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Double-toroidal, triple-toroidal and quadruple-toroidal nilpotent graph

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Abstract. In this paper, all finite non-nilpotent groups whose nilpotent graphs can be embedded on the double-torus, triple-torus or quadruple-torus are classified.

1. Introduction

Let G be a group and $nil(G) = \{y \in G \mid \langle x, y\rangle$ is nilpotent for all $x \in G\}$. The *nilpotent graph* of G, denoted by $\Gamma_{nil}(G)$, is a simple undirected graph in which the vertex set is $G \setminus nil(G)$, and two vertices x and y are adjacent if and only if $\langle x, y\rangle$ is a nilpotent subgroup of G. This graph is precisely the complement of the non-nilpotent graph of a group considered in [1], and may be regarded as a generalization of the commuting graph $\Gamma_c(G)$. In [5], Das and Nongsiang, studied the nilpotent graphs of groups. They determine (up to isomorphism) all finite non-nilpotent groups whose nilpotent graphs are planar or toroidal.

In the present paper, we deal with a topological aspect, namely, the genus of the nilpotent graphs of finite non-nilpotent groups. The primary objective of this paper is to determine (up to isomorphism) all finite non-nilpotent groups whose nilpotent graphs are double-toroidal, triple-toroidal or quadruple toroidal.

2. Some prerequisites

In this section, we recall certain graph theoretic terminologies (see, for example, [9] and [10]) and some well-known results which have been used extensively in the forthcoming sections. All graphs in this paper are undirected, with no loops or multiple edges.

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Let $x, y \in V(\Gamma)$. Then x and y are said to be *adjacent* if $x \neq y$ and there is an edge $x - y$ in $E(\Gamma)$ joining x and y. A *walk* between x and y is a sequence of adjacent vertices, often written as $x - x_1 - x_2 - \cdots - x_n - y$. A walk between x and y is called a *path* if the vertices in it are all distinct. The graph Γ is said to be *connected* if there is a path between every pair of distinct vertices in Γ. If in a path $x - x_1 - x_2 - \cdots - x_n - y$, x and y are adjacent in Γ, then the walk $x - x_1 - x_2 - \cdots - x_n - y - x$ is called a *cycle*. The number of edges in a path or a cycle, is called its *length*.

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The *girth* of a graph Γ is the minimum of lengths of all cycles in Γ, and is denoted by $\mathrm{gr}(\Gamma)$. We now have the following inequality (see [2, Section 2.3]).

Lemma 2.1. *Let Γ be a connected graph (but not acyclic) having n vertices and m edges. If the girth of Γ is equal to k, then we have,*

$$\gamma(\Gamma) \geq \frac{m(k-2)}{2k} - \frac{n}{2} + 1.$$

A graph Γ is said to be *complete* if there is an edge between every pair of distinct vertices in Γ. We denote the complete graph with n vertices by K_n. The *complete bipartite graph* is the one whose vertex set can be partitioned into two disjoint parts and two vertices are adjacent if and only if they lie in different parts. The complete bipartite graph, with parts of size m and n, is denoted by $K_{m,n}$.

Given a graph Γ, let U be a nonempty subset of $V(\Gamma)$. Then the *induced subgraph* of Γ on U is defined to be the graph $\Gamma[U]$ in which the vertex set is U and the edge set consists precisely of those edges in Γ whose endpoints lie in U. If $\{\Gamma_\alpha\}_{\alpha\in\Lambda}$ is a family of subgraphs of a graph Γ, then the union $\underset{\alpha\in\Lambda}{\cup}\Gamma_\alpha$ denotes the subgraph of Γ whose vertex set is $\underset{\alpha\in\Lambda}{\cup}V(\Gamma_\alpha)$ and the edge set is $\underset{\alpha\in\Lambda}{\cup}E(\Gamma_\alpha)$. The graph obtained by taking the union of graphs Γ_1 and Γ_2 with disjoint vertex sets is the disjoint union or sum, written $\Gamma_1 + \Gamma_2$. In general, $m\Gamma$ is the graph consisting of m pairwise disjoint copies of Γ. Further, given a graph Γ, its *complement* $\overline{\Gamma}$, is defined to be the graph in which the vertex set is the same as the one in Γ and two distinct vertices are adjacent if and only if they are not adjacent vertices in Γ. The join of two graphs Γ_1 and Γ_2, denoted by $\Gamma_1 \vee \Gamma_2$, is the graph obtained from $\Gamma_1 + \Gamma_2$ by joining each vertex of Γ_1 to each vertex of Γ_2.

The *genus* of a graph Γ, denoted by $\gamma(\Gamma)$, is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. Clearly, if $\tilde{\Gamma}$ is a subgraph of Γ, then $\gamma(\tilde{\Gamma}) \leq \gamma(\Gamma)$. The surface with one, two, three and four handles is the torus, double-torus, triple-torus and quadruple-torus respectively. The graphs embeddable on the surfaces of genus 0, 1, 2, 3 or 4 are the *planar*, *toroidal*, *double-toroidal*, *triple-toroidal* or *quadruple-toroidal* graphs, respectively.

A *block* of a graph Γ is a connected subgraph B of Γ that is maximal with respect to the property that removal of a single vertex (and the incident edges) from B does not make it disconnected, that is, the graph $B \setminus \{v\}$ is connected for all $v \in V(B)$. Given a graph Γ, there is a unique finite collection $\mathfrak{B}$ of blocks of Γ, such that $\Gamma = \underset{B\in\mathfrak{B}}{\cup} B$. The collection $\mathfrak{B}$ is called the *block decomposition* of Γ. In [3, Corollary 1], it has been proved that the genus of a graph is the sum of the genera of its blocks.

We conclude the section with the following two useful results.

Lemma 2.2 ([10], Theorem 6-38)**.** *If $n \geq 3$, then*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Lemma 2.3 ([10], Theorem 6-37)**.** *If $m, n \geq 2$, then*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

3. Nilpotent Graph

An ascending series $1 = G_0 \lhd G_1 \lhd \ldots G_\beta = G$ in a group G is said to be *central* if $G_\alpha \lhd G$ and $G_{\alpha+1}/G_\alpha$ lies in the center of G/G_α for every $\alpha < \beta$. A group which possesses a central ascending series is called *hypercentral*. If G is any group and α an ordinal, the terms $Z_\alpha(G)$ of the *upper central series* of G are defined by the rules

$$Z_0(G) = \{1\} \text{ and } Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G))$$

together with the completeness condition

$$Z_\lambda(G) = \bigcup_{\alpha<\lambda} Z_\alpha(G)$$

where λ is a limit ordinal. Since the cardinality of G cannot be exceeded, there is an ordinal β such that $Z_\beta(G) = Z_{\beta+1}(G) =$, etc., a terminal subgroup called the *hypercenter* of G and is denoted by $Z^*(G)$.

Given a group G with $x \in G$, the nilpotentizer of x in G is defined as $nil_G(x) = \{y \in G \mid \langle x, y\rangle \text{ is nilpotent}\}$. As in [5], a group G is said to be an $\mathfrak{nn}$-*group* if $nil_G(x)$ is a nilpotent subgroup of G for every $x \in G \setminus nil(G)$.

In this section, we shall determine all finite non-nilpotent groups whose nilpotent graphs are of genus at most 4. The following proposition of Das and Nongsiang gives all planar and toroidal nilpotent graphs.

Proposition 3.1. *([5, Proposition 5.1]). Let G be a finite non-nilpotent group. Then, the following assertions hold:*

1. *The nilpotent graph of G is planar if and only if G is isomorphic to S_3, D_{10}, D_{12}, Q_{12}, A_4, A_5, or $Sz(2)$.*
2. *The nilpotent graph of G is toroidal if and only if G is isomorphic to $SL(2,3)$, D_{14}, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_2 \times A_4$, or $\mathbb{Z}_3 \times S_3$.*

The following lemma enables us to use $Z^*(G)$ and $nil(G)$ interchangeably whenever the group G is finite.

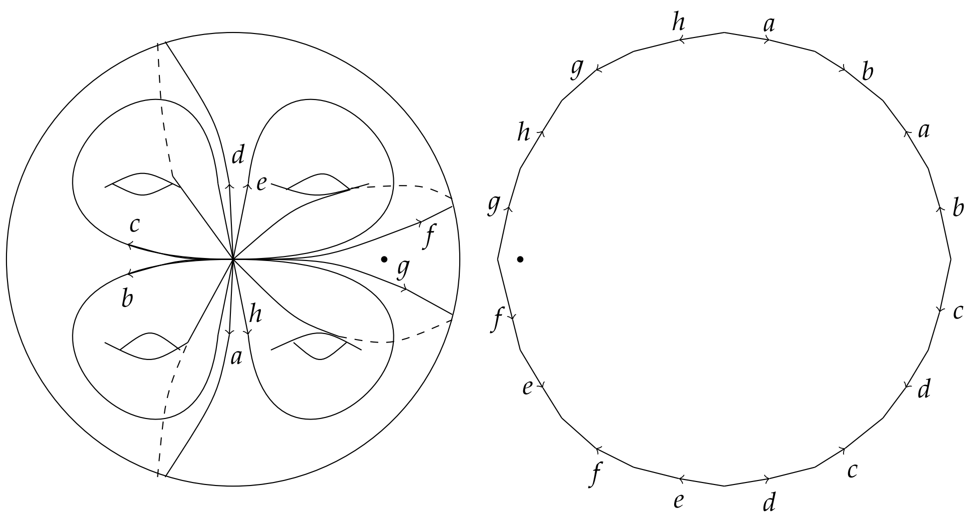
Lemma 3.2. *([5, Lemma 3.1]). Let G be a finite group. Then, the following assertions hold:*

1. *$\langle x, Z^*(G)\rangle$ is nilpotent for all $x \in G$;*
2. *$Z^*(G) = nil(G)$.*

Lemma 3.3. $\gamma(3K_3 \vee K_2) \geq 2$.

Proof: We know that $(3K_3 \vee \overline{K_2})$ is a subgraph of $(3K_3 \vee K_2)$. By [13, Theorem 3], $\gamma(3K_3 \vee \overline{K_2}) \geq 2$. Hence the result follows.

The quadruple torus can be constructed from a polygon of 16 sides by identifying pairs of edges ([12, p-5]) as shown in the following figure. The 16 edges of the polygon become a union of 8 circles in the surface, all intersecting at a single point.



Lemma 3.4. *The genus of the nilpotent graph of S_4 is 4.*

Proof. Note that in $\Gamma_{nil}(S_4)$, any vertex of the form (i, j, k) is only adjacent to (i, k, j). Therefore, these vertices does not contribute anything to the genus. Thus, they can be neglected.

Let

$$P = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\},$$
$$X_1 = \{(3,4), (1,2), (1,3,2,4), (1,4,2,3)\},$$
$$X_2 = \{(1,4), (2,3), (1,2,4,3), (1,3,4,2)\},$$
$$X_3 = \{(1,3), (2,4), (1,2,3,4), (1,4,3,2)\}.$$

Any two vertices in P are adjacent. The induced subgraph of $\Gamma_{nil}(S_4)$ by X_i is isomorphic to K_4 for $i = 1, 2, 3$. Any vertex from P is adjacent to any vertex from $X_1 \cup X_2 \cup X_3$ and any vertex from X_i is not adjacent to any vertex from X_j for $i \neq j$, $i, j \in \{1, 2, 3\}$. Thus $K_{3,12}$ is a subgraph of the nilpotent graph of S_4 and so genus of the nilpotent graph of S_4 is greater than or equal to 3.

If $\gamma(\Gamma_{nil}(S_4)) = 3$, then a triangular embedding can be found for $\Gamma_{nil}(S_4)$. By [14, Theorem 3], $\Gamma_{nil}(S_4)$ has a property that if x is a vertex of degree y in $\Gamma_{nil}(S_4)$, then there exists a wheel subgraph in $\Gamma_{nil}(S_4)$ of order y with center x. In $\Gamma_{nil}(S_4)$, the vertex $(1,2)(3,4)$ is of degree 14 but there does not exists a wheel subgraph in $\Gamma_{nil}(S_4)$ of order 14 with center $(1,2)(3,4)$, which is a contradiction. Hence, $\gamma(\Gamma_{nil}(S_4)) > 3$. Let $P = \{1, 2, 3\}$, $H = \{x_1, x_2, x_3, x_4\}$, $I = \{x_5, x_6, x_7, x_8\}$, and $J = \{x_9, x_{10}, x_{11}, x_{12}\}$. The graph $\Gamma_{nil}(S_4)$ can be embedded on the quadruple-torus as shown in Figure 1 and thus $\gamma(\Gamma_{nil}(S_4)) = 4$. □

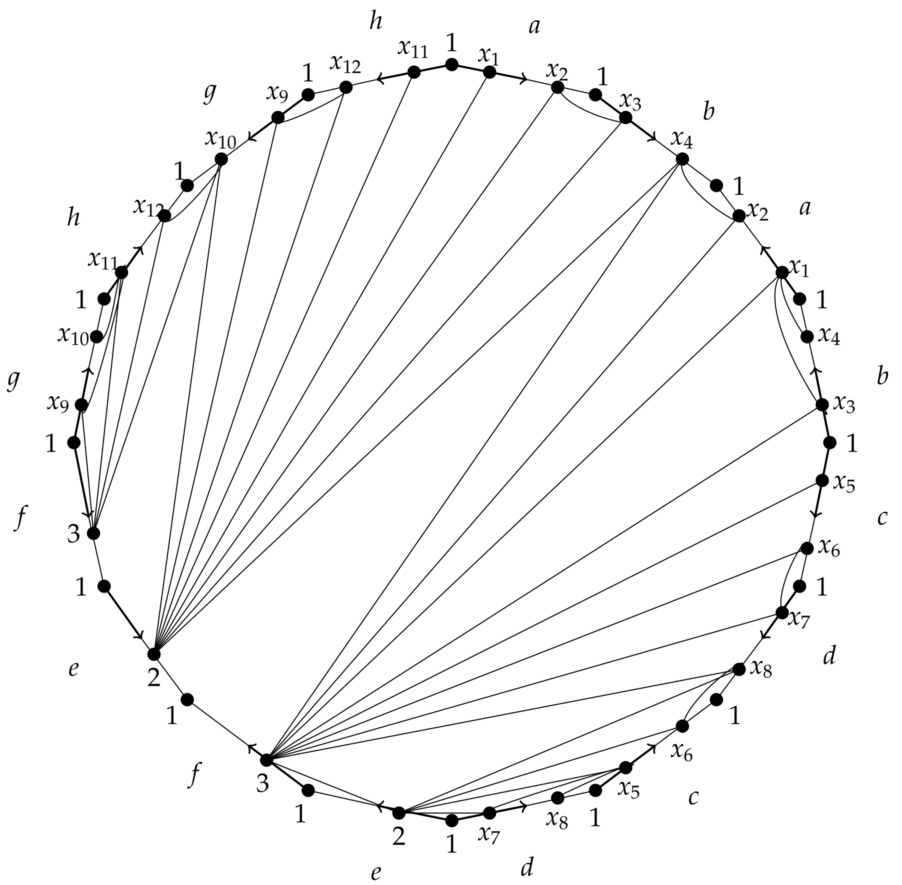


Figure 1: Embedding of $\Gamma_{nil}(S_4)$ on the quadruple-torus

Theorem 3.5. *Let G be a finite non-nilpotent group. Then the nilpotent graph of G is double-toroidal if and only if G is isomorphic to one of the following groups:*

1. $D_{18}, D_{20}, \mathbb{Z}_4 \times S_3, D_{24}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$,
2. $\langle x, y, z : x^3 = y^3 = z^2 = 1, [x, y] = 1, x^z = x^{-1}, y^z = y^{-1}\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$,
3. $\langle x, y : y^{10} = e, y^5 = x^2, xyx^{-1} = y^{-1}\rangle \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$,
4. $\langle x, y : y^3 = x^8 = 1, y^x = y^{-1}\rangle \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_8$,
5. $\langle x, y, z : x^6 = y^2 = z^2 = xyz = 1\rangle \cong \mathbb{Z}_3 \rtimes Q_8$,
6. $\langle x, y, z : x^6 = y^4 = z^2 = 1, x^3 = y^2, yxy^{-1} = x^{-1}, zx = xz, zy = yz\rangle \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$,
7. $\langle x, y, z : x^2 = y^2 = z^3 = (xz)^2 = (xy)^4 = 1, y^z = y^{-1}\rangle \cong (\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$,
8. $\langle x, y : x^4 = y^3 = (yx^2)^2 = [x^{-1}yx, y] = 1\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$.

Theorem 3.6. *There is no triple-toroidal nilpotent graph.*

Proof. [Proof of Theorem 3.5 and Theorem 3.6] Let G be a finite non-nilpotent group, whose nilpotent graph is double-toroidal or triple-toroidal. Then, $\Gamma_{nil}(G)$ has no subgraph isomorphic to K_{10}. Let $x \in G \setminus Z^*(G)$ such that $x^2 \notin Z^*(G)$. Such element exists; otherwise G is nilpotent. Then by [7, Lemma 3.13], $A = xZ^*(G) \cup x^2Z^*(G) \subset G \setminus Z^*(G)$ and any two elements of A generates a nilpotent subgroup. Therefore, $\Gamma_{nil}(G)[A] \cong K_{2|Z^*(G)|}$ and so $2|Z^*(G)| \leq 9$. Thus $|Z^*(G)| \leq 4$.

(1) $|Z^*(G)| = 4$. Suppose $p \mid |G|$, p a prime, $p \geq 5$. Let $x \in G$, such that $\circ(x) = p$. Then, by Lemma 3.2, $\langle x, Z^*(G)\rangle$ is a nilpotent subgroup of G of order at least 20, a contradiction to the fact that $\Gamma_{nil}(G)$ has no subgraph isomorphic to K_{10}. So $|G| = 2^m3^n$. Suppose, $n \geq 2$, then $\langle P_3, Z^*(G)\rangle$ is a nilpotent subgroup of G of order at least 36, a contradiction. Similarly, $m \geq 4$ is impossible. Therefore $|G| = 24$ and so G is isomorphic to one of the following groups:

- $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$,
- $\mathbb{Z}_3 \rtimes Q_8$,
- $\mathbb{Z}_4 \times S_3$,
- D_{24},
- $(\mathbb{Z}_3 \rtimes \mathbb{Z}_4) \times \mathbb{Z}_2$,
- $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$,
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$.

All these groups are $\mathfrak{n}$-groups, with one nilpotentizer of size 12 and three nilpotentizers of order 8. Thus by [5, Proposition 4.2], the nilpotent graphs of these groups are double-toroidal.

(2) $|Z^*(G)| \leq 3$. In this case, we have $Z^*(G) = Z(G)$. Thus the commuting graph $\Gamma_c(G)$ is a subgraph of the nilpotent graph $\Gamma_{nil}(G)$ and thus $\Gamma_c(G)$ is of genus at most 3. Suppose $\Gamma_c(G)$ is planar. Then by [4, Theorem 5.7], noting the fact that G is a non-nilpotent group, we have $G \cong S_3, D_{10}, A_4, D_{12}, Q_{12}, Sz(2), S_4, SL(2, 3), A_5$. By Proposition 3.1, the nilpotent graph of S_3, D_{10}, A_4, D_{12}, Q_{12}, $Sz(2)$ and A_5 are planar, whereas, the nilpotent graph of $SL(2, 3)$ is toroidal. By Lemma 3.4, the genus of the nilpotent graph of S_4 is equal to 4.

Again, if $\Gamma_c(G)$ is toroidal, then by [4, Theorem 6.6], noting the fact that G is a non-nilpotent group, we have $G \cong D_{14}, S_3 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2, \mathbb{Z}_7 \rtimes \mathbb{Z}_3$. By Proposition 3.1, the nilpotent graphs of all these groups are toroidal. Hence, we are left with the case when $\Gamma_c(G)$ is double-toroidal or triple-toroidal. If $\Gamma_c(G)$ is double-toroidal, then by [6, Theorem 3.6] and since $|Z(G)| \leq 3$, G is isomorphic to one of the following groups:

- D_{18}, D_{20}, Q_{20}
- $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$,
- $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$,
- $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$.

If G is one of the groups D_{18} or $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$, then G is an $\mathfrak{nn}$-group, with $|Z^*(G)| = 1$ and G consist of one nilpotentizer of order 9 and nine nilpotentizers of order 2. If G is one of the groups D_{20} or Q_{20}, then G is an $\mathfrak{nn}$-group, with $|Z^*(G)| = 2$ and G consist of one nilpotentizer of order 10 and five nilpotentizers of order 4. If $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$, then G is an $\mathfrak{nn}$-group, with $|Z^*(G)| = 1$ and G consist of one nilpotentizer of order 9 and nine nilpotentizers of order 4. Thus if G is any of these groups, then by [5, Proposition 4.2], the nilpotent graphs of any of these groups is double-toroidal. The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$ is an $\mathfrak{nn}$-group, with $Z^*((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) = \{1\}$, one nilpotentizer of size 9 and nine nilpotentizers of size 8. The intersection of any two of these nilpotentizers is trivial. Thus by [5, Proposition 4.2], the nilpotent graph of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$ is of genus 11.

If $\Gamma_c(G)$ is triple-toroidal, then by [6, Theorem 3.7] $G \cong SL(2,3) \circ \mathbb{Z}_2$ or $GL(2,3)$. Both these groups has a nilpotent subgroup of order 16. This completes the proof. □

Lemma 3.7. *Let G be a finite non-abelian group whose nilpotent graph is quadruple-toroidal. If $|G| = 7m$, where $m \geq 2$ and $7 \nmid m$, then $m = 2, 3$ or 6.*

Proof. Let H be a Sylow 7-subgroup of G. Note that if $|Z(G)| > 1$, then $K = \langle H, Z(G)\rangle$ is a nilpotent subgroups of G of order at least 14. Let $L = K \setminus Z^*(G)$. Then $|L| \geq 12$ and so $\Gamma_{nil}(G)[L]$ has a subgraph isomorphic to K_{11}, which is a contradiction to the fact that $\Gamma_{nil}(G)$ is quadruple-toroidal. Thus $Z(G) = 1$. Let n be the number of sylow 7−subgroups of G. Suppose $n \neq 1$. Then $n \geq 8$. Let $S_1, S_2, \ldots, S_8$, be sylow 7−subgroups of G. Then it is easy to see that the induced subgraph $\Gamma_c(G)[S_i \setminus Z(G)] \cong K_6$, for $i = 1, 2, \ldots, 8$. Thus $\gamma(\Gamma_{nil}(G)) \geq 8\gamma(K_6) = 8$, a contradiction. Therefore, H is the unique (hence, normal) Sylow 7-subgroup of G. Note that $C_G(H) = H$; otherwise $C_G(H)$ (hence, G) would have an element (hence, an abelian subgroup) of order at least 14, which would imply that $\Gamma_{nil}(G)[L]$ has a subgraph isomorphic to K_{11}, a contradiction. Therefore, by N/C Lemma [8, Theorem 7.1(i)], G/H is isomorphic to a subgroup of the cyclic group $\mathbb{Z}_6 \cong \mathrm{Aut}(H)$. Since $|G/H| = m$, it follows that $m|6$ and so $m = 2, 3$ or 6. □

Theorem 3.8. *Let G be a finite non-nilpotent group. Then, the nilpotent graph of G is quadruple-toroidal if and only if G is isomorphic to D_{22}, S_4 and $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$.*

Proof. Let G be a finite non-nilpotent group whose nilpotent graph has genus 4. Then $\Gamma_{nil}(G)$ has no subgraph isomorphic to K_{11}. Let $x \in G \setminus Z^*(G)$ such that $x^2 \notin Z^*(G)$. Such element exists; otherwise G is nilpotent. Then by [7, Lemma 3.13], $A = xZ^*(G) \cup x^2Z^*(G) \subset G \setminus Z^*(G)$ and any two elements of A generates a nilpotent subgroup. Therefore, $\Gamma_{nil}(G) \cong K_{2|Z^*(G)|}$ and so $2 \mid Z^*(G) |\leqslant 10$. Thus, $| Z^*(G) |\leqslant 5$.

(1). $|Z^*(G)| = 5$. Suppose $p \mid |G|$, p a prime and $p \geq 7$. Let $x \in G$ such that $\circ(x) = p$. Then by Lemma 3.2, $< x, Z^*(G) >$ is a nilpotent subgroup of G of order at least 35, a contradiction to the fact that $\Gamma_{nil}(G)$ has no subgraph isomorphic to K_{11}. So, $|G| = 2^p3^q5^r$. Suppose $r \geq 2$, then G has a nilpotent subgroup of order at least 25, a contradiction. So $r \in \{0, 1\}$. Suppose $q \geq 2$. Let P be a sylow 3−subgroup of G. By [7, Lemma 3.13], any two elements of $PZ^*(G) \setminus Z^*(G)$ are adjacent to each other. Thus $\Gamma_{nil}(G)$ has a subgraph isomorphic to K_{11}, a contradiction. So $q \in \{0, 1\}$. Similarly, $p \in \{0, 1\}$. Thus, $|G| = 30$ and so G is isomorphic to $\mathbb{Z}_5 \times S_3$. This is an *nn*-group with three nilpotentizers of size 10 and one nilpotentizer of size 15 and thus, by [5, Proposition 4.2], $\gamma(\Gamma_{nil}(\mathbb{Z}_5 \times S_3)) = 3\gamma(K_5) + \gamma(K_{10}) = 7$.

(2). $|Z^*(G)| = 4$. Suppose $p \mid |G|$, p a prime and $p \geq 5$. Let $x \in G$ such that $\circ(x) = p$. Then, $< x, Z^*(G) >$ is a nilpotent subgroup of G of order at least 20, a contradiction. So, $|G| = 2^m3^n$. Suppose $n \geq 2$. Let P be a sylow 3−subgroup of G. By [7, Lemma 3.13], any two elements of $PZ^*(G) \setminus Z^*(G)$ are adjacent to each other. Thus $\Gamma_{nil}(G)$ has a subgraph isomorphic to K_{11}, a contradiction. So $n \in \{0, 1\}$. Suppose $m \geq 4$, then G has a nilpotent subgroup of order at least 16, a contradiction. So, $m \in \{0, 1, 2, 3\}$. Therefore, $| G |= 24$ and so G is isomorphic to one of the following groups:

- $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$
- $\mathbb{Z}_3 \rtimes Q_8$
- $\mathbb{Z}_4 \times S_3$
- D_{24}
- $\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$
- $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$

By Theorem 3.5, the nilpotent graphs of these groups are double-toroidal.

(3). $|Z^*(G)| = 3$. Suppose $p \mid |G|$, p a prime and $p \geq 5$. Let $x \in G$ such that $\circ(x) = p$. Then, $< x, Z^*(G) >$ is a nilpotent subgroup of G of order at least 15, a contradiction. So, $|G| = 2^m 3^n$. Suppose $m \geq 3$. Let P be a sylow 2−subgroup of G. By [7, Lemma 3.13], any two elements of $PZ^*(G) \setminus Z^*(G)$ are adjacent to each other. Thus $\Gamma_{nil}(G)$ has a subgraph isomorphic to K_{11}, a contradiction. So, $m \in \{0, 1, 2\}$. Suppose $n \geq 3$, then G has a nilpotent subgroup of order at least 27, a contradiction. So $n \in \{0, 1, 2\}$. Therefore, $|G| \in \{18, 36\}$ and so G is isomorphic to one of the following groups:

- $\mathbb{Z}_3 \times S_3$
- $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$
- $\mathbb{Z}_3 \times A_4$

By Proposition 3.1, the nilpotent graph of $\mathbb{Z}_3 \times S_3$ is toroidal. The groups $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$ and $\mathbb{Z}_3 \times A_4$ are *nn*-groups with each having four nilpotentizers of size 9 and one nilpotentizer of size 12. Thus by [5, Proposition 4.2], the genus of the nilpotent graph of each these groups is $4\gamma(K_6) + \gamma(K_9) = 7$.

(4). $|Z^*(G)| = 2$. Suppose $p \mid |G|$ and $p \geq 7$. Let $x \in G$ such that $\circ(x) = p$. Then, $< x, Z^*(G) >$ is a nilpotent subgroup of G of order at least 14, a contradiction. So, $|G| = 2^p 3^q 5^r$. Suppose $r \geq 2$ and $q \geq 2$. Let P be a sylow 3−subgroup or a sylow 5−subgroup of G. By [7, Lemma 3.13], any two elements of $PZ^*(G) \setminus Z^*(G)$ are adjacent to each other. Thus $\Gamma_{nil}(G)$ has a subgraph isomorphic to K_{11}, a contradiction. So, $r, q \in \{0, 1\}$. Suppose $p \geq 4$, then G has a nilpotent subgroup of order at least 16, a contradiction. So, $p \in \{0, 1, 2, 3\}$. Therefore, $|G| \in \{12, 20, 24, 30, 40, 60, 120\}$.

There are two groups of order 12 with hypercenter of size 2. Namely D_{12} and Q_{12} and the nilpotent graph of these groups are planar. There are two groups of order 20 with hypercenter of size 2. Namely D_{20} and Q_{20} and the nilpotent graph of these groups are double-toroidal. There are two groups of order 24 with hypercenter of size 2. Namely $SL(2, 3)$ and $\mathbb{Z}_2 \times A_4$ and the nilpotent graph of these groups are toroidal.

If $|G| = 40$, then since G cannot have an abelian subgroup of order 20, we have, G is isomorphic to one of the following groups:

- $\mathbb{Z}_5 \rtimes \mathbb{Z}_8$
- $\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$

The groups $\mathbb{Z}_5 \rtimes \mathbb{Z}_8$ and $\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ are *nn*-groups with five nilpotentizers of size 8 and one nilpotentizers of size 10. Thus by [5, Proposition 4.2], the genus of the nilpotent graph of each these groups is $5\gamma(K_6) + \gamma(K_8) = 7$.

By [4, Lemma 5.4], if $|G| = 30$ or if G is a solvable group of order 60 or 120, then G has a nilpotent subgroup of order 15. Thus G is non-solvable and so G is isomorphic to one of the following groups:

- $SL(2, 5)$
- $\mathbb{Z}_2 \times A_5$

The groups $SL(2,5)$ and $\mathbb{Z}_2 \times A_5$ are *nn*-groups with six nilpotentizers of size 10, five nilpotentizers of size 8 and ten nilpotentizers of size 6. Thus by [5, Proposition 4.2], the genus of the nilpotent graph of each these groups is $6\gamma(K_8) + 5\gamma(K_6) + 10\gamma(K_4) = 17$.

(5). $|Z^*(G)| = 1$. Suppose $p \mid |G|$, p a prime and $p \geq 13$. Let $x \in G$ such that $\circ(x) = p$. Then, $< x >$ is a nilpotent subgroup of G of order at least 13, a contradiction. So, $\mid G \mid= 2^p 3^q 5^r 7^s 11^t$. Suppose $p \geq 4$,$q \geq 3$,$r \geq 2$,$s \geq 2$ or $t \geq 2$, then G has a nilpotent subgroup of order at least 16, 27, 25, 49 or 121 respectively, a contradiction. By Lemma 3.7, if $|G| = 7m$, then $m = 2$, $m = 3$ or 6. Suppose $11 \mid |G|$. Let K be a sylow 11−subgroup of G and H be a sylow p'−subgroup of G, $p' = 2,3,5$. Then $\Gamma_{nil}(G)[K \setminus Z^*(G)]$ and $\Gamma_{nil}(G)[H \setminus Z^*(G)]$ are two distinct subgraphs of $\Gamma_{nil}(G)$. Thus $\gamma(\Gamma_{nil}(G)[K \setminus Z^*(G)]) + \gamma(\Gamma_{nil}(G)[H \setminus Z^*(G)]) \leq \gamma(\Gamma_{nil}(G)) = 4$. It follows that $\gamma(\Gamma_{nil}(G)[H \setminus Z^*(G)]) = 0$ and so $|H| \leq 5$. Therefore if $11 \mid |G|$, then $p \leq 2, q \leq 1, r \leq 1$ and $s = 0$. Note that groups of order 15 and 33 are nilpotent. Thus $|G| \in \{6, 10, 12, 14, 18, 20, 21, 22, 24, 30, 36, 40, 42, 44, 45, 55, 60, 66, 72, 90, 110, 120, 132, 165, 180, 220, 330, 360, 660\}$.

The nilpotent graphs of the non-nilpotent groups of order 6, 10 and 12 are planar. The nilpotent graphs of the non-nilpotent groups of order 14 and 21 are toroidal. The nilpotent graphs of the non-nilpotent groups of order 20 are either planar or double-toroidal.

There are two non-nilpotent groups of order 18 with trivial center, namely D_{18} and $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. The nilpotent graphs of both these groups are double-toroidal.

There is only one non-nilpotent group of order 22 with trivial center, namely D_{22} and by [5, Proposition 4.4] the nilpotent graph of this group is quadruple-toroidal.

There is only one non-nilpotent group of order 24 with trivial center, namely S_4 and by Lemma 3.4, $\gamma(\Gamma_{nil}(S_4)) = 4$.

There is only one non-nilpotent group of order 30 with trivial center, namely D_{30} and by [5, Proposition 4.4] the nilpotent graph is of genus 10.

There are two non-nilpotent group of order 36 with trivial center, namely $S_3 \times S_3$ and

$$\langle x, y : x^4 = y^3 = (yx^2)^2 = [x^{-1}yx, y] = 1 \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4.$$

The non-nilpotent graph of the group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ is double-toroidal. Let $G' = S_3 \times S_3$. Since $Z(G') = 1$, the commuting graph $\Gamma_c(G')$ is a subgraph of $\Gamma_{nil}(G')$ and so $\gamma(\Gamma_c(G')) \leq \gamma(\Gamma_{nil}(G'))$. By [6, Remark 3.3] $\gamma(\Gamma_c(G')) \geq 4$. Let $a = (1, 2)$, $b = (4, 5)$, $c = (4, 5, 6)$ and $d = (1, 2, 3)$. Let $A = \{c, c^2, a, ac, ac^2, ad, acd, ac^2d, ad^2, acd^2, ac^2d^2\}$, $B = \{c, d, cd, c^2d, cd^2, c^2d^2\}$ and $C = \{d, d^2, b, bd, bd^2, bc, bcd, bc^2d, bc^2, bcd^2, bc^2d^2\}$. Note that $A \cap B = \{c\}$ and $B \cap C = \{d\}$. If $A' = \Gamma_{nil}(G')[A]$, $B' = \Gamma_{nil}(G')[B]$ and $C' = \Gamma_{nil}(G')[C]$, then $A' \cong C' \cong 3K_3 \vee K_2$ and $B' \cong K_6$. By Remark 3.3, $\gamma(A') = \gamma(C') \geq 2$. The graph $A' \cup B' \cup C'$, as depicted in Figure 2, has three blocks A', B' and C'. Thus, $\gamma(G') \geq \gamma(A' \cup B' \cup C') = \gamma(A') + \gamma(B') + \gamma(C') \geq 2 + 1 + 2 = 5$.

There are no groups of order 40, 44 and 45 with trivial center.

There are two non-nilpotent group of order 42 with trivial center, namely D_{42} and $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. By [5, Proposition 4.4], the nilpotent graph of D_{42} has genus greater than 4. The group $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ has 7 nilpotent subgroups of size 6 and the intersection of these subgroups is the trivial subgroup. Thus the nilpotent graph of $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ is not quadruple-toroidal

There is only one non-nilpotent group of order 55 group with trivial center, namely $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$. This is an *nn*-group with eleven nilpotentizers of size 5 and one nilpotentizer of size 11. Thus by [5, Proposition 4.2], $\gamma(\Gamma_{nil}(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5)) = 11\gamma(K_4) + \gamma(K_{10}) = 4$. Hence the nilpotent graph of this group is quadruple-toroidal.

There are three non-nilpotent group of order 60 with trivial center, namely A_5, $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ and $S_3 \times D_{10}$. The nilpotent graph of A_5 is planar. The group $\mathbb{Z}_{15} \rtimes \mathbb{Z}_4$ and $S_3 \times D_{10}$ has a cyclic group of order 15 and so their nilpotent graphs are not quadruple-toroidal.

There is only one non-nilpotent group of order 66 group with trivial center, namely D_{66}. By [5, Proposition 4.4], the nilpotent graph of this group has genus greater than 4.

There are six non-nilpotent groups of order 72, namely $((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_2$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$, $(S_3 \times S_3) \rtimes \mathbb{Z}_2$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$, $(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ and $A_4 \times S_3$.

The group $((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_2$, $(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ and $A_4 \times S_3$ has an abelian subgroup of order 12. The group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8$ and $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$ are *nn*-groups with nine nilpotentizers of size 8 and one nilpotentizers

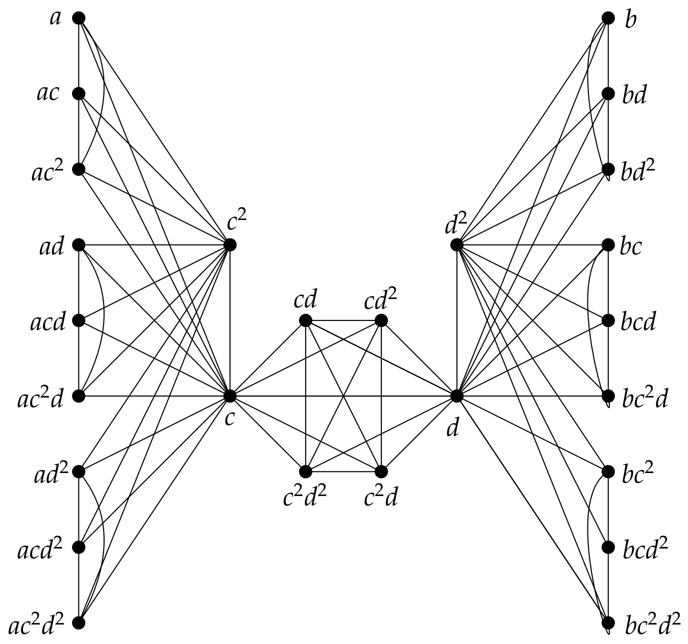


Figure 2: $A' \cup B' \cup C'$

of size 9. By [5, Proposition 4.2], the nilpotent graph of these groups has genus 12. The nilpotent graph of $S_3 \times S_3$ is a subgraph of the nilpotent graph of $(S_3 \times S_3) \rtimes \mathbb{Z}_2$ and thus $\gamma(\Gamma_{nil}((S_3 \times S_3) \rtimes \mathbb{Z}_2)) \geq \gamma(\Gamma_{nil}(S_3 \times S_3)) \geq 5$.

There are two non-nilpotent groups of order 90 with trivial center, namely D_{90} and $(\mathbb{Z}_{15} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. Both these groups has an abelian subgroup of order 45 and so their nilpotent graphs are not quadruple-toroidal.

There are two non-nilpotent groups of order 110 with trivial center, namely $(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$ and D_{110}. The group $(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$ is an *nn*-group with eleven nilpotentizers of size 10 and one nilpotentizer of size 11. Thus, by [5, Proposition 4.2] the nilpotent graph of this group is of genus 37. By [5, Proposition 4.4], the nilpotent graph of D_{110} is of genus 23.

Let G be a group of order 120 with trivial center. If G is solvable, then by Theorem of Hall (see [8, Theorem 5.28]), G has a subgroup of order 15, which is abelian. Thus G is not solvable and so $G \cong S_5$. The group S_5 has 10 abelian subgroups of order 6 and the intersection of any two of these subgroups is trivial. Thus the nilpotent graph of S_5 is not quadruple-toroidal.

There is only one non-nilpotent groups of order 132 with trivial center, namely $S_3 \times D_{22}$ and this group has an abelian subgroup of order 33. Thus its nilpotent graph is not quadruple-toroidal.

There are no non-nilpotent groups of order 165 with trivial center.

Let G be a group of order 180 with trivial center. Then G is solvable. So by [8, Theorem 5.28], G has a subgroup of order 45 which is abelian. Thus the nilpotent graph of G is not quadruple-toroidal.

Let G be a group of order 220. Then G is solvable and so G has a subgroup H of order 55. Thus $H \cong \mathbb{Z}_{55}$ or $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$. It follows that $\gamma(\Gamma_{nil}(G)) \geq \gamma(\Gamma_{nil}(G)[H \setminus \{1\}] > 4$. Thus the nilpotent graph of G is not quadruple-toroidal.

Let G be a group of order 330 with trivial center. Then G is solvable. Thus by [8, Theorem 5.28], G has a subgroup of order 15, which is abelian. Thus the nilpotent graph of G is not quadruple-toroidal.

Let G be a group of order 360. If G is solvable, then by [8, Theorem 5.28], G has a subgroup of order 45 which is abelian. It follows that G is not solvable. There are three non-solvable group of order 360 with trivial center, namely A_6, $GL(2,4) \rtimes \mathbb{Z}_2$ and $A_5 \times S_3$. The group $GL(2,4) \rtimes \mathbb{Z}_2$ and $A_5 \times S_3$ has an abelian subgroup of order 15. The group A_6 has three subgroups of order 9 whose intersection of any two is the trivial subgroup, namely $A = \langle(1,2,3),(4,5,6)\rangle$, $B = \langle(1,2,6),(3,4,5)\rangle$ and $C = \langle(1,2,5),(3,4,6)\rangle$. Thus $\gamma(\Gamma_{nil}(A_6)) \geq \gamma(\Gamma_{nil}(A_6)[(A \cup B \cup C) \setminus \{1\}]) \geq 6$. Thus $\Gamma_{nil}(A_6)$ is not quadruple-toroidal.

Let G be a group of order 660. If G is solvable then by [8, Theorem 5.28], G has a subgroup of order 15, which is abelian. Thus G is non-solvable and so $G \cong PSL(2,11)$. This group has 12 subgroups of order 11. Thus $\gamma(\Gamma_{nil}(PSL(2,11))) \geq 12\gamma(K_{10}) = 48$. Thus $\Gamma_{nil}(PSL(2,11))$ is not quadruple-toroidal.

This completes the proof. □

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