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Analyzing Boole's type inequalities for general convex functions via generalized fractional integrals with their computational analysis and applications

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Abstract. Fractional calculus extends the differentiation and integration of functions to non-integer order. This work presents a new identity to represent specific differentiable mappings through generalized fractional integrals. On the basis of the newly established identity, several Boole's type inequalities for differentiable generalized convex functions are obtained. Generalized fractional integrals are more versatile as compared to the traditional integral operators because they embody them. This method brings the connection between integer order calculus and fractional calculus, which gives more effective tools for solving non-singular problems where integer order calculus tools can be ineffective. The results provide further understanding of geometric properties of differentiable mappings and generalized convex functions which give rise to new identities and inequalities enriching the field of integral inequalities. Some numerical examples and applications are given in order to support these results as more feasible and relevant.

1. Introduction

In 1915, the term "numerical integration" first appeared in a publication under the title "A Course in Interpolation and Numerical Integration for the Mathematical Laboratory" by David Gibb [9]. In recent decades, numerical integration has become essential in scientific computing, engineering, and data analysis. Advanced techniques such as adaptive quadrature algorithms, numerical integration with error estimation, and high-dimensional integration methods have been developed to handle increasingly complex problems. In numerical integration, "quadrature" refers to the calculation of area and has historical mathematical significance. Constructing various interpolating polynomials allows one to generate a broad class of quadrature rules. Assuming a constant interpolating function (zero degree polynomial) is among the most straightforward techniques of this kind. The Midpoint or rectangle rule is thus the term used to describe it. If the interpolating function is linear (i.e., a straight line), we obtain the Trapezoidal rule. Simpson's rule applies in the case of a second-degree interpolating polynomial. Thomas Simpson, a mathematician, is credited with giving it the name Simpson's rule (1710–1761). Simpson's 1/3 rule is the most fundamental

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numerical method. In numerical integration with lower error bounds, the third Simpson's rule or Simpson's 2/45 rule, also called Boole's rule, is named after George Boole.

$$\int_{\alpha}^{\beta} F(\xi)\,d\xi = \frac{2h}{45}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]+E(F), \tag{1}$$

where $E(F)$ is an error term. One can see [8, 15] to learn more about numerical integration and its applications. The error bound for Boole's rule approximation is described as

$$E(F) = -\frac{8(\beta-\alpha)^{8}}{945}M,\ \left|F^{(vi)}(\xi)\right| \le M.$$

The history of convex functions is intertwined with mathematics and optimization theory development. Convexity can be traced back to ancient civilizations, where geometric ideas were explored. Early mathematicians, such as Euclid (300 BC), studied the properties of convex shapes like circles and triangles. However, the formalization of convex functions came much later. The study of convex functions began to take shape during this period. Mathematicians like Leonhard Euler (1707-1783) and Johann Radon (1887-1956) contributed significantly to understanding convex sets and functions. Euler, in particular, explored the properties of convex curves. The rigorous study of convex functions gained momentum with the development of convex analysis. Mathematicians like Hermann Minkowski (1864-1909) and Constantin Carathéodory (1873-1950) made foundational contributions to convex geometry and convex optimization theory. Convex optimization has become a cornerstone of applied mathematics, with applications spanning various fields such as engineering, economics, computer science, and machine learning. The development of efficient algorithms for solving convex optimization problems, such as interior-point methods have fueled its widespread adoption. For more details about the history of convexity, one can visit [5, 8, 15]. Convex set and function are defined as:

Definition 1.1 (Convex Set). *[5] A set $I \subset \mathbb{R}^n$ is convex, if for any two points $\alpha, \beta \in I$, the entire segment joining α and β lies in I. The points in the segment are of the form*

$$\delta\alpha + (1-\delta)\beta \in I,\ \forall \delta \in [0,1]. \tag{2}$$

Definition 1.2 (Convex Function). *[5] Let I be a convex subset of a real vector space and $F : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if*

$$\delta F(\alpha) + (1-\delta)F(\beta) \ge F(\delta\alpha + (1-\delta)\beta), \tag{3}$$

for all $\delta \in [0,1]$ and $\alpha, \beta \in I$.

The Hermite-Hadamard inequality was first presented by French mathematicians Charles Hermite (1822–1901) and Jacques Salomon Hadamard (1865–1963). C. Hermite and J. S. Hadamard made major contributions to mathematics in the fields of inequality theory, complex analysis, and much more; to learn more about these [16, 21]. The inequality states that: If a function $F : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ is convex then

$$\frac{F(\alpha)+F(\beta)}{2} \ge \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} F(\xi)d\xi \ge F\left(\frac{\alpha+\beta}{2}\right). \tag{4}$$

If the function F is concave, then the afore-mentioned inequalities hold in the opposite direction. Put another way, a function is convex if and only if its weighted average of functional values at its endpoints is larger than or equal to its value at the middle of any interval containing a set of real numbers. Akber *et al.* proved the generalization of quantum calculus and corresponding Hermite–Hadamard inequalities in [1].

Breckner was the first mathematician to introduce an s-convex function in 1979 [4], and the number of connections with s-convexity in the first sense is negotiated in [12]. Direct proof of Breckner's result was esteemed in 2001 by Pycia [19]. Many scholars primarily concentrated on s-convex functions because of the importance of convexity and s-convexity in studying optimality to resolve mathematical programming. For instance, earlier works by H. Hudzik *et al.* [12] presented two kinds of s-convexity $\{s \in (0,1]\}$. And demonstrated that whenever $\{s \in (0,1]\}$, the second sense is fundamentally stronger than the s-convexity in the first sense. In the second sense, we use the s-convexity of a function generally known as the s-convex function. Since $s \in (0,1]$, this class of functions is more important than the convex. We also observe in the main section that the results obtained by s-convexity are much better than the convexity. Secondly, s-convexity is the generalization of a convex function, so we can obtain the results for convex functions by using $s = 1$ in the results of s-convex functions.

Definition 1.3 (s-convex function). *[12] A function $F : [0,\infty) \to \mathbb{R}$ is said to be s-convex, if*

$$\delta^s F(\alpha) + (1-\delta)^s F(\beta) \geq F(\delta\alpha + (1-\delta)\beta), \tag{5}$$

for all $\delta \in [0,1]$, $s \in (0,1]$ and $\alpha, \beta \in [0,\infty)$.

S. S. Dragomir and Fitzpatrick also introduced Hadamard inequality for s-convex functions in the second sense in [6].

Theorem 1.4. *[6] Suppose that $F : [0,\infty) \subset \mathbb{R} \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1]$. If $F \in L_1[\alpha,\beta]$, then the following inequality holds:*

$$\frac{F(\alpha)+F(\beta)}{s+1} \geq \frac{1}{\beta-\alpha}\int_\alpha^\beta F(\xi)d\xi \geq 2^{s-1}F\left(\frac{\alpha+\beta}{2}\right). \tag{6}$$

Fractional calculus, an extension of classical calculus, focuses on the derivatives and integrals of non-integer orders. The formal development of fractional calculus started in the 18th century, but its origins may be seen in the work of mathematicians such as Leibniz and Euler. The Italian mathematician Joseph Liouville made significant contributions in the mid-19th century, introducing the notion of fractional derivatives. One of the key figures in advancing fractional calculus was the Polish mathematician Stanisław Saks, who introduced the concept of fractional integrals in the 1920s. He extended the classical integral operators to non-integer orders, providing a powerful tool for solving various differential equations and addressing problems in mathematical analysis. The notion of generalized fractional integral emerged as a natural extension of Saks' work, aiming to generalize integral operators to a broader class of functions and orders. This extension was motivated by the need to address more complex problems in mathematical analysis where classical integral operators were inadequate. The generalized fractional integrals are defined by Sarikaya and Ertugral in [22].

Let us define a function $\Psi : [0,\infty) \to [0,\infty)$ satisfying the following condition:

$$\int_0^1 \frac{\Psi(\delta)}{\delta} d\delta < \infty. \tag{7}$$

We examine these generalized fractional integral operators on the left and right sides.

$${}_\alpha I_\Psi F(\xi) = \int_\alpha^\xi \frac{\Psi(\xi-\delta)}{\xi-\delta} F(\delta)\, d\delta, \ \xi > \alpha, \tag{8}$$

and

$${}_\beta I_\Psi F(\xi) = \int_\xi^\beta \frac{\Psi(\delta-\xi)}{\delta-\xi} F(\delta)\, d\delta, \ \beta > \xi, \tag{9}$$

respectively. The key feature of generalized fractional integrals is their ability to extend various forms of fractional integrals, including Hadamard, Katugampola, conformable and Riemann-Liouville fractional integrals, among others. These important special cases of the integral operators (8) and (9) are mentioned below:

(1) If we choose $\Psi(\delta) = \delta$, the operators (8) and (9) reduce to the Riemann integral.

(2) Considering $\Psi(\delta) = \frac{\delta^{\varpi}}{\Gamma(\varpi)}$ and $\varpi > 0$, the operators (8) and (9) reduce to the Riemann-Liouville fractional integrals $J_{\alpha}^{\varpi}F(\xi)$ and $J_{\beta}^{\varpi}F(\xi)$ respectively. Here Γ is a Gamma function. Numerous articles have been published on inequalities involving generalized fractional integrals. Sarikaya and Ertuğral proved Hermite-Hadamard inequalities for these integrals in [22]. Additionally, Budak *et al.* established Midpoint-type inequalities in [2] and extended Hermite-Hadamard inequalities in [3]. Ertuğral and Sarikaya also presented Simpson-type inequalities for these fractional integral operators in [7]. Furthermore, Kara *et al.* proved Simpson-type inequalities for convex functions using generalized fractional integrals in [13]. Haider *et al.* recently proved to analyze Milne-type inequalities by using tempered fractional integrals in [10]. Meftah *et al.* in [17] proved some local fractional Maclaurin-type inequalities for generalized convex functions and their applications. For more information, readers are referred to [11, 14, 18, 20, 23, 24].

Inspired by the previously mentioned literature, we use the generalized fractional integral to establish a novel identity for differentiable mapping. Using generalized fractional integral, we show Boole's type inequality for differentiable general convex functions and analyze several exceptional examples using the recently obtained identity. We provide numerical examples to verify the accuracy of recently established results.

The article is structured as follows: Section 2 introduces a new identity for differentiable mappings using a generalized fractional integral. We use this identity to establish several Boole-type inequalities for differentiable general convex functions through generalized fractional integrals. Section 3 includes numerical examples and computational analyses to validate the newly established results. In Section 4, a discussion about graphical behaviour is given. In Section 5, we discuss applications to the quadrature formula. Finally, Section 6 offers conclusions and future research.

2. Main Results

This section uses a generalized fractional integral to provide a new identity for differentiable mapping. We utilize the recently obtained identity; we show several Boole's type inequalities for differentiable general convex functions using a generalized fractional integral. For conciseness, throughout this work, we define $\Phi : [0,\infty) \to [0,\infty)$ satisfying the following:

$$\Phi(\delta) := \int_0^{\delta} \frac{\Psi\left(\frac{\beta-\alpha}{4}u\right)}{u} du.$$

Lemma 2.1. *Assume that $F : [\alpha,\beta] \to \mathbb{R}$ is differentiable on (α,β). If $F \in L_1[\alpha,\beta]$, then we have the following identity for generalized fractional integral:*

$$\begin{aligned}
&\frac{1}{90}\left[7F(\alpha) + 32F\left(\frac{3\alpha+\beta}{4}\right) + 12F\left(\frac{\alpha+\beta}{2}\right) + 32F\left(\frac{\alpha+3\beta}{4}\right) + 7F(\beta)\right] \\
&-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_{\Phi}F(\alpha) + {}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_{\Phi}F\left(\frac{3\alpha+\beta}{4}\right) + {}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_{\Phi}F\left(\frac{\alpha+\beta}{2}\right) + {}_{\beta^-}I_{\Phi}F\left(\frac{\alpha+3\beta}{4}\right)\right] \\
= &\frac{\beta-\alpha}{16\Phi(1)}\left[I_1 + I_2 + I_3 + I_4\right],
\end{aligned} \tag{10}$$

where

$$I_1 := \int_0^1 \left(\Phi(\delta) - \frac{28}{90}\Phi(1)\right)F'\left(\frac{4-\delta}{4}\alpha + \frac{\delta}{4}\beta\right)d\delta,$$

$$I_2 := \int_0^1 \left(\Phi(\delta) - \frac{44}{60}\Phi(1)\right)F'\left(\frac{3-\delta}{4}\alpha + \frac{1+\delta}{4}\beta\right)d\delta,$$

$$I_3 := \int_0^1 \left(\Phi(\delta) - \frac{24}{90}\Phi(1)\right)F'\left(\frac{2-\delta}{4}\alpha + \frac{2+\delta}{4}\beta\right)d\delta,$$

and

$$I_4 := \int_0^1 \left(\Phi(\delta) - \frac{62}{90}\Phi(1)\right) F'\left(\frac{1-\delta}{4}\alpha + \frac{3+\delta}{4}\beta\right) d\delta.$$

Proof. Solve each integral I_1, I_2, I_3, *and* I_4 using integration by parts, and multiply each of them by $(\beta-\alpha)$, we have

$$\begin{aligned} I_1 &= \int_0^1 \left(\Phi(\delta) - \frac{28}{90}\Phi(1)\right) F'\left(\frac{4-\delta}{4}\alpha + \frac{\delta}{4}\beta\right) d\delta \\ &= 4\left(\left(\Phi(\delta) - \frac{28}{90}\Phi(1)\right) F\left(\frac{4-\delta}{4}\alpha + \frac{\delta}{4}\beta\right)\right)\Big|_0^1 - 4\int_0^1 \left(\frac{\Psi\left(\frac{\beta-\alpha}{4}\delta\right)}{\delta}\right) F\left(\frac{4-\delta}{4}\alpha + \frac{\delta}{4}\beta\right) d\delta \\ &= \frac{4\times 62\Phi(1)}{90} F\left(\frac{3\alpha+\beta}{4}\right) + \frac{4\times 28\Phi(1)}{90} F(\alpha) - 4\int_0^1 \left(\frac{\Psi\left(\frac{\beta-\alpha}{4}\delta\right)}{\delta}\right) F\left(\frac{4-\delta}{4}\alpha + \frac{\delta}{4}\beta\right) d\delta, \end{aligned} \tag{11}$$

$$\begin{aligned} I_2 &= \int_0^1 \left(\Phi(\delta) - \frac{44}{60}\Phi(1)\right) F'\left(\frac{3-\delta}{4}\alpha + \frac{1+\delta}{4}\beta\right) d\delta \\ &= 4\left(\left(\Phi(\delta) - \frac{44}{60}\Phi(1)\right) F\left(\frac{3-\delta}{4}\alpha + \frac{1+\delta}{4}\beta\right)\right)\Big|_0^1 - 4\int_0^1 \left(\frac{\Psi\left(\frac{\beta-\alpha}{4}\delta\right)}{\delta}\right) F\left(\frac{3-\delta}{4}\alpha + \frac{1+\delta}{4}\beta\right) d\delta \\ &= \frac{4\times 16\Phi(1)}{60} F\left(\frac{\alpha+\beta}{2}\right) + \frac{4\times 44\Phi(1)}{60} F\left(\frac{3\alpha+\beta}{4}\right) - 4\int_0^1 \left(\frac{\Psi(4\delta)}{\delta}\right) F\left(\frac{3-\delta}{4}\alpha + \frac{1+\delta}{4}\beta\right) d\delta, \end{aligned} \tag{12}$$

$$\begin{aligned} I_3 &= \int_0^1 \left(\Phi(\delta) - \frac{24}{90}\Phi(1)\right) F'\left(\frac{2-\delta}{4}\alpha + \frac{2+\delta}{4}\beta\right) d\delta \\ &= 4\left(\left(\Phi(\delta) - \frac{24}{90}\Phi(1)\right) F\left(\frac{2-\delta}{4}\alpha + \frac{2+\delta}{4}\beta\right)\right)\Big|_0^1 - 4\int_0^1 \left(\frac{\Psi\left(\frac{\beta-\alpha}{4}\delta\right)}{\delta}\right) F\left(\frac{2-\delta}{4}\alpha + \frac{2+\delta}{4}\beta\right) d\delta \\ &= \frac{4\times 66\Phi(1)}{90} F\left(\frac{\alpha+3\beta}{4}\right) + \frac{4\times 24\Phi(1)}{90} F\left(\frac{\alpha+\beta}{2}\right) - 4\int_0^1 \left(\frac{\Psi\left(\frac{\beta-\alpha}{4}\delta\right)}{\delta}\right) F\left(\frac{2-\delta}{4}\alpha + \frac{2+\delta}{4}\beta\right) d\delta, \end{aligned} \tag{13}$$

and

$$\begin{aligned} I_4 &= \int_0^1 \left(\Phi(\delta) - \frac{62}{90}\Phi(1)\right) F'\left(\frac{1-\delta}{4}\alpha + \frac{3+\delta}{4}\beta\right) d\delta \\ &= 4\left(\left(\Phi(\delta) - \frac{62}{90}\Phi(1)\right) F\left(\frac{1-\delta}{4}\alpha + \frac{3+\delta}{4}\beta\right)\right)\Big|_0^1 - 4\int_0^1 \left(\frac{\Psi\left(\frac{\beta-\alpha}{4}\delta\right)}{\delta}\right) F\left(\frac{1-\delta}{4}\alpha + \frac{3+\delta}{4}\beta\right) d\delta \\ &= \frac{4\times 28\Phi(1)}{90} F(\beta) + \frac{4\times 62\Phi(1)}{90} F\left(\frac{\alpha+3\beta}{4}\right) - 4\int_0^1 \left(\frac{\Psi\left(\frac{\beta-\alpha}{4}\delta\right)}{\delta}\right) F\left(\frac{1-\delta}{4}\alpha + \frac{3+\delta}{4}\beta\right) d\delta. \end{aligned} \tag{14}$$

Adding (11) to (14) also by change of variable and multiplying by $\frac{1}{16\phi(1)}$, we get the desired identity. The proof of Lemma 2.1 is completed. □

Theorem 2.2. *Assume that $F : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable on $I^\circ = (\alpha,\beta)$ such that $F' \in L_1[\alpha,\beta]$, where $\alpha,\beta \in I^\circ$. If $|F'|$ is s-convex on $[\alpha,\beta]$ for some fixed $s \in (0,1]$, then we have the following inequality:*

$$\left|\frac{1}{90}\left[7F(\alpha) + 32F\left(\frac{3\alpha+\beta}{4}\right) + 12F\left(\frac{\alpha+\beta}{2}\right) + 32F\left(\frac{\alpha+3\beta}{4}\right) + 7F(\beta)\right]\right.$$

$$-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\Bigg|$$

$$\leq \frac{\beta-\alpha}{16\Phi(1)}\left[\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|\left(\left(\frac{4-\delta}{4}\right)^s|F'(\alpha)|+\left(\frac{\delta}{4}\right)^s|F'(\beta)|\right)d\delta\right.$$

$$+\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|\left(\left(\frac{3-\delta}{4}\right)^s|F'(\alpha)|+\left(\frac{1+\delta}{4}\right)^s|F'(\beta)|\right)d\delta$$

$$+\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|\left(\left(\frac{2-\delta}{4}\right)^s|F'(\alpha)|+\left(\frac{2+\delta}{4}\right)^s|F'(\beta)|\right)d\delta$$

$$\left.+\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|\left(\left(\frac{1-\delta}{4}\right)^s|F'(\alpha)|+\left(\frac{3+\delta}{4}\right)^s|F'(\beta)|\right)d\delta\right].$$

Proof. Taking absolute value of Lemma 2.1, we have

$$\Bigg|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]$$

$$-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\Bigg|$$

$$\leq \frac{(\beta-\alpha)}{16\Phi(1)}\left[\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|\left|F'\left(\frac{4-\delta}{4}\alpha+\frac{\delta}{4}\beta\right)\right|d\delta\right.$$

$$+\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|\left|F'\left(\frac{3-\delta}{4}\alpha+\frac{1+\delta}{4}\beta\right)\right|d\delta$$

$$+\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|\left|F'\left(\frac{2-\delta}{4}\alpha+\frac{2+\delta}{4}\beta\right)\right|d\delta$$

$$\left.+\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|\left|F'\left(\frac{1-\delta}{4}\alpha+\frac{3+\delta}{4}\beta\right)\right|d\delta\right].$$

Since $|F'|$ is s-convex on $[\alpha,\beta]$, we get

$$\Bigg|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]$$

$$-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\Bigg|$$

$$\leq \frac{\beta-\alpha}{16\Phi(1)}\left[\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|\left(\left(\frac{4-\delta}{4}\right)^s|F'(\alpha)|+\left(\frac{\delta}{4}\right)^s|F'(\beta)|\right)d\delta\right.$$

$$+\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|\left(\left(\frac{3-\delta}{4}\right)^s|F'(\alpha)|+\left(\frac{1+\delta}{4}\right)^s|F'(\beta)|\right)d\delta$$

$$+\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|\left(\left(\frac{2-\delta}{4}\right)^s|F'(\alpha)|+\left(\frac{2+\delta}{4}\right)^s|F'(\beta)|\right)d\delta$$

$$\left.+\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|\left(\left(\frac{1-\delta}{4}\right)^s|F'(\alpha)|+\left(\frac{3+\delta}{4}\right)^s|F'(\beta)|\right)d\delta\right].$$

Hence, the proof of Theorem 2.2 is completed. □

Corollary 2.3. *If we replace $\Psi(\delta)=\delta$ in Theorem 2.2, we get the following Boole's type inequality:*

$$\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]-\frac{1}{(\beta-\alpha)}\int_\alpha^\beta F(\xi)\,d\xi\right|$$

$$\begin{aligned}
&\leq \frac{(\beta-\alpha)\times\left[|F'(\alpha)|+|F'(\beta)|\right]}{16(s+1)(s+2)}\Big\{2^{1-2s}45^{-s-2}\left(45^{s+1}\left(3^{s+1}+7\times4^{s+1}-1\right)s-19\times4^{s+2}45^{s+1}\right.\\
&\left.-47\times45^{s+1}-133\times135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)\Big\}.
\end{aligned}$$

Remark 2.4. *If we replace $\Psi(\delta)=\delta$ and $s=1$ in Theorem 2.2, we get classical Boole's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]-\frac{1}{(\beta-\alpha)}\int_\alpha^\beta F(\xi)\,d\xi\right|\\
&\leq \frac{239(\beta-\alpha)}{6480}\left[|F'(\alpha)|+|F'(\beta)|\right].
\end{aligned}$$

Corollary 2.5. *If we replace $\Psi(\delta)=\frac{\delta^{\varpi}}{\Gamma(\varpi)}$ and $s=1$ in Theorem 2.2, we get the following Boole's type inequality for Riemann–Liouville fractional integrals:*

$$\begin{aligned}
&\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]\right.\\
&\left.-\frac{2^{\varpi-1}\Gamma(\varpi+1)}{(\beta-\alpha)^{\varpi}}\left[J^{\varpi}_{\left(\frac{3\alpha+\beta}{4}\right)^-}F(\alpha)+J^{\varpi}_{\left(\frac{\alpha+\beta}{2}\right)^-}F\left(\frac{3\alpha+\beta}{4}\right)+J^{\varpi}_{\left(\frac{\alpha+3\beta}{4}\right)^-}F\left(\frac{\alpha+\beta}{2}\right)+J^{\varpi}_{\beta^-}F\left(\frac{\alpha+3\beta}{4}\right)\right]\right|\\
&\leq \frac{(\beta-\alpha)}{64(\varpi+1)(\varpi+2)}\Big[|F'(\alpha)|\Big\{45^{-\frac{\varpi+2}{\varpi}}\Big(-31^{\frac{\varpi+2}{\varpi}}\varpi(\varpi+1)+26\times3^{\frac{\varpi+4}{\varpi}}5^{\frac{\varpi+2}{\varpi}}+2\times31^{\frac{1}{\varpi}+1}45^{1/\varpi}\varpi(\varpi+2)\\
&-\varpi\Big(-2^{\frac{1}{\varpi}+4}7^{\frac{1}{\varpi}+1}45^{1/\varpi}\varpi+11\times3^{\frac{\varpi+4}{\varpi}}5^{\frac{\varpi+2}{\varpi}}\varpi+12^{\frac{\varpi+2}{\varpi}}\varpi+14^{\frac{\varpi+2}{\varpi}}\varpi+9^{\frac{\varpi+2}{\varpi}}25^{\frac{1}{\varpi}+1}-2^{\frac{1}{\varpi}+5}7^{\frac{1}{\varpi}+1}45^{1/\varpi}\\
&+12^{\frac{\varpi+2}{\varpi}}+14^{\frac{\varpi+2}{\varpi}}+33^{\frac{\varpi+2}{\varpi}}(\varpi+1)-2\times3^{\frac{3}{\varpi}+2}5^{1/\varpi}11^{\frac{1}{\varpi}+1}(\varpi+2)\Big)\\
&+48\varpi(\varpi+2)\left(\sinh\left(\frac{\log(540)}{\varpi}\right)+\cosh\left(\frac{\log(540)}{\varpi}\right)\right)\Big)\Big\}\\
&+|F'(\beta)|\Big\{45^{-\frac{\varpi+2}{\varpi}}\Big(2\times3^{\frac{\varpi+3}{\varpi}}5^{1/\varpi}\varpi\left(11^{\frac{1}{\varpi}+1}\varpi+2^{\frac{2}{\varpi}+3}(\varpi+2)\right)+22\times3^{\frac{\varpi+4}{\varpi}}5^{\frac{\varpi+2}{\varpi}}\\
&+\varpi\Big(-13\times3^{\frac{\varpi+4}{\varpi}}5^{\frac{\varpi+2}{\varpi}}\varpi+12^{\frac{\varpi+2}{\varpi}}\varpi+14^{\frac{\varpi+2}{\varpi}}\varpi+33^{\frac{\varpi+2}{\varpi}}\varpi+4\times3^{\frac{\varpi+3}{\varpi}}5^{1/\varpi}11^{\frac{1}{\varpi}+1}-3^{\frac{4}{\varpi}+3}5^{\frac{\varpi+2}{\varpi}}\\
&+12^{\frac{\varpi+2}{\varpi}}+14^{\frac{\varpi+2}{\varpi}}+33^{\frac{\varpi+2}{\varpi}}+31^{\frac{\varpi+2}{\varpi}}(\varpi+1)+2\times3^{\frac{\varpi+2}{\varpi}}5^{1/\varpi}31^{\frac{1}{\varpi}+1}(\varpi+2)\Big)\Big)\Big\}\Big].
\end{aligned}$$

Theorem 2.6. *Assume that $F: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable on I° such that $F'\in L_1[\alpha,\beta]$, where $\alpha,\beta\in I^\circ$. If $|F'|^q$ is s-convex on $[\alpha,\beta]$ for some fixed $s\in(0,1]$ and $q>1$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]\right.\\
&\left.-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\right|\\
&\leq \frac{(\beta-\alpha)}{16\Phi(1)}\left[\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\frac{4-3^{1+s}\times4^{-s}}{1+s}|F'(\alpha)|^q+\frac{4^{-s}}{1+s}|F'(\beta)|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(-2^{1+s}+3^{1+s}\right)}{1+s}|F'(\alpha)|^q+\frac{4^{-s}\left(-1+2^{1+s}\right)}{1+s}|F'(\beta)|^q\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(-1+2^{1+s}\right)}{1+s}|F'(\alpha)|^q+\frac{4^{-s}\left(-2^{1+s}+3^{1+s}\right)}{1+s}|F'(\beta)|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1 \left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\frac{4^{-s}}{1+s}\left|F'(\alpha)\right|^q+\frac{4-3^{1+s}\times 4^{-s}}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}\Bigg],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\Bigg|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]\\
&-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\Bigg|\\
\le\ &\frac{\beta-\alpha}{16\Phi(1)}\Bigg[\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left(\frac{4-\delta}{4}\alpha+\frac{\delta}{4}\beta\right)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left(\frac{3-\delta}{4}\alpha+\frac{1+\delta}{4}\beta\right)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left(\frac{2-\delta}{4}\alpha+\frac{2+\delta}{4}\beta\right)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left|F'\left(\frac{1-\delta}{4}\alpha+\frac{3+\delta}{4}\beta\right)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Since $|F'|^q$ is s-convex function on $[\alpha,\beta]$, we get

$$\begin{aligned}
&\Bigg|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]\\
&-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\Bigg|\\
\le\ &\frac{\beta-\alpha}{16\Phi(1)}\Bigg[\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{4-\delta}{4}\right)^s\left|F'(\alpha)\right|^q d\delta+\int_0^1\left(\frac{\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{3-\delta}{4}\right)^s\left|F'(\alpha)\right|^q d\delta+\int_0^1\left(\frac{1+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{2-\delta}{4}\right)^s\left|F'(\alpha)\right|^q d\delta+\int_0^1\left(\frac{2+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1-\delta}{4}\right)^s\left|F'(\alpha)\right|^q d\delta+\int_0^1\left(\frac{3+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg]\\
=\ &\frac{\beta-\alpha}{16\Phi(1)}\Bigg[\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\frac{4-3^{1+s}\times 4^{-s}}{1+s}\left|F'(\alpha)\right|^q+\frac{4^{-s}}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(-2^{1+s}+3^{1+s}\right)}{1+s}\left|F'(\alpha)\right|^q+\frac{4^{-s}\left(-1+2^{1+s}\right)}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(-1+2^{1+s}\right)}{1+s}\left|F'(\alpha)\right|^q+\frac{4^{-s}\left(-2^{1+s}+3^{1+s}\right)}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1 \left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|^p d\delta\right)^{\frac{1}{p}}\left(\frac{4^{-s}}{1+s}\left|F'(\alpha)\right|^q+\frac{4-3^{1+s}\times 4^{-s}}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}\Bigg].$$

Hence, the proof of Theorem 2.6 is completed. □

Corollary 2.7. *If we replace $\Phi(\delta)=\delta$ in Theorem 2.6, we get the following Boole's type of inequality:*

$$\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]-\frac{1}{(\beta-\alpha)}\int_\alpha^\beta F(\xi)\,d\xi\right|$$

$$\leq \frac{(\beta-\alpha)}{16}\Bigg[\left(\frac{45^{-1-p}\left(14^{1+p}+31^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\left(\frac{4-3^{1+s}\times 4^{-s}}{1+s}\left|F'(\alpha)\right|^q+\frac{4^{-s}}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{15^{-1-p}\left(14^{1+p}+11^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(-2^{1+s}+3^{1+s}\right)}{1+s}\left|F'(\alpha)\right|^q+\frac{4^{-s}\left(-1+2^{1+s}\right)}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{15^{-1-p}\left(14^{1+p}+11^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\left(\frac{4^{-s}\left(-1+2^{1+s}\right)}{1+s}\left|F'(\alpha)\right|^q+\frac{4^{-s}\left(-2^{1+s}+3^{1+s}\right)}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{45^{-1-p}\left(14^{1+p}+31^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\left(\frac{4^{-s}}{1+s}\left|F'(\alpha)\right|^q+\frac{4-3^{1+s}\times 4^{-s}}{1+s}\left|F'(\beta)\right|^q\right)^{\frac{1}{q}}\Bigg].$$

Corollary 2.8. *If we replace $\Psi(\delta)=\delta$ and $s=1$ in Theorem 2.6, we get the following Boole's type inequality:*

$$\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]-\frac{1}{(\beta-\alpha)}\int_\alpha^\beta F(\xi)\,d\xi\right|$$

$$\leq \frac{(\beta-\alpha)}{16}\Bigg[\left(\frac{45^{-1-p}\left(14^{1+p}+31^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\times\left(\frac{7\left|F'(\alpha)\right|^q+\left|F'(\beta)\right|^q}{8}\right)^{\frac{1}{q}}$$

$$+\left(\frac{15^{-1-p}\left(14^{1+p}+11^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\times\left(\frac{5\left|F'(\alpha)\right|^q+3\left|F'(\beta)\right|^q}{8}\right)^{\frac{1}{q}}$$

$$+\left(\frac{15^{-1-p}\left(14^{1+p}+11^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\times\left(\frac{3\left|F'(\alpha)\right|^q+5\left|F'(\beta)\right|^q}{8}\right)^{\frac{1}{q}}$$

$$+\left(\frac{45^{-1-p}\left(14^{1+p}+31^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\times\left(\frac{\left|F'(\alpha)\right|^q+7\left|F'(\beta)\right|^q}{8}\right)^{\frac{1}{q}}\Bigg].$$

Remark 2.9. *For different values of Ψ in Theorem 2.6, we can find other results as finding in Theorem 2.2.*

Theorem 2.10. *Assume that $F: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable on I° such that $F'\in L_1[\alpha,\beta]$, where $\alpha,\beta\in I^\circ$. If $|F'|^q$ is s-convex on $[\alpha,\beta]$ for some fixed $s\in(0,1]$ and $q\geq 1$, then we have the following inequality:*

$$\Bigg|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]$$

$$-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\Bigg|$$

$$
\begin{aligned}
\leq\ & \frac{\beta-\alpha}{16\Phi(1)}\left[\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|\left(\frac{4-\delta}{4}\right)^s|F'(\alpha)|^q d\delta+\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|\left(\frac{\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|\left(\frac{3-\delta}{4}\right)^s|F'(\alpha)|^q d\delta+\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|\left(\frac{1+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|\left(\frac{2-\delta}{4}\right)^s|F'(\alpha)|^q d\delta+\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|\left(\frac{2+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|\left(\frac{1-\delta}{4}\right)^s|F'(\alpha)|^q d\delta+\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|\left(\frac{3+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. From Lemma 2.1 and applying the power mean inequality, we get

$$
\begin{aligned}
&\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]\right.\\
&\left.-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\right|\\
\leq\ & \frac{\beta-\alpha}{16\Phi(1)}\left[\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|\left|F'\left(\frac{4-\delta}{4}\alpha+\frac{\delta}{4}\beta\right)\right|^q d\delta\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|\left|F'\left(\frac{3-\delta}{4}\alpha+\frac{1+\delta}{4}\beta\right)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|\left|F'\left(\frac{2-\delta}{4}\alpha+\frac{2+\delta}{4}\beta\right)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|\left|F'\left(\frac{1-\delta}{4}\alpha+\frac{3+\delta}{4}\beta\right)\right|^q d\delta\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Since $|F'|^q$ is s-convex function on $[\alpha,\beta]$, we get

$$
\begin{aligned}
&\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]\right.\\
&\left.-\frac{1}{(\beta-\alpha)\Phi(1)}\left[{}_{\left(\frac{3\alpha+\beta}{4}\right)^-}I_\Phi F(\alpha)+{}_{\left(\frac{\alpha+\beta}{2}\right)^-}I_\Phi F\left(\frac{3\alpha+\beta}{4}\right)+{}_{\left(\frac{\alpha+3\beta}{4}\right)^-}I_\Phi F\left(\frac{\alpha+\beta}{2}\right)+{}_{\beta^-}I_\Phi F\left(\frac{\alpha+3\beta}{4}\right)\right]\right|\\
&\leq\frac{\beta-\alpha}{16\Phi(1)}\left[\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}\right.
\end{aligned}
$$

$$\times\left(\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|\left(\frac{4-\delta}{4}\right)^s|F'(\alpha)|^q d\delta+\int_0^1\left|\Phi(\delta)-\frac{28}{90}\Phi(1)\right|\left(\frac{\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|\left(\frac{3-\delta}{4}\right)^s|F'(\alpha)|^q d\delta+\int_0^1\left|\Phi(\delta)-\frac{44}{60}\Phi(1)\right|\left(\frac{1+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|\left(\frac{2-\delta}{4}\right)^s|F'(\alpha)|^q d\delta+\int_0^1\left|\Phi(\delta)-\frac{24}{90}\Phi(1)\right|\left(\frac{2+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|^q d\delta\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|\left(\frac{1-\delta}{4}\right)^s|F'(\alpha)|^q d\delta+\int_0^1\left|\Phi(\delta)-\frac{62}{90}\Phi(1)\right|\left(\frac{3+\delta}{4}\right)^s\left|F'(\beta)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg].$$

Hence, the proof of Theorem 2.10 is completed. □

Remark 2.11. *By special choices of Ψ in Theorem 2.10, one can obtain new Boole's type inequalities. These are left to the readers.*

3. Numerical Examples

In this section, we present numerical examples and their graphical analyses to illustrate the behaviour of the functions and validate the newly proved results.

Example 3.1. *Assume that $F(\xi)=\xi^{5s+1}$ is differentiable s-convex function on $[1,2]$ for all $s\in(0,1]$, then we have different cases:*

Case-i $\Psi(\delta)=\delta,\ s=0.1$;

$$\left|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]-\frac{1}{(\beta-\alpha)}\int_\alpha^\beta F(\xi)\,d\xi\right|$$
$$=\ 4.60394\times10^{-7}. \tag{15}$$

$$\frac{(\beta-\alpha)\times\left[|F'(\alpha)|+\left|F'(\beta)\right|\right]}{16(s+1)(s+2)}\Big\{2^{1-2s}45^{-s-2}\Big(45^{s+1}\Big(3^{s+1}+7\times4^{s+1}-1\Big)s-19\times4^{s+2}45^{s+1}$$
$$-47\times45^{s+1}-133\times135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\Big)\Big\}$$
$$=\ 0.0704243. \tag{16}$$

From (15) and (16) Case-i is verified.

Case-ii $\Psi(\delta)=\frac{\delta^{\varpi}}{\Gamma(\varpi)}$, for all $\varpi=1.2,\ s=1$.

$$\Bigg|\frac{1}{90}\left[7F(\alpha)+32F\left(\frac{3\alpha+\beta}{4}\right)+12F\left(\frac{\alpha+\beta}{2}\right)+32F\left(\frac{\alpha+3\beta}{4}\right)+7F(\beta)\right]$$
$$-\frac{2^{\varpi-1}\Gamma(\varpi+1)}{(\beta-\alpha)^{\varpi}}\left[J^{\varpi}_{\left(\frac{3\alpha+\beta}{4}\right)^-}F(\alpha)+J^{\varpi}_{\left(\frac{\alpha+\beta}{2}\right)^-}F\left(\frac{3\alpha+\beta}{4}\right)+J^{\varpi}_{\left(\frac{\alpha+3\beta}{4}\right)^-}F\left(\frac{\alpha+\beta}{2}\right)+J^{\varpi}_{\beta^-}F\left(\frac{\alpha+3\beta}{4}\right)\right]\Bigg|$$

$$= \quad 1.74889. \tag{17}$$

$$\begin{aligned}
&\frac{(\beta-\alpha)}{64(\varpi+1)(\varpi+2)}\Big[|F'(\alpha)|\Big\{45^{-\frac{\varpi+2}{\varpi}}\Big(-31^{\frac{\varpi+2}{\varpi}}\varpi(\varpi+1)+26\times 3^{\frac{\varpi+4}{\varpi}}5^{\frac{\varpi+2}{\varpi}}\\
&+2\times 31^{\frac{1}{\varpi}+1}45^{1/\varpi}\varpi(\varpi+2)-\varpi\Big(-2^{\frac{1}{\varpi}+4}7^{\frac{1}{\varpi}+1}45^{1/\varpi}\varpi+11\times 3^{\frac{\varpi+4}{\varpi}}5^{\frac{\varpi+2}{\varpi}}\varpi+12^{\frac{\varpi+2}{\varpi}}\varpi\\
&+14^{\frac{\varpi+2}{\varpi}}\varpi+9^{\frac{\varpi+2}{\varpi}}25^{\frac{1}{\varpi}+1}-2^{\frac{1}{\varpi}+5}7^{\frac{1}{\varpi}+1}45^{1/\varpi}+12^{\frac{\varpi+2}{\varpi}}+14^{\frac{\varpi+2}{\varpi}}+33^{\frac{\varpi+2}{\varpi}}(\varpi+1)\\
&-2\times 3^{\frac{3}{\varpi}+2}5^{1/\varpi}11^{\frac{1}{\varpi}+1}(\varpi+2)\Big)\ +48\varpi(\varpi+2)\Big(\sinh\Big(\frac{\log(540)}{\varpi}\Big)+\cosh\Big(\frac{\log(540)}{\varpi}\Big)\Big)\Big)\Big\}\\
&+\Big|F'(\beta)\Big|\Big\{45^{-\frac{\varpi+2}{\varpi}}\Big(2\times 3^{\frac{\varpi+3}{\varpi}}5^{1/\varpi}\varpi\Big(11^{\frac{1}{\varpi}+1}\varpi+2^{\frac{2}{\varpi}+3}(\varpi+2)\Big)+22\times 3^{\frac{\varpi+4}{\varpi}}5^{\frac{\varpi+2}{\varpi}}\\
&+\varpi\Big(-13\times 3^{\frac{\varpi+4}{\varpi}}5^{\frac{\varpi+2}{\varpi}}\varpi+12^{\frac{\varpi+2}{\varpi}}\varpi+14^{\frac{\varpi+2}{\varpi}}\varpi+33^{\frac{\varpi+2}{\varpi}}\varpi+4\times 3^{\frac{\varpi+3}{\varpi}}5^{1/\varpi}11^{\frac{1}{\varpi}+1}-3^{\frac{4}{\varpi}+3}5^{\frac{\varpi+2}{\varpi}}\\
&\ +12^{\frac{\varpi+2}{\varpi}}+14^{\frac{\varpi+2}{\varpi}}+33^{\frac{\varpi+2}{\varpi}}+31^{\frac{\varpi+2}{\varpi}}(\varpi+1)+2\times 3^{\frac{\varpi+2}{\varpi}}5^{1/\varpi}31^{\frac{1}{\varpi}+1}(\varpi+2)\Big)\Big)\Big\}\Big]\\
=\ &\ 7.54074.
\end{aligned} \tag{18}$$

From (17) and (18) Case-ii is verified.

Remark 3.2. *In Example 3.1, we discuss different cases for Theorem 2.2, other Theorems can be verified similarly and left for the readers.*

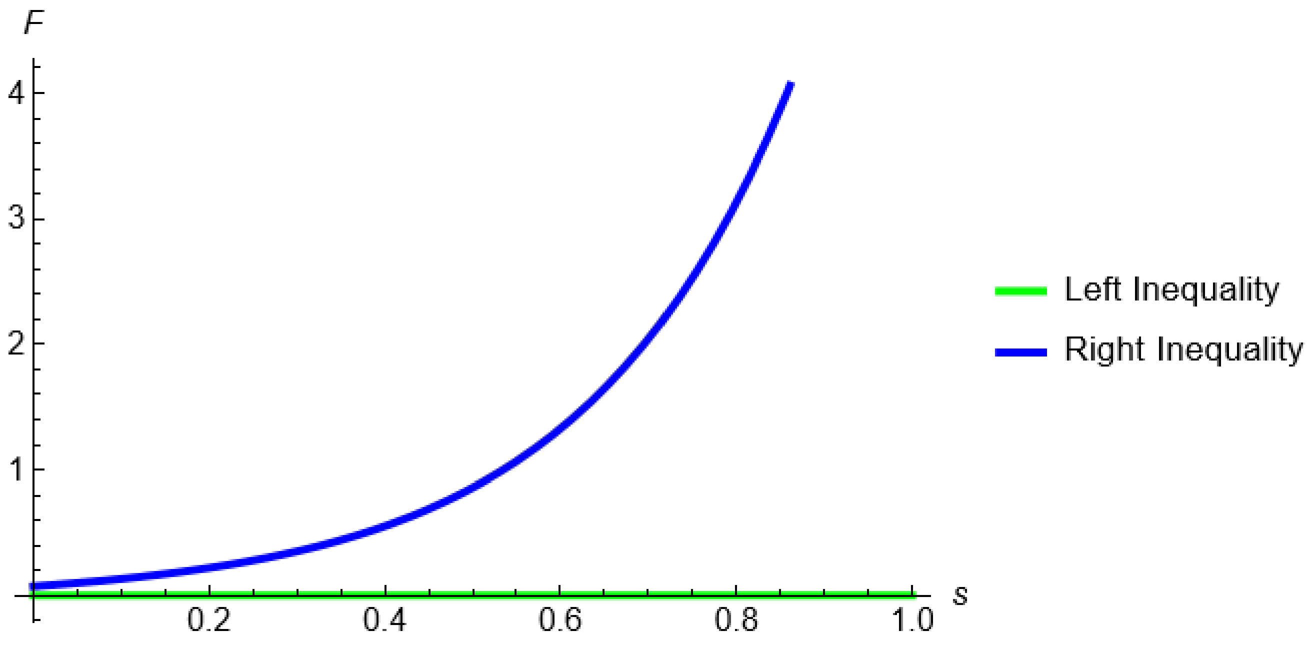

Figure 1: Comparative analysis of the inequalities of Theorem 2.2, of Example 3.1: 2-D plot when $\varpi > 0$ is fixed and s lies between 0 and 1.

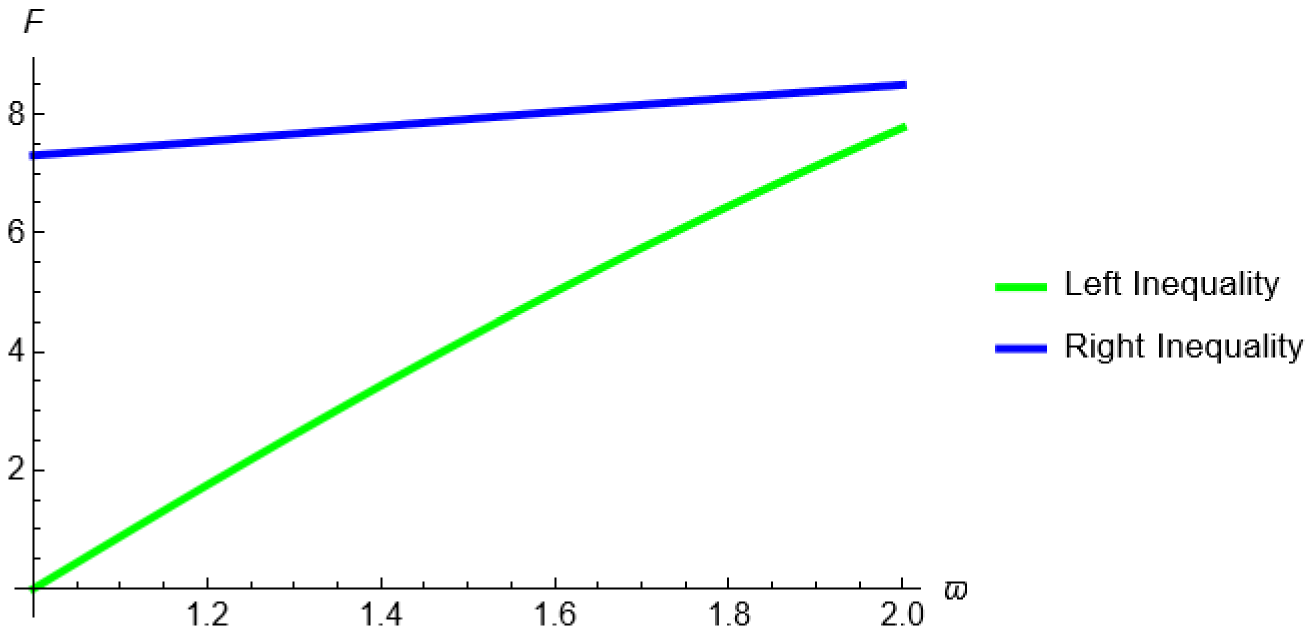

Figure 2: Comparative analysis of the inequalities of Theorem 2.2, of Example 3.1: 2-D plot when $\varpi > 0$ and $s = 1$ is fixed.

s	Left inequality	Right inequality
0.1	4.60394×10^{-7}	0.105636
0.3	4.56165×10^{-7}	0.183396
0.5	9.79689×10^{-7}	0.308539
0.7	4.16966×10^{-6}	0.527502
0.9	0.00006882	0.917985

Table 1: ϖ is fixed and s varies

ϖ	Left inequality	Right inequality
1.0	0.000372024	7.30278
1.2	1.74889	7.54074
1.4	3.42718	7.79003
1.6	4.9993	8.03575
1.8	6.44838	8.27106
2.0	7.76903	8.49309

Table 2: s is fixed and ϖ varies

4. Discussion

This article focuses on generalized fractional integrals and general convex functions. Using Mathematica 13.3.1 for graphical analysis, we present 2D and 3D plots involving the parameters s and ϖ to illustrate the newly established results. In Figure 1, we perform a comparative analysis of the inequalities from Theorem 2.2 and Example 3.1, with a fixed $\varpi > 0$ and s ranging between 0 and 1. Figure 2, presents a 2D plot for the same inequalities when $\varpi > 0$ and s is fixed at 1. Additionally, Figure 3 displays a 3D plot for Theorem 2.2 and Example 3.1, with $\varpi > 0$ and s varying between 0 and 1. These visualizations help to better understand and compare the behaviors of the inequalities under different conditions.

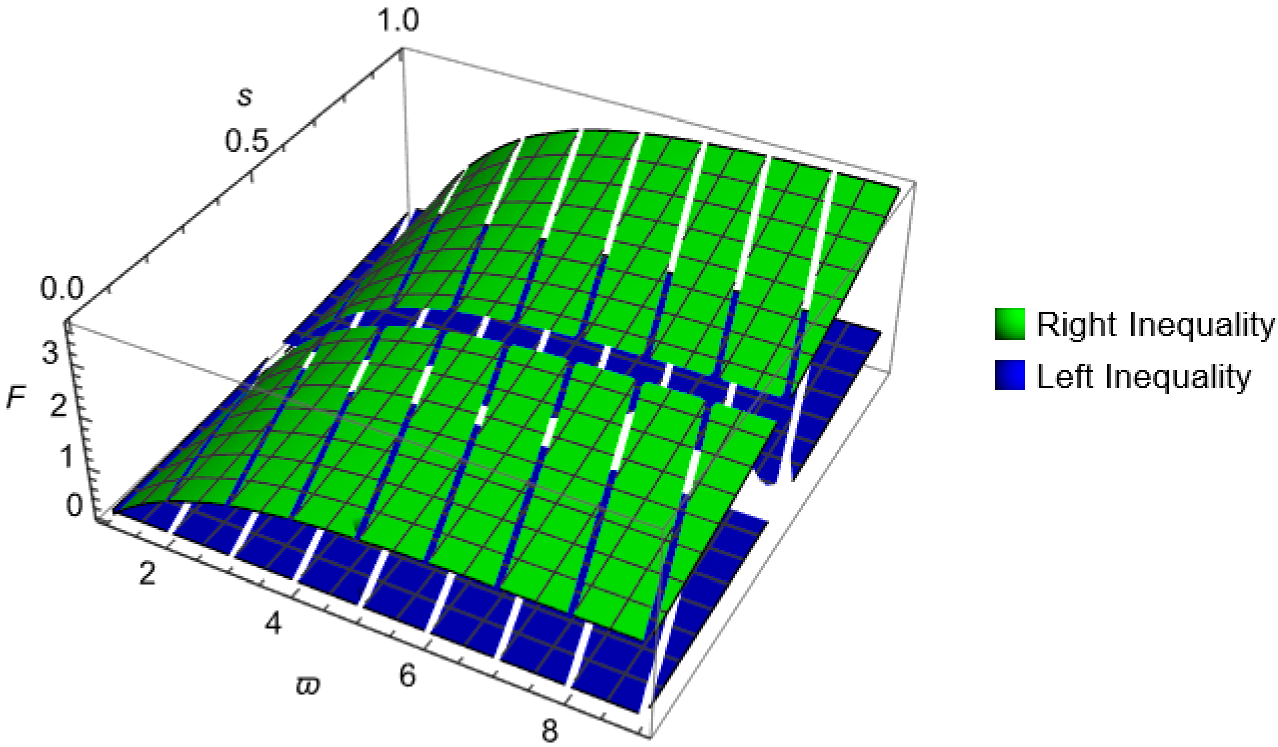

Figure 3: 3-D plot for Theorem 2.2, of Example 3.1, when $\varpi > 0$ and s lies between 0 and 1.

5. Application to the quadrature formula

The newly established results derived using generalized fractional integrals apply to the quadrature formula and have significant applications in various fields requiring precise numerical integration. This includes improving the accuracy of numerical solutions in computational physics, optimizing complex systems in engineering, and enhancing financial models for better risk assessment and prediction. The flexibility of fractional integrals in extending classical integral operators to non-integer orders provides a robust tool for handling complex systems, offering deeper insights and improved computational techniques. This advancement supports theoretical research and facilitates practical applications in scientific and engineering domains.

Assume that I_n be the partition given by

$$I_n : a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b,$$

$$h_i = \frac{x_{i+1} - x_i}{\mathbf{n}}, \; i = 0, 1, 2, \ldots, n-1,$$

where **n** must be divisible by 4. Then, we have

$$\int_a^b F(x)\, dx = S_n(I_n, F) + \mathbb{R}_n(I_n, F),$$

where

$$S_n(I_n,F) = \frac{1}{90}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left[7F(x_i)+32F(x_i+h)+12F(x_i+2h)\right.$$
$$\left.+32F(x_i+3h)+7F(x_{i+1})\right],$$

and remainder term satisfies the estimation

$$\mathbb{R}_n(I_n,F) \le \frac{239}{6480}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{\mathbf{n}}\left[|F'(x_i)|+|F'(x_{i+1})|\right].$$

We prove the following Proposition for the error bounds of Boole's rule.

Proposition 5.1. *Suppose that $F:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) then we have*

$$\int_a^b F(x)\,dx = S_n(I_n,F)+\mathbb{R}_n(I_n,F),$$

where

$$S_n(I_n,F) = \frac{1}{90}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left[7F(x_i)+32F(x_i+h)+12F(x_i+2h)\right.$$
$$\left.+32F(x_i+3h)+7F(x_{i+1})\right],$$

I_n be the partition given by

$$I_n: a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b,$$
$$h_i=\frac{x_{i+1}-x_i}{4},\ i=0,1,2,\ldots,n-1.$$

The remainder term satisfies the condition

$$|\mathbb{R}_n(I_n,F)| \le \frac{239}{6480}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{4}\left[|F'(x_i)|+|F'(x_{i+1})|\right],$$

for all $i=0,1,2,\ldots,n-1$.

Proof. Let us set things according to the situation

$$a=x_i, b=x_{i+1}, h_i=\frac{x_{i+1}-x_i}{4},$$

where $i=0,1,2,\ldots,n-1$. Then we have the following estimation

$$\left|\frac{1}{90}(x_{i+1}-x_i)\left[7F(x_i)+32F(x_i+h)+12F(x_i+2h)\right.\right.$$
$$\left.\left.+32F(x_i+3h)+7F(x_{i+1})\right]-\int_{x_i}^{x_{i+1}}F(t)\,dt\right|$$
$$\le \frac{239}{6480}\left(\frac{(x_{i+1}-x_i)^2}{4}\right)\left[|F'(x_i)|+|F'(x_{i+1})|\right], \text{ for all } i=0,1,2,\ldots,n-1.$$

After summing and by triangular inequality, we have

$$\left|\frac{1}{90}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left[7F(x_i)+32F(x_i+h)+12F(x_i+2h)\right.\right.$$

$$+32F\left(x_i+3h\right)+7F\left(x_{i+1}\right)\Big]-\int_{x_i}^{x_{i+1}}F\left(t\right)dt\Bigg|$$
$$\leq \frac{239}{6480}\sum_{i=0}^{n-1}\frac{\left(x_{i+1}-x_i\right)^2}{6}\left[\left|F'\left(x_i\right)\right|+\left|F'\left(x_{i+1}\right)\right|\right].$$

Which is the required proof of the Proposition 5.1. □

Remark 5.2. *Other propositions can be proved similarly, left for the interested reader.*

6. Conclusion

In conclusion, this article has demonstrated the effectiveness of generalized fractional integrals in establishing new identities and Boole's type inequalities for differentiable mappings via generalized convex functions. The newly proven identity is a valuable tool for further research in differential mapping, providing insights into the behaviour of differentiable functions under fractional integration. Moreover, the derived Boole's type inequalities offer practical implications for studying generalized convex functions, showcasing the utility of fractional calculus in addressing complex mathematical problems. Numerical examples are provided to validate the newly established results, demonstrating their practical significance. Fractional calculus extends the traditional concepts of differentiation and integration to non-integer orders, offering a powerful tool for modelling complex systems in various scientific fields. By applying generalized fractional integrals, this study reveals new inequalities for convex functions, providing deeper insights and enhancing the computational techniques available for real-world applications. Researchers are encouraged to explore the extension of generalized fractional integrals to multi-dimensional spaces and investigate their applications in optimizing complex systems in applied mathematics and engineering.

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