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Three-branching transmission irregular graphs

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Abstract. The transmission of a vertex v in a graph G is the sum of distances from v to other vertices in G. If any two vertices of G have different transmissions, then G is transmission irregular. A vertex in a graph is a branching vertex if its degree is at least 3. A graph G is three-branching if G contains exactly three branching vertices. It is shown in this paper that, for any natural number $n \geq 11$ with $n \notin \{12, 14\}$, there exists a three-branching transmission irregular graph of order n. In particular, there exists a three-branching transmission irregular tree for each odd n or $n \equiv 4 \pmod 6$.

1. Introduction

Throughout this paper we only consider the undirected, finite, connected and simple graphs unless stated otherwise. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is just $n(G) = |V(G)|$. For any vertex $v \in V(G)$, the degree $d_G(v)$ of v is the number of vertices adjacent to it in G. For a graph G with $x, y \in V(G)$, we denote by $d_G(x, y)$ (or $d(x, y)$ if G is clear from the context) the shortest-path distance between x and y in G. As usual, we denote by $P_n = v_1v_2 \cdots v_{n-1}v_n$ a path of order n with the natural adjacency relation, that is, $v_iv_{i+1} \in E(P_n)$ for each $i \in \{1, 2, \ldots, n-1\}$. Moreover, $C_n = v_1v_2 \ldots v_nv_1$ is a cycle of order $n \geq 3$. Other undefined notations and terminology on graph theory can be found in [8].

As a fundamental parameter of a graph in pure graph theory, the distance (between vertices) plays an important role in chemical graph theory. As an oldest and most well-known distance-based graph invariant (also known as topological index in chemical graph theory), the Wiener index $W(G)$ of a graph G introduced in 1947 [17] is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)$$

for exploring chemical graphs, which in turn reflect the physico-chemical properties of the corresponding (organic) compounds. The area is still very active; for a survey on graphs extremal with respect to distance-based topological indices see [20], and for a selection of recent developments with a focus on applications see [12].

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As a basic concept in metric graph theory, the *transmission* $\mathrm{Tr}_G(v)$ of vertex v in a connected graph G is the sum of all distances from v to other vertices in G, that is,

$$\mathrm{Tr}_G(v) = \sum_{u \in V(G) \setminus \{v\}} d_G(v, u) .$$

Moreover, the transmission of a vertex is also known by several other names, such as the status of a vertex [1, 16] and the total distance of a vertex [9, 15]. The *transmission set* of a graph G is $\mathrm{Tr}(G) = \{\mathrm{Tr}_G(v) | v \in V(G)\}$. If $|\mathrm{Tr}(G)| = n(G)$ holds, then G is *transmission irregular*, or TI for short. Since the Wiener complexity [2] of a graph G is the number of different transmissions of its vertices, see also [18], TI graphs are the graphs with maximum Wiener complexity.

It was shown [2] that almost all graphs are not TI. Thus TI graphs are rare, and the characterization of TI graphs in various classes of graphs is an interesting and challenging task. Many nice results are reported on this topic. Al-Yakoob and Stevanović [4] provided a complete characterization of TI starlike trees with maximum degree 3. Recently Damnjanović [10] extended this result to all the starlike trees and double starlike trees and gave the complete, but complicated, equivalent conditions of these two classes of TI graphs, respectively. Dobrynin [13] constructed an infinite family of TI trees of even order. Other excellent relevant results on this topic can be found in [5, 11, 14, 19, 21] and the references therein.

A vertex v in a tree T is a *branching vertex* in T if $d_T(v) \geq 3$. In general, a vertex of degree at least 3 in a graph G, which is not necessarily a tree, is also called a branching vertex in G. A graph G is *three-branching* if G contains exactly three branching vertices. We denote by $\mathcal{GB}_n^3$ the set of three-branching graphs of order n. In particular, $\mathcal{TB}_n^3$ is the set of three-branching trees of order n. Clearly, $\mathcal{TB}_n^3 \subseteq \mathcal{GB}_n^3$. For a tree $T \in \mathcal{TB}_n^3$ with three branching vertices v_1, v_2 and v_3, if $d(v_1, v_3) = d(v_1, v_2) + d(v_2, v_3)$, then v_2 is the *central branching vertex* of T. Clearly, we have $n \geq 8$ in $\mathcal{TB}_n^3$. Moreover, any TI tree T in $\mathcal{TB}_n^3$ must satisfy $n \geq 10$ since the removal of the central branching vertex will result in at least three components of total order at least 9. Moreover, it can be checked by computer that there is no TI tree in $\mathcal{TB}_{10}^3$. Therefore we only need to consider the TI trees in $\mathcal{TB}_n^3$ with $n \geq 11$.

Motivated by the characterization of starlike TI trees and double starlike TI trees [10], in the present paper we consider the TI property of graphs with exactly three branching vertices, that is, those graphs in $\mathcal{GB}_n^3$. This paper is organized as follows. In Section 2 we list or prove some preliminary results which will be used in the subsequent proofs. In Section 3 we construct a TI tree $T \in \mathcal{TB}_n^3$ for each odd $n \geq 11$. In Section 4 we prove that there exists a TI tree from $\mathcal{TB}_n^3$ for each $n \geq 16$ with $n \equiv 4 \pmod 6$. Moreover, a TI graph in $\mathcal{GB}_n^3$ is presented for each $n \geq 18$ with $n \equiv \ell \pmod 6$ where $\ell \in \{0, 2\}$, in particular, a tree in $\mathcal{TB}_n^3$ is constructed for $n \in \{18, 20\}$. In Section 5 we conclude the paper with some open problems.

2. Preliminaries

For $X \subseteq V(G)$ of a graph G, let $G - X$ be the subgraph of G obtained from G by removing the vertices from X and the edges incident with them. In particular, $G - \{v\}$ will be briefly denoted by $G - v$. Similarly, for $F \subseteq E(G)$, $G - F$ is the spanning subgraph of G obtained by removing the edges of F and if $e \in E(G)$, then we will write $G - e$ for $G - \{e\}$. The *eccentricity* $\mathrm{ecc}_G(v)$ of a vertex $v \in V(G)$ is the maximum distance from v to all other vertices in G. If $uv \in E(G)$, then n_u (or $n_G(u)$ if the graph G is necessarily mentioned) is the number of vertices in G closer to u than to v and n_v (or $n_G(v)$ for completeness) can be similarly defined.

For a set A of at least three real numbers, we denote by $M(A)$, $m(A)$ the maximum and the minimum of A, by $SM(A)$, $sm(A)$ the second maximum and the second minimum of A, and by $TM(A)$, $tm(A)$ the third maximum and the third minimum of A, respectively. For any positive integer k, we set $[k] = \{1, 2, \ldots, k\}$ and $[k]_0 = [k] \cup \{0\}$. For a set A of integers and an integer t, we denote by $A + t$ the usual coset, that is, $A + t = \{a + t | a \in A\}$. A set of positive integers is odd (even, resp.) if it consists of odd (even, resp.) integers.

A vertex v with $\deg_G(v) = 1$ is called a *pendent vertex* (also *leaf* when G is a tree) in G, and the edge incident with a pendent vertex is called a *pendent edge*. A path $P := u_k u_{k-1} \cdots u_2 u_1$ with the natural adjacency relation in a graph G is a *proper pendent path* in G if $d_G(u_k) \geq 3$, $d_G(u_1) = 1$, and $d_G(u_i) = 2$ for $i \in \{2, 3, \ldots, k-1\}$, where u_k is its root. If both u_k and u_1 in P have degrees at least 3 and each of u_j with $j \in \{2, 3, \dots, k-1\}$ has degree 2, then P is an *internal path* in G with two *terminals* u_k and u_1. In particular, if u_1 and u_k have degrees at least 2, then the above P is a *weak internal path* [21] with two weak terminals u_1 and u_k. A tree with a unique branching vertex v is *starlike*. A starlike tree T with the branching vertex v will be denoted by $T = T(n_1, \dots, n_k)$ if $T - v$ consists of k disjoint paths of orders $n_1, \dots, n_k$, respectively, where v is called the center of T. Also, a pendent path of length n_i from v is called an *n_i-arm* in T.

Lemma 2.1. ([6]) *If G is a graph with $n(G) > 2$ and $uv \in E(G)$, then* $\mathrm{Tr}(u) - \mathrm{Tr}(v) = n_v - n_u$.

From Lemma 2.1, the following two results are obvious.

Corollary 2.2. *Let T be a tree with $u \in V(T)$ and $uv \in E(T)$.*

(1) *If $T - u = \bigcup_{k=1}^{t} T_k$ with $n(T_i) = n(T_j)$ for two distinct numbers $i, j \in [t]$, then T is not TI.*

(2) *If $T - uv = T' \cup T''$ with $n(T') = n(T'')$, then T is not TI.*

Corollary 2.3. *Let G_1, G_2 be two vertex-disjoint graphs of orders $n_1 \le n_2$ with $v_i \in V(G_i)$ for $i \in [2]$. Suppose that $G = G_{1,2}(v_1 - v_2)$ is the graph obtained from G_1 and G_2 by joining vertices v_1 and v_2 with a new edge. Then $Tr_G(v_1) - Tr_G(v_2) = n_2 - n_1$.*

Lemma 2.4. ([19]) *Let G be a graph with $n(G) = n$ and $v \in V(G)$ with* $\deg(v) \ge 3$. *If $P = uv_1v_2 \cdots v_{x-1}v$ is a pendent path with natural adjacency relation attached at v where $d(u) = 1$ and $x < \frac{n}{2}$, then* $\mathrm{Tr}(v_{x-1}) - \mathrm{Tr}(v) = n - 2x$.

Lemma 2.5. ([21]) *Let G be a graph with $n(G) = n$ and $P = vv_1v_2 \cdots v_kv^*$ be a weak internal path in G with two weak terminals v and v^* such that each edge in P is a cut edge. If* $\mathrm{Tr}(v_1) - \mathrm{Tr}(v) = a > 0$, *then* $\mathrm{Tr}(v_j) - \mathrm{Tr}(v) = j(a + j - 1)$ *for any $j \in [k]$.*

3. Three-branching TI graphs of odd order

For $i \ge 2$, we denote by $M(i)$ a tree obtained from a star with center v and three leaves u_1, u_2 and u_3 by attaching at u_1 a pendent edge, at u_2 two pendent paths of respective lengths $i-1$ and i and at u_3 two pendent paths of respective lengths i and $i+1$. See in Figure 1 the structure of $M(i)$. Clearly, $n(M(i)) = 4i + 5$.

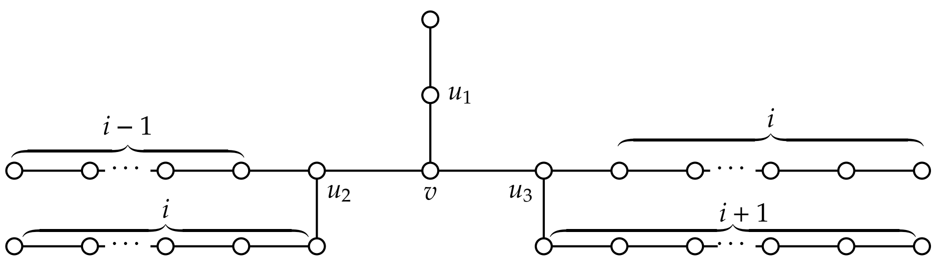

Figure 1: Tree $M(i)$.

Lemma 3.1. *Let $M(i)$ with $i \ge 2$ be a tree defined as above. If $i \notin \{6, 9, 11, 14\}$, then $M(i)$ is TI.*

Proof. Assume that $\mathrm{Tr}(v) = x$. By Lemma 2.1, we have $\mathrm{Tr}(u_1) = 4i + 1 + x$, $\mathrm{Tr}(u_2) = 5 + x$ and $\mathrm{Tr}(u_3) = 1 + x$. By Corollary 2.3 and Lemma 2.4, we have $\mathrm{Tr}(M(i)) = B \cup (\bigcup_{i=0}^{3} A_i)$ with

$$\begin{aligned} B &= \{0, 4i + 1, 8i + 4\} + x, \\ A_0 &= \{(k + 2)^2 + 2ik - 3 | k \in [i]_0\} + x, \\ A_1 &= A_0 + 4, \\ A_2 &= A_0 + 2i + 3, \\ A_3 &= A^* - (2i + 1) \end{aligned}$$

where $A^* = A_0 \setminus \{1, 2i + 6\}$.

We first prove that $A_k \cap A_j = \emptyset$ for any $k, j \in [3]_0$ with $k \neq j$. Note that the minimum positive difference between two elements in A_0 is $\min\{2k + 2i + 5 | k \in [i - 1]_0\} > 4$. Therefore $A_0 \cap A_1 = \emptyset$. If there are two numbers $p, q \in [i]_0$ such that $(p+2)^2 + 2ip - 3 = (q + 2)^2 + 2i(q + 1)$, then $p > q$ with $(p - q)(p + q + 4 + 2i) = 2i + 3$, which implies that $p - q \geq 1$ is odd. So it follows that $2i + 3 \geq 2i + 4 + p + q \geq 2i + 5$ as a clear contradiction. Hence $A_0 \cap A_2 = \emptyset$. If there exist $p, q \in [i]_0$ such that $(p + 2)^2 + 2ip + 1 = (q + 2)^2 + 2i(q + 1)$, then $p > q$ with $(p - q)(p + q + 4 + 2i) = 2i - 1$. Similarly as above, we have $A_1 \cap A_2 = \emptyset$.

Now we claim that $(q+2)^2 + 2i(q-1) - 4 \notin A_0 \cup A_1 \cup A_2$ for any $q \in [i] \setminus [1]$. Otherwise, there exists a $p \in [i]_0$ such that $(q + 2)^2 + 2i(q - 1) - 4 \in \{-3, 1, 2i\} + (p + 2)^2 + 2ip$. If $(q + 2)^2 + 2i(q - 1) - 4 \in \{-3, 1\} + (p + 2)^2 + 2ip$, then $(q - p)(p + q + 2i + 4) \in \{1, 5\} + 2i$, each of which will result in a similar contradiction as above. If $(q + 2)^2 + 2i(q - 1) - 4 = (p + 2)^2 + 2i(p + 1)$, then $(q - p)(q + p + 2i + 4) = 4i + 4$, implying that $q - p \geq 2$ is even. But $4i + 4 \geq 2(2i + 6)$, this is a contradiction again. All three above contradictions show that $A_3 \cap A_s = \emptyset$ for any $s \in [2]_0$.

Next it suffices to show that $B \cap A_t = \emptyset$ for any $t \in [3]_0$. Clearly, $x \notin A_t$ for any $t \in [3]_0$. So we only need to prove that $A_t \cap (\{4i + 1, 8i + 4\} + x) = \emptyset$ for any $t \in [3]_0$. Since $4i + 1 \notin \{4, 6, 10, 12\} + 2i$ with $4i + 1 < 4i + 9 = \min\{9, 13, 17, 21\} + 4i$, we have $4i + 1 \notin \bigcup_{k=0}^{3} A_k$. Moreover, we have $8i + 4 < 8i + 25 = \min\{25, 33, 37, 45\} + 8i$ and $8i + 4 \notin \{16, 22, 26, 32\} + 6i$ since $i \notin \{6, 9, 11, 14\}$. Therefore $8i + 4 \notin \bigcup_{k=0}^{3} A_k$. Thus $B \cap \bigcup_{k=0}^{3} A_k = \emptyset$. Thus our result holds as desired. □

Denote by $N(i)$ with $i \geq 3$ a tree obtained from a starlike tree $T(i - 1, i, i, i + 1)$ with branching vertex v by attaching a pendent edge at the vertex w_1 on an i-arm with $d(v, w_1) = i - 2$ and another pendent edge at the vertex w_2 on the $(i + 1)$-arm with $d(v, w_2) = i - 2$. See the structure of $N(i)$ in Figure 2.

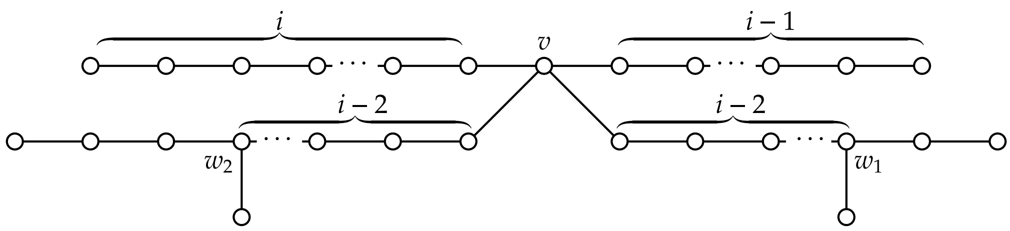

Figure 2: Tree $N(i)$.

Lemma 3.2. *Let $N(i)$ with $i \geq 3$ be a tree defined as above. Then $N(i)$ is TI if and only if $i \neq 3$.*

Proof. Note that $n(N(i)) = 4i + 3$. Thus $n(N(3)) = 15$. From the structure of $N(3)$ (see Figure 2), we have $\mathrm{Tr}(z) = \mathrm{Tr}(u) = \mathrm{Tr}(v) + 20$ where u is the vertex lying on the pendent path of length 3 in $N(3)$ with $d(u, v) = 2$, while z is the pendent vertex attached to w_1. Therefore $N(3)$ is not TI. This completes the only if part of the proof. Next we turn to the if part of the proof. Assume that $\mathrm{Tr}(v) = x$. By the structure of $N(i)$ and Lemma 2.4, we have $\mathrm{Tr}(N(i)) - x = \{0\} \cup (\bigcup_{k=1}^{2} B_k) \cup (\bigcup_{k=1}^{4} A_k)$ with

$$B_1 = \{-4i + 5, -4i + 3, 4\} + 3i^2,$$
$$B_2 = \{-6i + 9, -6i + 5, -2i + 4, 2i + 5\} + 3i^2,$$
$$A_1 = \{2mi + m^2 + 4m | m \in [i - 1]\},$$
$$A_2 = \{2mi + m^2 + 2m | m \in [i]\},$$
$$A_3 = \{2mi + m^2 | m \in [i - 2]\},$$
$$A_4 = \{2mi + (m - 2)m | m \in [i - 2]\}.$$

Let $B = B_1 \cup B_2$. It can be routinely checked that $B_1 \cap B_2 = \emptyset$. We first prove that $A_3 \cap B = \emptyset$, $A_4 \cap B = \emptyset$. Clearly, $B_1 \cap A_3 = \emptyset = B_2 \cap A_4$ since $m(B_1) > M(A_3)$ and $m(B_2) > M(A_4)$. Similarly, we have $m(B_1) = 3i^2 - 4i + 3 > 3i^2 - 10i + 8 = M(A_4)$ and $m(B_2) = 3i^2 - 6i + 5 > 3i^2 - 8i + 4 = M(A_3)$, which yield $B_1 \cap A_4 = \emptyset$ and $B_2 \cap A_3 = \emptyset$, respectively. Thus $A_k \cap B = \emptyset$ for $k \in \{3, 4\}$.

Now we claim that $A_k \cap A_j = \emptyset$ for any distinct $k, j \in [4]$. If there are two numbers $m \in [i - 1]$, $n \in [i]$ such that $m^2 + (2i + 4)m = n^2 + (2i + 2)n$, then $n > m$ with $(n - m)(n + m + 2i + 2) = 2m$, which implies that $n - m \geq 2$ is even. But $2m \geq 2(n + m + 2i + 2) \geq 4m + 4i + 8$ is a clear contradiction. Hence $A_1 \cap A_2 = \emptyset$. If there are two numbers $m \in [i - 1]$, $p \in [i - 2]$ with $m^2 + (2i + 4)m = p^2 + 2ip$, then $p - m \geq 2$ is even with $(p - m)(p + m + 2i) = 4m$. Thus $4m \geq 2(p + m + 2i) \geq 4m + 4i + 4$ as a clear contradiction, which implies that $A_1 \cap A_3 = \emptyset$. If $m^2 + (2i + 4)m = q^2 + (2i - 2)q$ with $m \in [i - 1]$ and $q \in [i - 2]$, then $q - m \geq 2$ must be even with $(q - m)(q + m + 2i - 2) = 6m$. Then it follows that $6m \geq 2(q + m + 2i - 2) \geq 4m + 4i$, contradicting with the range of m. Hence $A_1 \cap A_4 = \emptyset$. If $n^2 + (2i + 2)n = p^2 + 2ip$ with $n \in [i]$, $p \in [i - 2]$, then $p - n \geq 2$ must be even with $(p - n)(p + n + 2i) = 2n$. So it follows that $2n \geq 2(p + n + 2i) \geq 4n + 4i + 4$ as a clear contradiction. Hence $A_2 \cap A_3 = \emptyset$. If there are two numbers $n \in [i]$, $q \in [i - 2]$ with $n^2 + (2i + 2)n = q^2 + (2i - 2)q$, then $q - n \geq 2$ is even with $(q - n)(q + n + 2i - 2) = 4n$. So $4n \geq 2(q + n + 2i - 2) \geq 4n + 4i$ holds as a clear contradiction. Hence $A_2 \cap A_4 = \emptyset$. If there are two numbers $p \in [i - 2]$, $q \in [i - 2]$ with $p^2 + 2ip = q^2 + (2i - 2)q$, then $q - p \geq 2$ is even with $(q - p)(q + p + 2i - 2) = 2p$. Therefore we conclude that $2p \geq 2(q + p + 2i - 2) \geq 4p + 4i$ as a clear contradiction. Hence $A_3 \cap A_4 = \emptyset$.

Next it suffices to show that $A_k \cap B = \emptyset$ with $k \in [2]$. Note that $M(A_1) = 3i^2 - 3$ with integer $i \geq 3$. Thus $M(A_1) \notin \{-4i + 5, -4i + 3, 4, -6i + 9, -6i + 5, -2i + 4, 2i + 5\} + 3i^2 = B$. Similarly, we have $SM(A_1) = 3i^2 - 4i - 4 \notin B$, while $TM(A_1) = 3i^2 - 8i - 3 < 3i^2 - 6i + 5 = m(B)$. Hence $A_1 \cap B = \emptyset$. Similarly as above, we have $M(A_2) = 3i^2 + 2i \notin \{-4i + 5, 4i + 3, 4, -6i + 9, -6i + 5, -2i + 4, 2i + 5\} + 3i^2 = B$. Moreover, we have $SM(A_2) = 3i^2 - 2i - 1 \notin B$ since $i \neq 3$. Clearly, $TM(A_2) = 3i^2 - 6i < 3i^2 - 6i + 5 = m(B)$. Hence $A_2 \cap B = \emptyset$. Therefore our result holds clearly. □

In what follows some figures will be presented with a specific vertex v being given with $\mathrm{Tr}_G(v) = x$ and $Tr(G) = \{a_u : u \in V(G)\} + x$ for all the values of a_u being labelled. Next we provide a result on the existence of TI trees in $\mathcal{TB}_n^3$ with $n \geq 11$.

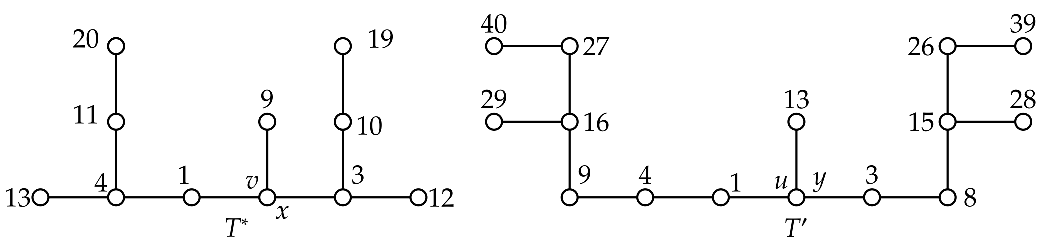

Figure 3: Trees T^* and T'.

Theorem 3.3. *There is a TI tree in $\mathcal{TB}_n^3$ for each odd integer $n \geq 11$.*

Proof. For $n = 11, 15$ in $\mathcal{TB}_n^3$, two TI trees T^* of order 11 with the special vertex v and T' of order 15 with the special vertex u (with $\mathrm{Tr}(u) = y$) are shown, respectively, in Figure 3. By Lemma 3.2, there is a TI tree from $\mathcal{TB}_n^3$ for any odd integer $n > 15$ with $n \equiv 3 \pmod 4$. In view of Lemma 3.1, we only need to prove the existence of a TI tree from $\mathcal{TB}_n^3$ with $n \in \{29, 41, 49, 61\}$. Denote by $T_{a,b,c}^{r,s,t}$ with $a \leq b \leq c$ the tree obtained from a starlike tree $T_0 = T(a, b, c)$ by attaching three pendent paths of lengths r, s and t to each leaf of the pendent paths of lengths b and c in T_0. It can be routinely checked that $T_{3,3,4}^{2,3,4} \in \mathcal{TB}_{29}$, $T_{3,3,4}^{4,5,6} \in \mathcal{TB}_{41}$, $T_{3,4,5}^{5,6,7} \in \mathcal{TB}_{49}$ and $T_{3,4,5}^{7,8,9} \in \mathcal{TB}_{61}$ are all TI trees. This completes the proof. □

From Theorem 3.3, the following result is obvious.

Corollary 3.4. *There is a TI graph in $\mathcal{GB}_n^3$ for each odd integer $n \geq 11$.*

4. Three-branching TI graphs of even order

In this section we turn to the determination of TI graphs of even order. For an integer $k \geq 2$, we denote by $X(k)$ a tree obtained from $T(2k, 2k, 2k+1)$ by attaching a pendent vertex at each of the two vertices v' and v'' lying on a $(2k)$-arm and the $(2k+1)$-arm of $T(2k, 2k, 2k+1)$ with $d(v', v) = 2 = d(v'', v)$ where v is the center of $T(2k, 2k, 2k+1)$. See in Figure 4 the structure of $X(k)$. Below we characterize the TI property of $X(k)$.

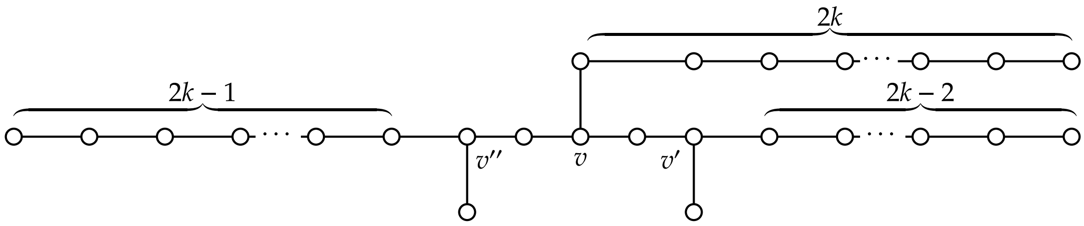

Figure 4: Tree $X(k)$.

Lemma 4.1. *Let $X(k)$ be a tree defined as above with $k \geq 2$. Then $X(k)$ is TI if and only if $k \notin \{4, 6, 8, 10, 12\}$.*

Proof. Assume that v with $\mathrm{Tr}(v) = x$ is the vertex in $X(k)$ at which a pendent path of length $2k$ is attached. Note that $n(X(k)) = 6k + 4$. Let

$$A_1 = \{2k\} \cup \{2(p+2)k + p^2 + 5p + 2 | p \in [2k-1]_0\},$$
$$A_2 = \{2k+2\} \cup \{2(p+2)k + p^2 + 7p + 6 | p \in [2k-2]_0\},$$
$$A_3 = \{2pk + p(p+3) | p \in [2k]\}.$$

By Lemmas 2.4 and 2.5, we have $\mathrm{Tr}(X(k)) - x = B \cup \bigcup_{j=1}^{3} A_j$ with $B = \{0, 10k+4, 10k+8\}$.

Now we prove that $\bigcup_{j=1}^{3} A_j$ is a partition set. We first prove that $A_1 \cap A_2 = \emptyset$. Clearly, $2k \notin \{2(p+2)k + p^2 + 7p + 6 | p \in [2k-2]_0\}$ and $2k+2 \notin \{2(p+2)k + p^2 + 5p + 2 | p \in [2k-1]_0\}$. Moreover, if there exist two distinct numbers $i \in [2k-1]_0$ and $j \in [2k-2]_0$ such that $2(i+2)k + i^2 + 5i + 2 = 2(j+2)k + j^2 + 7j + 6$, then $i > j$ with $(i-j)(i+j+2k+5) = 2j+4$. Consequently, $2j+4 \geq i+j+2k+5 \geq j+2k+6$, which is impossible. Thus $A_1 \cap A_2 = \emptyset$. If there are two distinct numbers $i_1 \in [2k-1]_0$ and $j_1 \in [2k]$ with

$2(i_1+2)k+i_1^2+5i_1+2=2j_1k+j_1^2+3j_1$, then $j_1>i_1$ with $4k+2i_1+2=(j_1-i_1)(2k+j_1+i_1+3)$. If $j_1-i_1=1$, we have $4k+2i_1+2=2k+2i_1+4$, that is, $k=1$, contradicting with the assumption $k\geq 2$. Therefore $j_1-i_1\geq 2$. It follows that $4k+2i_1+2\geq 2(2k+2i_1+5)=4k+4i_1+10$, that is, $2i_1+8\leq 0$, as a clear contradiction. Therefore $A_1\cap A_3=\emptyset$. If there are two distinct numbers $i_2\in[2k-2]_0$ and $j_2\in[2k]$ with $2(i_2+2)k+i_2^2+7i_2+6=2j_2k+j_2^2+3j_2$, then $j_2>i_2$ with $4k+4i_2+6=(j_2-i_2)(2k+j_2+i_2+3)$. If $j_2-i_2=1$, we have $4k+4i_2+6=2k+2i_2+4$, that is, $k+i_2=-1$, which is impossible. Thus $j_2-i_2\geq 2$, which implies that $4k+4i_2+6\geq 2(2k+2i_2+5)$, that is, $6\geq 10$ as a contradiction. Note that $m(A_3)=2k+4>2k+2>2k$. Thus $A_2\cap A_3=\emptyset$. Therefore $\bigcup_{j=1}^{3}A_j$ is a partition set as desired.

Next we consider the partition property of the set $B\cup(\bigcup_{j=1}^{3}A_j)$. Let $A=\bigcup_{j=1}^{3}A_j$. For $k=2$, we have $A=\{4,6,8,10,14,18,20,26,30,32,40,44,46\}$ and $B=\{0,24,28\}$. Then $B\cap A=\emptyset$ as desired. In the following we assume that $k\geq 3$. For the set A_1, we have $6k+8<10k+4<10k+8<10k+26$. Moreover, $8k+16\in B$ if and only if $k\in\{4,6\}$. Equivalently, $B\cap A_1=\emptyset$ if and only if $k\notin\{4,6\}$. Similarly as above, we have $B\cap A_2=\emptyset$ if and only if $k\notin\{8,10\}$, and $B\cap A_3=\emptyset$ if and only if $k\notin\{10,12\}$. Thus we conclude that $B\cap A=\emptyset$ if and only if $k\notin\{4,6,8,10,12\}$. This completes the proof. □

To further determine the three-branching TI trees, we first prove the following result. Let $T^*=T(2,k-2,k-3)$ with v as its center. Denote by $H(k)$ with $k\geq 7$ a tree obtained from T^* by attaching a pendent vertex at the vertex w_1 with distance 3 to the leaf on the $(k-2)$-arm of T^* and another pendent vertex at the vertex w_2 with distance 3 to the leaf on the $(k-3)$-arm of T^*. See Figure 5 for the structure of $H(9)$.

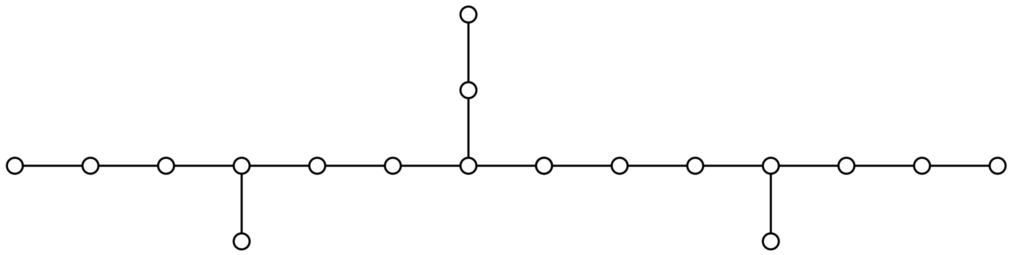

Figure 5: Tree $H(9)$.

Lemma 4.2. *Let $H(k)$ be a tree defined as above with $k\geq 10$. If neither $2k-4$ nor $4k-6$ is of the form m^2+m or m^2+3m where m is a positive integer, then $H(k)$ is TI.*

Proof. By the structure of $H(k)$, we have $n(H(k))=2k$. Assume that v is the central branching vertex in $H(k)$ with $\mathrm{Tr}(v)=x$. Let $P'=vu_1u_2\cdots u_{k-6}w_1$ and $P''=vv_1v_2\cdots v_{k-7}w_2$ be two internal paths of $H(k)$ with natural adjacency relations. By Lemma 2.5, we have $\mathrm{Tr}(u_i)=x+i^2+i$ with $i\in[k-6]$, $\mathrm{Tr}(w_1)=x+(k-5)(k-4)$ and $\mathrm{Tr}(v_j)=x+j^2+3j$ with $j\in[k-7]$, $\mathrm{Tr}(w_2)=x+(k-6)(k-3)$. Then $\mathrm{Tr}(H(k))-x=B\cup\bigcup_{\ell=1}^{2}(A_\ell\cup A_\ell^*)$ with $B=\{0,2k-4,4k-6\}$, $A_1=\{i^2+i|i\in[k-5]\}$, $A_1^*=\{2k-6,2k-2,4k-10,6k-12\}+(k-5)(k-4)$, $A_2=\{j^2+3j|j\in[k-6]\}$ and $A_2^*=A_1^*-2$. Next it suffices to prove that $\mathrm{Tr}(H(k))-x$ is a partition set.

Set $A=\bigcup_{\ell=1}^{2}(A_\ell\cup A_\ell^*)$. Obviously, we have $A_\ell^*\cap A_\ell=\emptyset$ for $\ell\in\{1,2\}$ because of the fact that $m(A_\ell^*)>M(A_\ell)$. Furthermore, we claim that $A_1\cap A_2=\emptyset$ since $m^2+m=r^2+3r$ for two positive integers m and r will result in a clear contradiction $2r=(m-r)(m+r+1)\geq 2r+1$. Moreover, $m(A_1^*)=k^2-7k+14>k^2-9k+18=M(A_2)$, $m(A_2^*)=k^2-7k+12>k^2-9k+20=M(A_1)$ yield that $A_1^*\cap A_2=\emptyset=A_2^*\cap A_1$. Note that the minimum difference between any numbers in A_1^* is $4>2$. Combining this fact with $A_2^*=A_1^*-2$, we have $A_1^*\cap A_2^*=\emptyset$. Thus A is a partition set.

From the assumption, we have $B \cap (A_1 \cup A_2) = \emptyset$. Further, by the definitions of A_1^* and A_2^*, we have $m(A_1^* \cup A_2^*) = k^2 - 7k + 12 > 4k - 6$ for $k \geq 10$. Then $B \cap (A_1^* \cup A_2^*) = \emptyset$. Therefore $B \cap A = \emptyset$, implying that $B \cup A$ is a partition set. This completes the proof. □

Theorem 4.3. *There is a TI tree in $\mathcal{TB}_n^3$ for each $n \geq 16$ with $n \equiv 4 \pmod 6$.*

Proof. By Lemma 4.1, we only need to prove the existence of TI trees of order $n \in \{28, 40, 52, 64, 76\}$. For $n \in \{28, 40, 52, 64\}$, we assume that $n = 2k$, that is, $k \in \{14, 20, 26, 32\}$. By Lemma 4.2, we conclude that $H(k)$ of order $2k$ with exactly three branching vertices is TI for $k \in \{14, 20, 26, 32\}$. It suffices to construct a TI tree of order 76 with exactly three branching vertices.

Let T be a tree obtained from $T_0 = T(2, 35, 36)$ with center v by attaching a pendent vertex to the vertex on the 35-arm with distance 9 to v in T_0 and another pendent vertex to the vertex on the 36-arm with distance 7 to v in T_0. Assume that $\mathrm{Tr}(v) = x$ in T. From the structure of T, we have $\mathrm{Tr}(T) - x = \{0, 72, 146, 130, 182\} \cup A_1 \cup A_2 \cup B \cup (B + 8)\}$ with $A_1 = \{x^2 + 3x | x \in [9]\}$, $A_2 = \{y^2 + y | y \in [7]\} \cup \{74, 94, 116\}$ and $B = \{x^2 + 23x | x \in [26]\} + 108$. It can be routinely checked that $\mathrm{Tr}(T) - x$ is a partition set, that is, $T \in \mathcal{TB}_{76}^3$ is TI. This completes the proof. □

In general, the determination of TI trees of even order is more difficult since their respective transmissions have the same parity. Although constructing a TI tree of even order is tricky work, it turns more reachable if we extend to the general TI graphs of even order. Let BK be the bowknot graph which consists of two triangles sharing one vertex (which is called the universal vertex in it). Let $BK_{a,b,c}$ be the graph obtained by attaching a pendent path of length a at the universal vertex and attaching a pendent path of length b at one vertex u of degree 2 in BK and attaching at another vertex v of degree 2 in BK with $d_{BK}(u, v) = 2$ a pendent path of length c. The structure of $BK(a, b, c)$ is shown in Figure 6. Denote by T^* the tree obtained from $T_0 = T(2, 6, 7)$ with the center w by attaching a pendent vertex to each vertex in T_0 with distance 4 to w. By a routine check we find that $T^* \in \mathcal{TB}_{18}^3$ is a TI tree. Moreover, $H(10) \in \mathcal{TB}_{20}^3$ is a TI tree from Lemma 4.2. So we assume that $n \geq 24$ in the following result.

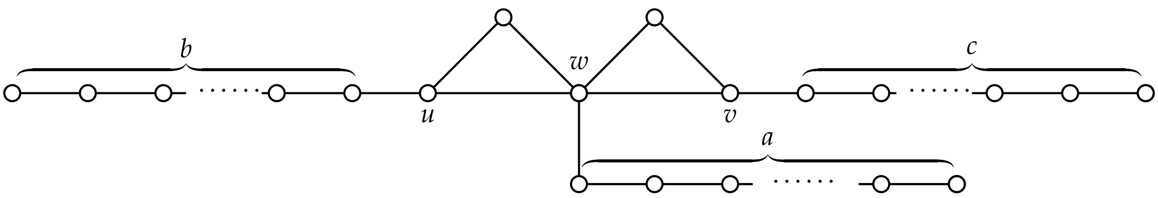

Figure 6: Graph $BK(a, b, c)$.

Theorem 4.4. *Let $n \geq 24$ be an even integer. If $n \equiv t \pmod 6$ with $t \in \{0, 2\}$, then there is a TI graph in $\mathcal{GB}_n^3$.*

Proof. We first consider the case when $n \equiv 0 \pmod 6$. Assume that $n = 6k + 6$ with $k \geq 3$. Let $G_1 = BK(2k, 2k, 2k+1)$ with $n(G_1) = 6k+6$ where the vertex w is of degree 5 and two vertices u and v are of degrees 3 such that there exist a pendent path of length $2k$ attached at u and a pendent path of length $2k+1$ attached at v.

Assume that $\mathrm{Tr}(w) = x$ in $G_1 \in \mathcal{GB}_n^3$. By the structure of G_1 and Lemma 2.1, we have $\mathrm{Tr}(u) = x + 2k + 3$, $\mathrm{Tr}(v) = x+2k+1$ and $\mathrm{Tr}(G_1) - x = \bigcup_{i=0}^{3} A_i$ where $A_0 = \{0, 2k+1, 2k+3, 4k+2, 4k+3\}$, $A_1 = \{y^2 + (2k+5)y | y \in [2k]\}$, $A_2 = A_1 + (2k + 3)$ and $A_3 = \{y^2 + (2k + 3)y | y \in [2k + 1]\} + (2k + 1)$. Since $k \geq 3$, we have $A_0 \cap (\bigcup_{j=1}^{3} A_j) = \emptyset$. Next we prove that $A_i \cap A_j = \emptyset$ for any two distinct $i, j \in [3]$. Note that A_1 is even, while A_2 and A_3 are both odd. So we have $A_1 \cap (A_2 \cup A_3) = \emptyset$, that is, $A_1 \cap A_2 = \emptyset = A_1 \cap A_3$. If there exist $p \in [2k]$ and $q \in [2k + 1]$ such that $p^2 + (2k + 5)p + (2k + 3) = q^2 + (2k + 3)q + (2k + 1)$, then $q > p$ with $(q - p)(q + p + 2k + 3) = 2p + 2$.

But $(q-p)(q+p+2k+3) \geq 2p+2k+4 > 2p+2$ as a clear contradiction. So $A_2 \cap A_3 = \emptyset$. Thus $\mathrm{Tr}(G_1) - x$ is a partition set, that is, G_1 is TI as desired.

For $n \equiv 2 \pmod 6$, we assume that $n = 6k+2$ with $k \geq 4$. Let $G_2 = BK(2k-2, 2k-1, 2k)$. Then $G_2 \in \mathcal{GB}_n^3$ with $n(G_2) = 6k+2$. Assume that $\mathrm{Tr}(r) = z$ where r is the vertex of degree 5 in G_2. Similarly as above, we have $\mathrm{Tr}(G_2) - z = \bigcup_{i=0}^{3} B_i$ where $B_0 = \{0, 2k-1, 2k+1, 4k-1, 4k, 4k+5\}$, $B_1 = \{y^2+(2k+5)y \mid y \in [2k-2]\}$, $B_2 = B_1 + (4k+5)$ and $B_3 = \{y^2+(2k+1)y \mid y \in [2k]\} + (2k-1)$. Note that $k \geq 4$, which implies that $B_0 \cap (\bigcup_{j=1}^{3} B_j) = \emptyset$. Since B_1 is even, while B_2 and B_3 are both odd, we have $B_1 \cap (B_2 \cup B_3) = \emptyset$, that is, $B_1 \cap B_2 = \emptyset = B_1 \cap B_3$. If there exist $p \in [2k-2]$ and $q \in [2k]$ such that $p^2 + (2k+5)p + (4k+5) = q^2 + (2k+1)q + (2k-1)$, then $(q-p)(q+p+2k+1) = 4p+2k+6$. Clearly, we have $q > p$. If $q - p = 1$, we get $2p+2k+2 = 4p+2k+6$, that is, $p = -2$ as a clear contradiction. If $q - p \geq 2$, we have $(q-p)(q+p+2k+1) \geq 2(2p+2k+3) = 4p+4k+6 > 4p+2k+6$ as a contradiction again. Therefore $B_2 \cap B_3 = \emptyset$. Thus $\mathrm{Tr}(G_2) - z$ is a partition set, that is, G_2 is TI as desired. □

Combining Theorems 4.3, 4.4 with the existence of TI trees in $\mathcal{TB}_n^3$ with $n \in \{18, 20\}$, we arrive at the following result.

Corollary 4.5. *There is a TI graph in $\mathcal{GB}_n^3$ for each even integer $n \geq 16$.*

5. Concluding remarks

In this paper we prove the existence of TI trees in $\mathcal{TB}_n^3$ for each odd $n \geq 11$ and even $n \geq 16$ with $n \equiv 4 \pmod 6$, respectively. Also, the TI graphs $\mathcal{GB}_n^3$ are constructed for each $n \geq 16$ with $n \equiv \ell \pmod 6$ where $\ell \in \{0, 2\}$. Thus there is a TI graph in $\mathcal{GB}_n^3$ for each $n \geq 11$ such that $n \notin \{12, 14\}$. What about the existence of TI trees in $\mathcal{TB}_n^3$ for $n \equiv \ell \pmod 6$ with $\ell \in \{0, 2\}$? Before formally presenting the corresponding problem, we prove the following result.

Lemma 5.1. *There is no TI tree in $\mathcal{TB}_{14}^3$.*

Proof. To the contrary, we suppose that $T \in \mathcal{TB}_{14}^3$ is TI. Assume that $v \in V(T)$ is the central branching vertex with other two branching vertices u and w of T. Let $T - v = \bigcup_{k=1}^{t} T_k$ with $n(T_k) = n_k$ for $k \in [t]$ such that $n_t \geq n_{t-1} \geq \cdots \geq n_2 \geq n_1$. Then $\sum_{k=1}^{t} n_k = 13$. By Corollary 2.2 (1) and the structure of T, we have $n_t > n_{t-1} \geq 4$. Taking into account that v is a branching vertex, we have $3 \leq t \leq 4$.

If $t = 4$, we have $n_1 = 1$. Otherwise, we have $\sum_{k=1}^{4} n_t \geq 14$ as a contradiction. Moreover, $2 \leq n_2 \leq 3$. If $n_2 = 2$, then $(n_3, n_4) = (4, 6)$. Assume that $u \in V(T_4)$ with $n(T_4) = 6$. Then $w \in T_3$ with $n(T_3) = 4$. If $d(u, v) = 1$, then, from the structure of T, there is a vertex $x \in V(T) \setminus V(T_4)$ and $y \in V(T_4)$ with $\mathrm{Tr}(x) = \mathrm{Tr}(y)$. If $d(v, w) \geq 2$, then there is a vertex $z \in V(T_4)$ with $\mathrm{Tr}(z) = \mathrm{Tr}(w)$. These are both contradictions to the fact that T is TI. If $n_2 = 3$, we have $(n_3, n_4) = (4, 5)$. Then there is a pendent path $P_4 = vv_1v_2v_3$ with the natural adjacency relation attached at v with $w \in V(T_3)$, $u \in V(T_4)$. Moreover, there is a leaf w' attached at w. Regardless of the structure of T_4, we have $\mathrm{Tr}(w') = \mathrm{Tr}(v_2)$ as a contradiction again.

For $t = 3$, we have $n_3 \neq 7$ from Corollary 2.2 (2). Then $n_3 \in \{6, 8\}$. Assume that $u \in V(T_3)$ and $w \in V(T_2)$. If $n_3 = 8$, then $(n_1, n_2) = (1, 4)$. If $d(u, v) \geq 2$, then there is a vertex $x \in V(T_3)$ with $\mathrm{Tr}(x) = \mathrm{Tr}(v)$. If $d(u, v) = 1$, based on the degree of u in T, we conclude that there is a vertex $y \in V(T_3)$ and $z \in V(T) \setminus V(T_3)$ with $\mathrm{Tr}(y) = \mathrm{Tr}(z)$. These two contradictions show that $n_3 \neq 8$. If $n_3 = 6$, then $(n_1, n_2) \in \{(2, 5), (3, 4)\}$. If $(n_1, n_2) = (2, 5)$, then a pendent path $P_3 = vv_1v_2$ is attached at v and another pendent path $P_4 = ww_1w_2w_3$ is attached at w in T. From the structure of T, we have $\mathrm{Tr}(v_2) = \mathrm{Tr}(w_2)$ independent of the subtree T_3. For $(n_1, n_2) = (3, 4)$, there must be a pendent path P^* of length 3 attached at v and a leaf w' attached at w. Let $v^* \in V(P^*)$ with $d(v, v^*) = 2$. Then, regardless of the structure of subtree T_3, we have $\mathrm{Tr}(w') = \mathrm{Tr}(v^*)$. These above contradictions complete the proof. □

By a similar but shorter reasoning than that in the proof of Lemma 5.1, we can get the following result, but omit its proof here.

Remark 5.2. *There is no TI tree in* $\mathcal{TB}_{12}^3$.

In [13] Dobrynin constructed an infinite family of TI trees of order $n = m(m+1)+6$ with $m = 3$ or $m \geq 8$. Taking the above fact into account with Theorem 4.3, we only need to consider the case when $n \equiv \ell \pmod 6$ with $\ell \in \{0, 2\}$. Despite the fact that we have some examples of TI trees of order n with $n \equiv \ell \pmod 6$ where $\ell \in \{0, 2\}$, combining Lemma 5.1 with Remark 5.2, we would like to pose the following problem.

Problem 5.3. *Is there a TI tree in* $\mathcal{TB}_n^3$ *for each* $n \geq 24$ *with* $n \equiv k \pmod 6$ *where* $k \in \{0, 2\}$ *such that* $n \neq m(m+1)+6$ *for any* $m \geq 8$*?*

It can be checked via computer that there are at least 6 TI trees in $\mathcal{TB}_{24}^3$. We end this paper with the following natural problem.

Problem 5.4. *Determine the number of TI trees in* $\mathcal{TB}_n^3$ *with* $n \geq 11$.

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