Filomat, 37 (18) , 5979-5992, (2023).
[16] E. Pişkin, Finite time blow up of solutions for a strongly damped nonlinear Klein-Gordon equation with variable exponents, Honam Mathematical J., 40(4) (2018) 771-783.
[17] E. Pişkin, G. Butakın, Blow-up phenomena for a $p(x)$-biharmonic heat equation with variable exponent, Math. Morav., 27(2) (2023) 25-32.
[18] E. Pişkin, G. Butakın, Existence and Decay of solutions for a parabolic-type Kirchhoff equation with variable exponents, J. Math. Sci. Model., 6 (1) (2023) 32-41.
[19] E. Pişkin, B. Okutmuştur, An Introduction to Sobolev Spaces, Bentham Science, 2021.
[20] M. A. Ragusa A.Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, Adv. Nonlinear Anal. 9 (2020) 710-728.
[21] M. A. Ragusa A.Tachikawa, Boundary regularity of minimizers of $p(x)$-energy functionals, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 33(2) (2016) 451-476.
[22] A. Rahmoune, Lower and upper bounds for the blow-up time to a viscoelastic Petrovsky wave equation with variable sources and memory term, J. Appl. Anal., 2022(1), 1-29.
[23] M. Shahrouzi, J. Ferreira, E. Pişkin, K. Zennir, On the behavior of solutions for a class of nonlinear viscoelastic fourth-order $p(x)$-Laplacian equation, Mediterr. J. Math., 20 (2023) 1-28.
[24] M. Shahrouzi, J. Ferreira, E. Pişkin, General decay and blow up of solutions for a plate viscoelastic $p(x)$-Kirchhoff type equation with variable exponent nonlinearities and boundary feedback, Quaest. Math., (2023), 1-18.
[25] S. Tian, Bounds for blow-up time in a semilinear parabolic problem with viscoelastic term, Comput. Math. Appl. 74(4), (2017) 736-743.
[26] X. Wu, X. Yang, Y. Zhao, The Blow-Up of Solutions for a Class of Semi-linear Equations with p-Laplacian Viscoelastic Term Under Positive Initial Energy, Mediterr. J. Math., 20 (2023) 272.