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Hyers–Ulam stability of non-linear Volterra-Fredhlom integro-differential equations via successive approximation method

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Abstract. In this work, we examine the Hyers–Ulam and Hyers–Ulam–Rassias stability of non-linear Volterra-Fredhlom integro-differential equations of fractional order with boundary conditions using the successive approximation approach. This study investigates a boundary value problem involving a non-linear Volterra-Fredholm integro-differential equations of fractional order, along with its boundary conditions. The main results are demonstrated through several examples.

1. Introduction

Hyers-Ulam stability is a concept in functional analysis that focuses on the stability of functional equations. S. M. Ulam [1] expanded on this idea by studying how approximate solutions behave, following the groundwork laid by D. H. Hyers [2] in 1941. The main objective is to identify the conditions under which a functional equation is stable. Essentially, if a function closely approximates the equation, we want to determine whether there exists an exact solution that is "near" to this approximation. The traditional Hyers-Ulam stability theorem states that there is a unique solution to the functional equation that remains uniformly close to any approximate solution given specific conditions. Applications of fractional calculus in numerical analysis and many applied disciplines, including engineering and physics, have sparked a great deal of interest in the field in recent years ([3–6]).

The study of linear viscoelasticity is frequently the focus of fractal phenomena. The fact that fractional calculus is essentially an enhanced version of integer-order calculus is one of its main benefits. For this reason, fractional calculus can potentially do what integer-order calculus is unable to. The integro-differential equation, which combines the differential and Volterra-Fredholm integral equations, has garnered increased attention in recent years.In numerous fields of linear and non-linear functional analysis, as well as in the theory of engineering, mechanics, physics, chemistry, biology, economics, and statistics, integral-differential equations are crucial.Since the aforementioned integro-differential equations are typically challenging to

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solve analytically, approximation techniques are needed to find the solutions to the linear and non-linear integro-differential equations [7].

Numerous reseaarchers have examined and deliberated upon the linear Volterra-Fredholm integro-differential equations. Boabolian, et al.[8] devised a novel, direct approach for resolving non-linear Volterra - Fredholm integral and integro-differential equations, employing operational matrix block-pulse functions. Volterra integral equations are important because of their depth of theory and applicability in real-world situations. The modelling of systems that display memory and reliance on previous states is made possible by these equations, which reflect relationships in which a quantityú's future value is influenced by its prior behaviour. This temporal feature is similar to many situations found in natural and artificial systems, ranging from species population predictions in biological systems to the behaviour of materials under time-dependent stresses. Solving Volterra integral equations is still a difficult task, nonetheless, despite its amazing application and value.Due to the complex interactions between integral and differential components and the nonlinear nature of many real-world systems, these equations are difficult to solve analytically. Numerous numerical methods, iterative techniques, and approximation schemes have been developed in an attempt to solve these equations; however, an all-encompassing and reliable analytical method has not yet been discovered [9, 10].

Important properties and widespread applications in mathematics may be found in the Fredholm, Volterra, and integro-differential equations. Generating functions, combinatorial sums of certain polynomials, and integro-differentia lequations in particular have been studied by several mathematicians. Since such integral issues may be found in a variety of mathematical models, computer algorithms, physics, engineering difficulties, and fractional calculus theory (see other articles [11–17]). Boundary value issues for nonlinear fractional differential equations have recently attracted the attention of several academics. Fractional derivatives, in fact, provide a useful tool for describing the memory and genetic properties of various materials and processes [18], making fractional-order models more realistic and grounded than their traditional integer-order equivalents. Fractional differential equations have implications for a wide range of scientific and technological domains, such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, bloodflow phenomena, aerodynamics, and fitting experimental data [18, 19].

In mathematical studies on fractional differential equations, the notion of a fractional order derivative $\gamma = 0$ is reached via the Riemann - Liouville technique. As the left inverse of the equivalent fractional integral, the fractional Riemann-Liouville derivative is a natural generalisation of the Cauchy formula for the antiderivative function u(t). Michele Caputo developed a new concept of the fractional order derivative in 1967 to address boundary value difficulties in viscoelasticity theory [20]. The Caputo technique is advantageous because the initial and boundary conditions for differential equations with the Caputo fractional derivative are similar to those for integer order differential equations, allowing for consistent interpretation. As a result, it is widely used in practical applications. A revised version of fixed point theory and fixed point method for functional equations and control theory in Caputo contexts, which is thought to be a generalization of the classical fundamental theorem of calculus, was presented in [21, 22], where the authors also recovered the concepts of fractional integrals and fractional derivatives in alternative forms. To satisfy the physical limitations, Caputo created an altered definition of the fractional differential derivative. Caputo picked it up with Mainard.For integro-differential equations of non-integer orders, this notion has the advantage of offering a more comprehensible solution to the beginning conditions problem. For $0 < \gamma < 1$, the Caputo derivative situations were referred to as the regularised fractional derivative.

There has been an increase in interest in Hyers-Ulam and Hyers-Ulam-Rassias stability for differential equations and integro-differential equations in recent years (see [23, 28–31]). The authors of [23–31] established many types of Hyers-Ulam-Rassias stability to the volterra integro-differential equations by applying the fixed point theorem techniques.

Within the constraints of the following boundary value conditions, this work studies a nonlinear Volterra-Fredhlom integro-differential equation of fractional order:

$$^{c}D^{\alpha}u(t) = g(t) + \lambda_1 \int_0^t k_1(t,s)f_1(s,u(s))ds + \lambda_2 \int_0^1 k_2(t,s)f_2(s,u(s))ds, \tag{1}$$

$au(0) + bu(1) = 0.$

where $t \in J = [0,1]$, $0 < \alpha \leq 1$ and $k_1, k_2 : J \to J \to \mathbb{R}$ are known continuous functions, $f_1, f_2 : J \to \mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions, $g : J \to \mathbb{R}$ is a continuous functions, λ_1, λ_2 are parameters, and a,b,are real constants with $a + b \neq 0$, and D^α is Caputo fractional derivative .

2. Preliminaries

Some basic concepts and lemmas that will be utilised throughout the article are presented in this section.

Lemma 2.1. *Let* $h \in C[0,1]$, $a + b \neq 0$, *then the fractional boundary value problem*

$${}^cD^\alpha u(t) = h(t), 0, \alpha \leq 1, t \in J = [0,1],$$

$$au(0) + bu(1) = 0.$$

has a unique solution

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds. \tag{2}$$

The boundary value problem (1) and equation (2) is equivalent to the fractional integral equation

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) \\ & f_2(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) \\ & f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u(\tau))d\tau\Big]ds. \end{aligned}$$

The remainder of this work is organized as follows: Section 3 presents the Hyers-Ulam stability of (2), whereas Section 5 provides the Hyers-Ulam-Rassias stability of (2). The definitions of the Hyers-Ulam stability categories that will be utilised in this article are now provided. Let first $\varepsilon > 0$, $\psi \in C(J, R+)$, and $\sigma \in C(J, R+)$ be taken into consideration. We take into account the following disparities:

$$|v'(t) - \rho(t)| \leq \varepsilon, t \in J, \tag{3}$$

and

$$|v'(t) - \rho(t)| \leq \varepsilon\psi(t), t \in J, \tag{4}$$

also

$$|v'(t) - \rho(t)| \leq \varepsilon\sigma(t), t \in J, \tag{5}$$

where

$$\begin{aligned} \rho(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) \\ & f_2(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) \\ & f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u(\tau))d\tau\Big]ds. \end{aligned}$$

Definition 2.2. *The problem (2) is Hyers-Ulam stable if there is a constant $K_f > 0$ such that for each $\epsilon > 0$ and for each solution $\upsilon \in C^1(J, \mathbb{R})$ of (3) there is a solution u of (2) satisfying*

$$|\upsilon(t) - u(t)| \leq K_f \varepsilon.$$

Definition 2.3. *The problem (2) is Hyers-Ulam-Rassias stable concerning $\phi \in C(J, \mathbb{R}+)$ if there is a constant $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\upsilon \in C^1(J, \mathbb{R})$ of (4) there is a solution u of (2) satisfying*

$$|\upsilon(t) - u(t)| \leq C_f \varepsilon \phi.$$

Definition 2.4. *The problem (2) is σ-semi-Hyers-Ulam stable if there is a constant $K_f > 0$ and σ be a non-decreasing function and for each solution $\upsilon \in C^1(J, R)$ of (5) there is a solution u of (2) satisfying*

$$|\upsilon(t) - u(t)| \leq K_f \varepsilon \sigma(t).$$

3. Hyers-Ulam stability of non-linear Volterra-Fredhlom integro-differential equations

The Hyers-Ulam stability for non-linear Volterra Fredhlom integro-differential equation of fractional order with boundary conditions (2) will be presented in this section using the successive approximation approach.

Remark 3.1. *We note that there is a continuous function $\delta(t)$ on J such that $|\delta(t)| \leq \varepsilon$ and that if the function υ is a solution of 3.*

$$\upsilon'(t) = \rho(t) + \delta(t).$$

Let $f_1 : J \to R \to R$ and $f_2 : J \to R \to R$ are non-linear continuous functions. We consider the following hypotheses:
$(H1)$ There exist positive constants L_1, L_2 such that for each $\tau \in J$ and $w_1, w_2 \in R$ one has

$$|f_1(t, w_1) - f_1(t, w_2)| \leq L_1 |w_1 - w_2|,$$

$$|f_2(t, w_1) - f_2(t, w_2)| \leq L_2 |w_1 - w_2|.$$

$(H2)$ Let us consider the inequality (4) where $\psi \in C(J, R+)$. Assume that $C > 0$ is a constant such that $k\, C^k = (1 - 0)\, C^{(k-1)}$, $\forall\, k \geq 1$, and $0 < CL < 1$, and that, for $t \in J$, the following hypothesis is met.

$$\int_0^t \psi(s) ds \leq C \psi(t).$$

Theorem 3.2. *Assume that f_1 and f_2 satisfy the (H1).Then , for each $\varepsilon > 0$ if the function v satisfies (3), there exists a unique solution u of (2) provided $u_0 = \upsilon_0$ and u satisfies the following estimate*

$$|u(t) - \upsilon(t)| \leq \varepsilon(1) exp((1 - 0)(1 + L)). \tag{6}$$

Proof. For each $\varepsilon > 0$ and let the function υ satisfy 3, then basing on Remark 3.1,one has that then there is a continuous function $\delta(t)$ on J such that $|\delta(t)| \leq \varepsilon$ and υ' (t)= $\rho(t)$+$\delta(t)$. This yields that the function υ satisfies the integral equation

$$\upsilon(t) = \upsilon_0 + \int_0^t \rho(s) ds + \int_0^t \delta(s) ds, \tag{7}$$

where

$$\int_0^t \rho(s)ds = \int_0^t \Big[\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u(\tau))d\tau\Big]ds\Big]ds.$$

We consider the sequence $(u_n)_{n\geq 0}$ defined as follows: $u_0(t) = v(t)$ and for $n = 1, 2, \cdots$,

$$u_n(t) = v_0 + \int_o^t \rho_{n-1}(s)ds, \tag{8}$$

where

$$\int_0^t \rho_{n-1}(s)ds = \int_0^t \Big[\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u_{n-1}(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u_{n-1}(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u_{n-1}(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u_{n-1}(\tau))d\tau\Big]ds\Big]ds.$$

by (7) and (8) , for n=1 one has

$$\begin{aligned} |u_1(t) - u_0(t)| &= \Big|v_0 + \int_0^t \rho_0(s)ds - v(t)\Big| \\ &= \Big|v_0 + \int_0^t \rho_0(s)ds - v_0 - \int_0^t \rho_0(s)ds - \int_o^t \delta(s)ds\Big| \end{aligned}$$

$$= \Big|\int_o^t \delta(s)ds\Big| \leq \epsilon(t-0), \forall t \in J. \tag{9}$$

For $n = 1, 2, \cdots$, from the hypothesis (H1) one has

$$\begin{aligned} |u_{n+1}(t) - u_n(t)| &= \Big|\int_0^t \rho_0(s)ds - \int_0^t \rho_{n-1}(s)ds\Big| \\ &\leq L \int_0^t \int_0^s |u_n(s) - u_{n-1}(s)|dsds + L \int_0^t \int_o^1 |u_n(r) - u_{n-1}(r)|drds, \end{aligned}$$

where $L = \max\{L_1, L_2\}$. In particular, for n =1 and by (9) one gets

$$\begin{aligned} |u_2(t) - u_1(t)| &\leq \varepsilon L \int_0^t \int_0^s (s-0)dsds + \varepsilon L \int_0^t \int_o^1 (r-0)drds \\ &= \varepsilon L\Big(\frac{(t-o)^3}{3!} + \frac{(t-o)^2}{2!}\Big) \\ &= \varepsilon L\Big(\frac{(t-o)^2}{2!} + \frac{(t-o)^3}{3!}\Big) \\ &\leq 2\varepsilon L\Big(\frac{(t-o)^2}{2!} + \frac{(t-o)^3}{3!}\Big) \end{aligned}$$

and so, for n =2, one also obtains

$$|u_3(t) - u_2(t)| \leq \varepsilon L^2 \int_0^t \int_0^s \Big(\frac{(s-0)^2}{2!} + \frac{(s-0)^3}{3!}\Big) ds ds + \varepsilon L^2 \int_0^t \int_o^1 \Big(\frac{(r-0)^2}{2!} + \frac{(r-0)^3}{3!}\Big) dr ds$$
$$= \varepsilon L^2 \Big(\frac{(t-o)^3}{3!} + \frac{(t-o)^4}{4!} + \frac{(t-o)^5}{5!}\Big)$$
$$\leq 3\varepsilon L^2 \Big(\frac{(t-o)^3}{3!} + \frac{(t-o)^4}{4!} + \frac{(t-o)^5}{5!}\Big)$$

and for $n \geq 4$ we have

$$|u_n(t) - u_{n-1}(t)| \leq \varepsilon n L^{n-1} \Big(\frac{(t-o)^n}{n!} + \frac{(t-o)^{n+1}}{(n+1)!} + \cdots + \frac{(t-o)^{2n}}{(2n)!} + \frac{(t-o)^{2n+1}}{(2n+1)!}\Big). \tag{10}$$

Then, the estimation (10) can be rewritten by:

$$|u_n(t) - u_{n-1}(t)| \leq \frac{\varepsilon(t-0)(L(t-0))^{n-1}}{(n-1)!}\Big(1 + \frac{(t-0)}{n+1} + \frac{(t-0)^2}{(n+1)(n+2)}$$
$$+ \cdots + \frac{(t-0)^n}{(n+1)(n+2)\cdots} + \frac{(t-0)^{n+1}}{(n+1)(n+2)\cdots 2n(2n+1)}\Big)$$
$$\leq \frac{\varepsilon(1)(L(t-0))^{n-1}}{(n-1)!}\Big(1 + \frac{(t-0)}{1!} + \frac{(t-0)^2}{2!} + \cdots + \frac{(t-0)^n}{n!} + \frac{(t-0)^{n+1}}{(n+1)!}\Big)$$
$$\leq \frac{\varepsilon(1)(L(t-0))^{n-1}}{(n-1)!} exp(t-0).$$

Furthermore, if we assume that

$$|u_n(t) - u_{n-1}(t)| \leq \frac{\varepsilon(1)(L(t-0))^{n-1}}{(n-1)!} exp(t-0), \tag{11}$$

then one also gets

$$|u_{n+1}(t) - u_n(t)| \leq \frac{\varepsilon(1)(L(t-0))^n}{n!} exp(t-0), \forall t \in J.$$

This yields that

$$\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \leq \varepsilon(1) exp(1-0) \sum_{n=0}^{\infty} \frac{(L(t-0))^n}{n!}. \tag{12}$$

Since, the right-hand series is convergent to the function exp(L(t-0)), for each $\varepsilon > 0$ we deduce the series $u_0(t) + \sum_{n=1}^{\infty}[u_{n+1}(t) - u_n(t)]$ is uniformly convergent concerning the norm |.| and

$$\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \leq \varepsilon(1) exp((1-0)(1+L)). \tag{13}$$

Assume that

$$\overline{u}(t) = u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t) - u_n(t)]. \tag{14}$$

Then,

$$u_j(t) = u_0(t) + \sum_{n=0}^{j}[u_{n+1}(t) - u_n(t)] \tag{15}$$

is the jth partial of the series (14). From (14) and (15), we obtain

$$\lim_{j\to\infty} |\overline{u}(t) - u_j(t)| = 0, \forall t \in J.$$

Define:= $u(t) = \overline{u}(t), \forall t \in J$. We observe that the limit of the above sequence is a solution to the following equation:

$$u(t) = v_0 + \int_0^t \rho(s)ds, \forall t \in J, \tag{16}$$

where

$$\begin{aligned}\rho(t) :=& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) \\ & f_2(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) \\ & f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u(\tau))d\tau\Big]ds.\end{aligned}$$

By (8), (16) and the hypothesis (H1), one has that

$$\begin{aligned}|u(t) - v_0 - \int_0^t \rho(s)ds| &= |\overline{u}(t) - (u_j(t) - \int_0^t \rho_{j-1}(s)ds - \int_o^t \rho(s)ds| \\ &\leq |\overline{u}(t) - u_j(t)| + \int_0^t |\rho_{j-1}(s)ds - \rho(s)|ds.\end{aligned}$$

$$\leq |\overline{u}(t) - u_j(t)| + L \int_0^t \int_0^s |u_{j-1}(s) - u(s)|dsds + L \int_0^t \int_0^1 |u_{j-1}(r) - u(r)|drds \tag{17}$$

Combining (14) and (15), we get

$$|\overline{u}(t) - u_j(t)| \leq \sum_{n=j+1}^{\infty} |u_{n+1}(t) - u_n(t)|$$

and by the estimation (12), one has

$$|u(t) - u_j(t)| \leq \varepsilon(1)exp(1-0) \sum_{n=j+1}^{\infty} \frac{(L(t-0))^n}{n!}, \forall t \in J. \tag{18}$$

Hence, it follows from the inequalities (17) and (18) that

$$\begin{aligned}|u(t) - v_0 - \int_0^t \rho(s)ds| \leq& \varepsilon(1)e^{(1-o)} \sum_{n=j+1}^{\infty} \frac{(L(t-0))^n}{n!} + \varepsilon L(1)e^{(1-o)}\Big(\int_0^t \int_0^s \\ & \sum_{n=j+1}^{\infty} \frac{(L(s-0))^n}{n!}dsds + \int_0^t \int_0^1 \sum_{n=j+1}^{\infty} \frac{(L(r-0))^n}{n!}drds\Big)\end{aligned}$$

$$\leq \varepsilon(1)e^{(1-o)}\Big[\sum_{n=j+1}^{\infty} \frac{(L(t-0))^n}{n!}ds + \sum_{n=j+1}^{\infty} L^{n+1}\Big(\frac{(t-0)^{n+1}}{(n+1)!} + \frac{(t-0)^{n+2}}{(n+2)!}\Big)\Big]. \tag{19}$$

Taking limit as n $\to \infty$, we see that the right-hand series of (19) is convergent. Therefore, one deduces that

$$|u(t) - v_0 - \int_0^t \rho(s)ds| \leq 0, \forall t \in J.$$

This means that

$$u(t) = v_0 + \int_0^t \rho(s)ds, \forall t \in J, \tag{20}$$

which is a solution of (2). In addition, from the estimation (13), we have the estimate as follows:

$$|u(t) - v(t)| \leq \varepsilon(1)exp((1-0)(1+L)).$$

To show the uniqueness of solution to the problem (2), we assume that $\overline{u}(t)$ is another solution of (2), which has the form

$$\overline{u}(t) = v_0 + \int_0^t \rho(s)ds, \forall t \in J, \tag{21}$$

where

$$\begin{aligned}\overline{\rho}(t) :=& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau)\overline{f_1}(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau)\\ & \overline{f_2}(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau)\\ & \overline{f_1}(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau)\overline{f_2}(\tau, u(\tau))d\tau\Big]ds.\end{aligned}$$

By using the hypothesis (H1), one obtains

$$w(t) \leq L \int_0^t \int_0^s w(s)dsds + L \int_0^t \int_0^1 w(r)dsds, \forall t \in J.$$

where $w(t) = |u(t) - \overline{u}(t)|$. Then by applying Grownwall′ s lemma (see Theorem 2.1 in [13]) we infer that w(t)=0 on J. So, $u(t) = \overline{u}(t)$. This completes the proof. □

4. σ -semi-Hyers-Ulam stability of non-linear Volterra-Fredhlom integro-differential equations

We will introduce the σ-semi-Hyers-Ulam stability for non-linear Volterra Fredhlom integro-differential equation of fractional order with boundary conditions (2) in this section using the successive approximation approach.

Theorem 4.1. *Assume that f_1 and f_2 satisfy the (H1). Next, for any $\varepsilon > 0$ and $\sigma : [a,b] \to (o,\infty)$, if the function v satisfies (5), then there is a unique solution. Utilising u from (2), we derive $u_0 = v_0$, and u satisfies the provided constraint.*

$$|u(t) - v(t)| \leq \varepsilon(1)exp((1-0)(1+L))\frac{\sigma(t)}{\sigma(o)}. \tag{22}$$

Proof. According to Remark (3.1), for any $\varepsilon > 0$ and given the function v satisfying (5), there exists a continuous function $\delta(t)$ on J such that $|\delta(t)| \leq \varepsilon$ and v' (t)= $\rho(t)$+$\delta(t)$. This indicates that the integral equation is satisfied by the function v.

$$v(t) = v_0 + \int_0^t \rho(s)ds + \int_0^t \delta(s)ds, \tag{23}$$

where

$$\int_0^t \rho(s)ds = \int_0^t \Big[\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau)f_1(\tau,u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau)$$
$$f_2(\tau,u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau)$$
$$f_1(\tau,u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau)f_2(\tau,u(\tau))d\tau\Big]ds\Big]ds.$$

We consider the sequence $(u_n)_{n\geq 0}$ defined as follows: $u_0(t) = v(t)$ and for $n = 1, 2, \cdots,$

$$u_n(t) = v_0 + \int_o^t \rho_{n-1}(s)ds, \tag{24}$$

where

$$\int_0^t \rho_{n-1}(s)ds = \int_0^t \Big[\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau)f_1(\tau,u_{n-1}(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau)$$
$$f_2(\tau,u_{n-1}(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau)$$
$$f_1(\tau,u_{n-1}(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau)f_2(\tau,u_{n-1}(\tau))d\tau\Big]ds\Big]ds.$$

by (23) and (24) , for n=1 one has

$$\begin{aligned}|u_1(t) - u_0(t)| &= \Big|v_0 + \int_0^t \rho_0(s)ds - v(t)\Big| \\ &= \Big|v_0 + \int_0^t \rho_0(s)ds - v_0 - \int_0^t \rho_0(s)ds - \int_o^t \delta(s)ds\Big|\end{aligned}$$

$$= \Big|\int_o^t \delta(s)ds\Big| \leq \varepsilon(t-0)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J. \tag{25}$$

For $n = 1, 2, \cdots,$ from the hypothesis (H1) one has

$$\begin{aligned}|u_{n+1}(t) - u_n(t)| &= \Big|\int_0^t \rho_0(s)ds - \int_0^t \rho_{n-1}(s)ds\Big| \\ &\leq L\int_0^t\int_0^s |u_n(s) - u_{n-1}(s)|dsds + L\int_0^t\int_o^1 |u_n(r) - u_{n-1}(r)|drds,\end{aligned}$$

where $L = \max L_1, L_2$. In particular, for n =1 and by (25) one gets

$$\begin{aligned}|u_2(t) - u_1(t)| &\leq \varepsilon L\int_0^t\int_0^s (s-0)dsds\frac{\sigma(t)}{\sigma(0)} + \varepsilon L\int_0^t\int_o^1 (r-0)drds\frac{\sigma(t)}{\sigma(0)} \\ &= \varepsilon L\Big(\frac{(t-o)^3}{3!} + \frac{(t-o)^2}{2!}\Big)\frac{\sigma(t)}{\sigma(0)} \\ &\leq 2\varepsilon L\Big(\frac{(t-o)^2}{2!} + \frac{(t-o)^3}{3!}\Big)\frac{\sigma(t)}{\sigma(0)}\end{aligned}$$

and so, for n =2, one also obtains

$$|u_3(t)-u_2(t)| \leq \varepsilon L^2 \int_0^t \int_0^s \Big(\frac{(s-0)^2}{2!}+\frac{(s-0)^3}{3!}\Big)dsds\frac{\sigma(t)}{\sigma(0)} + \varepsilon L^2 \int_0^t \int_o^1 \Big(\frac{(r-0)^2}{2!}+ \frac{(r-0)^3}{3!}\Big)drds\frac{\sigma(t)}{\sigma(0)}$$
$$= \varepsilon L^2\Big(\frac{(t-o)^3}{3!}+\frac{(t-o)^4}{4!}+\frac{(t-o)^5}{5!}\Big)\frac{\sigma(t)}{\sigma(0)}$$
$$\leq 3\varepsilon L^2\Big(\frac{(t-o)^3}{3!}+\frac{(t-o)^4}{4!}+\frac{(t-o)^5}{5!}\Big)\frac{\sigma(t)}{\sigma(0)}$$

and for $n \geq 4$ we have

$$|u_n(t)-u_{n-1}(t)| \leq \varepsilon n L^{n-1}\Big(\frac{(t-o)^n}{n!}+\frac{(t-o)^{n+1}}{(n+1)!}+\cdots+\frac{(t-o)^{2n}}{(2n)!}+\frac{(t-o)^{2n+1}}{(2n+1)!}\Big)\frac{\sigma(t)}{\sigma(0)} \tag{26}$$

Then, the estimation (26) can be rewritten by:

$$|u_n(t)-u_{n-1}(t)| \leq \frac{\varepsilon(t-0)(L(t-0))^{n-1}}{(n-1)!}\Big(1+\frac{(t-0)}{n+1}+\frac{(t-0)^2}{(n+1)(n+2)} + \cdots + \frac{(t-0)^n}{(n+1)(n+2)\cdots}+\frac{(t-0)^{n+1}}{(n+1)(n+2)\cdots 2n(2n+1)}\Big)\frac{\sigma(t)}{\sigma(0)}$$
$$\leq \frac{\varepsilon(1)(L(t-0))^{n-1}}{(n-1)!}\Big(1+\frac{(t-0)}{1!}+\frac{(t-0)^2}{2!}+\cdots+\frac{(t-0)^n}{n!}+ \frac{(t-0)^{n+1}}{(n+1)!}\Big)\frac{\sigma(t)}{\sigma(0)}$$
$$\leq \frac{\varepsilon(1)(L(t-0))^{n-1}}{(n-1)!}exp(t-0)\frac{\sigma(t)}{\sigma(0)}.$$

Furthermore, if we assume that

$$|u_n(t)-u_{n-1}(t)| \leq \frac{\varepsilon(1)(L(t-0))^{n-1}}{(n-1)!}exp(t-0)\frac{\sigma(t)}{\sigma(0)}, \tag{27}$$

then one also gets

$$|u_{n+1}(t)-u_n(t)| \leq \frac{\varepsilon(1)(L(t-0))^n}{n!}exp(t-0)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J.$$

This yields that

$$\sum_{n=0}^{\infty}|u_{n+1}(t)-u_n(t)| \leq \varepsilon(1)\frac{\sigma(t)}{\sigma(0)}exp(1-0)\sum_{n=0}^{\infty}\frac{(L(t-0))^n}{n!}. \tag{28}$$

Since the right-hand series is convergent to the function exp(L(t-0)), for each $\epsilon > 0$ we deduce the series $u_0(t)+\sum_{n=1}^{\infty}[u_{n+1}(t)-u_n(t)]$ is uniformly convergent concerning the norm |.| and

$$\sum_{n=0}^{\infty}|u_{n+1}(t)-u_n(t)| \leq \varepsilon(1)exp((1-0)(1+L))\frac{\sigma(t)}{\sigma(0)}. \tag{29}$$

Assume that

$$\overline{u}(t) == u_0(t)+\sum_{n=0}^{\infty}[u_{n+1}(t)-u_n(t)]. \tag{30}$$

Then,

$$u_j(t) = u_0(t) + \sum_{n=0}^{j}[u_{n+1}(t) - u_n(t)] \tag{31}$$

is the jth partial of the series (30). From (30) and (31), we obtain

$$\lim_{j\to\infty} |\overline{u}(t) - u_j(t)| = 0, \forall t \in J.$$

Define:= $u(t) = \overline{u}(t), \forall t \in J$. We observe that the limit of the above sequence is a solution to the following equation:

$$u(t) = v_0 + \int_0^t \rho(s)ds, \forall t \in J, \tag{32}$$

where

$$\begin{aligned}\rho(t) :=& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau)\\ & f_2(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau)\\ & f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u(\tau))d\tau\Big]ds.\end{aligned}$$

By (25), (32) and the hypothesis (H1), one has that

$$\begin{aligned}\Big|u(t) - v_0 - \int_0^t \rho(s)ds\Big| &= \Big|\overline{u}(t) - \Big(u_j(t) - \int_0^t \rho_{j-1}(s)ds\Big) - \int_o^t \rho(s)ds\Big|\\ &\le |\overline{u}(t) - u_j(t)| + \int_0^t |\rho_{j-1}(s)ds - \rho(s)|ds.\end{aligned}$$

$$\le |\overline{u}(t) - u_j(t)| + L\int_0^t \int_0^s |u_{j-1}(s) - u(s)|dsds + L\int_0^t \int_0^1 |u_{j-1}(r) - u(r)|drds \tag{33}$$

Combining (30) and (31), we get

$$|\overline{u}(t) - u_j(t)| \le \sum_{n=j+1}^{\infty} |u_{n+1}(t) - u_n(t)|$$

and by the estimation (28), one has

$$|u(t) - u_j(t)| \le \varepsilon(1)exp(1-0) \sum_{n=j+1}^{\infty} \frac{(L(t-0))^n}{n!}\frac{\sigma(t)}{\sigma(0)}, \forall t \in J. \tag{34}$$

Hence, it follows from the inequalities (33) and (34) that

$$\begin{aligned}\Big|u(t) - v_0 - \int_0^t \rho(s)ds\Big| \le& \varepsilon(1)e^{(1-o)} \sum_{n=j+1}^{\infty} \frac{(L(t-0))^n}{n!}\frac{\sigma(t)}{\sigma(0)} + \varepsilon L(1)e^{(1-o)}\Big(\int_0^t \int_0^s\\ & \sum_{n=j+1}^{\infty} \frac{(L(s-0))^n}{n!}dsds + \int_0^t \int_0^1 \sum_{n=j+1}^{\infty} \frac{(L(r-0))^n}{n!}drds\Big)\frac{\sigma(t)}{\sigma(0)}\end{aligned}$$

$$\leq \varepsilon(1)e^{(1-o)}\Big[\sum_{n=j+1}^{\infty}\frac{(L(t-0))^{n}}{n!}ds+\sum_{n=j+1}^{\infty}L^{n+1}\Big(\frac{(t-0)^{n+1}}{(n+1)!}+\frac{(t-0)^{n+2}}{(n+2)!}\Big)\Big]\frac{\sigma(t)}{\sigma(0)}. \tag{35}$$

Taking limit as n $\to \infty$, we see that the right-hand series of (35) is convergent. Therefore, one deduces that

$$\Big|u(t)-v_0-\int_0^t \rho(s)ds\Big| \leq 0, \forall t \in J.$$

This means that

$$u(t)=v_0+\int_0^t \rho(s)ds, \forall t \in J, \tag{36}$$

which is a solution of (2). In addition, from the estimation (29), we have the estimate as follows:

$$|u(t)-v(t)| \leq \varepsilon(1)exp((1-0)(1+L))\frac{\sigma(t)}{\sigma(0)}.$$

To show the uniqueness of solution to the problem (2), we assume that $\overline{u}(t)$ is another solution of (2), which has the form

$$\overline{u}(t)=v_0+\int_0^t \overline{\rho}(s)ds, \forall t \in J, \tag{37}$$

where

$$\begin{aligned}\overline{\rho}(t):=&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[g(s)+\lambda_1\int_0^s k_1(s,\tau)\overline{f_1}(\tau,u(\tau))d\tau+\lambda_2\int_0^1 k_2(s,\tau)\\ &\overline{f_2}(\tau,u(\tau))d\tau\Big]ds-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\Big[g(s)+\lambda_1\int_0^s k_1(s,\tau)\\ &\overline{f_1}(\tau,u(\tau))d\tau+\lambda_2\int_0^1 k_2(s,\tau)\overline{f_2}(\tau,u(\tau))d\tau\Big]ds.\end{aligned}$$

By using the hypothesis (H1), one obtains

$$w(t) \leq L\int_0^t\int_0^s w(s)dsds+L\int_0^t\int_0^1 w(r)drds, \forall t \in J.$$

where $w(t)=|u(t)-\overline{u}(t)|$. Then by applying Grownwall ′s lemma (see Theorem 2.1 in [13]) we infer that w(t)=0 on J. So, $u(t)=\overline{u}(t)$. This completes the proof. □

5. Hyers-Ulam-Rassias stability of non-linear Volterra-Fredhlom integro-differential equations

The Hyers-Ulam-Rassias stability for non-linear Volterra Fredhlom integro-differential equation of fractional order with boundary conditions 2 will be presented in this section using the successive approximation approach.

Remark 5.1. *We note that there exists a continuous function $\xi(t)$ on J such that $|\xi(t)| \leq \varepsilon\psi(t)$ and that if the function v is a solution of (4).*

$$v'(t)=\rho(t)+\xi(t).$$

Theorem 5.2. *Assume (H1) and (H2) are the true hypotheses. There exists a unique solution u of (2) with $u_0=v_0$ and u fulfils the following estimate, for $t \in J$, for each $\varepsilon > 0$ if the function v satisfies (4).*

$$|v(t)-u(t)| \leq \varepsilon\frac{1}{(1-C)(1-CL)}\psi(t). \tag{38}$$

Proof. Based on Remark (5.1), we have that for any $\varepsilon > 0$, the function v must fulfil (4). This means that there exists a continuous function $\xi(t)$ on J such that $|\xi(t)| \leq \varepsilon\psi(t)$ and $v'(t) = \rho(t) + \xi(t)$. As a result, the integral equation is satisfied by the function v in the following way:

$$v(t) = v_0 + \int_0^t \rho(s)d(s) + \int_0^t \xi(s)d(s), \tag{39}$$

where

$$\int_0^t \rho(s)ds = \int_0^t \Big[\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau)$$
$$f_2(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau)$$
$$f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u(\tau))d\tau\Big]ds\Big]ds.$$

Similar to the proof of Theorem (3.2), we also reconsider the sequence $(u_n)_n \geq 0$ defined as in (8) with $u_0(t) = v(t), \forall t \in J$. Now, by (8), the hypothesis (H3) and (39), for n=1 one has

$$|u_1(t) - u_0(t)| = \Big|v_0 + \int_0^t \rho_0(s)ds - v(t)\Big| \leq \varepsilon \int_0^t \psi(s)d(s) \leq \varepsilon C\psi(t), \forall t \in J.$$

For $n = 1, 2, \cdots$, and from the hypothesis (H1), one has

$$|u_{n+1}(t) - u_n(t)| \leq L \int_0^t \Big(\int_0^s |u_n(s) - u_{n-1}(s)|ds + \int_0^1 |u_n(r) - u_{n-1}(r)|dr\Big)ds,$$

where $L = max\{L_1, L_2\}$. In particular for n=1 one has

$$\begin{aligned} |u_2(t) - u_1(t)| &\leq \varepsilon LC \int_0^t \int_0^s \psi(s)dsds + \varepsilon LC \int_0^t \int_0^1 \psi(r)drds \\ &= \varepsilon L(C^2 + C^3)\psi(t), \forall t \in J, \\ &\leq 2\varepsilon L(C^2 + C^3)\psi(t), \forall t \in J \end{aligned}$$

and so, for n=2, we also obtain

$$\begin{aligned} |u_3(t) - u_2(t)| &\leq L \int_0^t \int_0^s |u_2(s) - u_1(s)|dsds + L \int_0^t \int_0^1 |u_2(r) - u_1(r)|drds \\ &\leq 3\varepsilon L^2(C^3 + C^4 + C^5)\psi(t). \end{aligned}$$

and for $n \geq 4$ we have

$$|u_n(t) - u_{n-1}(t)| \leq n\varepsilon(C^n + C^{n+1} + ... + C^{2n} + C^{2n+1})L^{n-1}\psi(t). \tag{40}$$

Then by the hypothesis (H3), the estimation (10) is rewritten by:

$$\begin{aligned} |u_n(t) - u_{n-1}(t)| &\leq \varepsilon(1-0)(CL)^{n-1}(1 + C^1 + ... + C^{n+1})\psi(t) \\ &\leq \varepsilon(1)\Big(\frac{1 - C^{n+1}}{1 - C}\Big)(CL)^{n-1}\psi(t), \forall t \in J. \end{aligned}$$

In addition, if the assumption

$$|u_n(t) - u_{n-1}(t)| \leq \varepsilon(1)\Big(\frac{1 - C^{n+1}}{1 - C}\Big)(CL)^{n-1}\psi(t), \forall t \in J, \tag{41}$$

is satisfied, then by using the mathematical induction we also get

$$|u_{n+1}(t) - u_n(t) \leq \varepsilon(1)\Big(\frac{1-C^{n+2}}{1-C}\Big)(CL)^n\psi(t), \forall t \in J.$$

This yields that

$$\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \leq \varepsilon(1)\Big(\frac{1}{1-C}\Big)\sum_{n=0}^{\infty}(CL)^n\psi(t). \tag{42}$$

By the hypothesis (H3), we observe that $\sum_{n=0}^{\infty}(CL)^n \to \frac{1}{1-CL}$ as $n \to \infty$. Hence for every $\varepsilon > 0$ we infer that the series $u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t) - u_n(t)]$ is uniformly convergent on J and

$$\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \leq \varepsilon\frac{1}{(1-C)(1-CL)}\psi(t), \forall t \in J. \tag{43}$$

With the same manner as in the proof of theorem (3.2), we also can show that u(t) is a solution of (2) which has form

$$u(t) = v_0 + \int_0^t \rho(s)d(s), \forall t \in J,$$

where

$$\begin{aligned}\rho(t) :=& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[g(s) + \lambda_1\int_0^s k_1(s,\tau)f_1(\tau,u(\tau))d\tau + \lambda_2\int_0^1 k_2(s,\tau)\\ &f_2(\tau,u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\Big[g(s) + \lambda_1\int_0^s k_1(s,\tau)\\ &f_1(\tau,u(\tau))d\tau + \lambda_2\int_0^1 k_2(s,\tau)f_2(\tau,u(\tau))d\tau\Big]ds.\end{aligned}$$

In addition, the following estimate is also satisfied

$$|u(t) - v(t)| \leq \varepsilon\frac{1}{(1-C)(1-CL)}\psi(t), \forall t \in J.$$

This completes the proof. □

6. σ-semi-Hyers-Ulam-Rassias stability of non-linear Volterra-Fredhlom integro-differential equations

Using the successive approximation method, this section will provide the σ-semi-Hyers-Ulam-Rassias stability for non-linear Volterra Fredhlom integro-differential equation of fractional order with boundary conditions (2).

Remark 6.1. *We note that there exists a continuous function $\xi(t)$ on J such that $|\xi(t)| \leq \varepsilon\sigma(t)|$ and that if the function v is a solution of (4).*

$$v'(t) = \rho(t) + \xi(t).$$

Theorem 6.2. *Assume that both hypothesis (H1) and hypothesis (H2) are true. There is a single solution u of (2) with $u_0 = v_0$ and u fulfils the following estimate, for $t \in J$, for each $\varepsilon > 0$ and the $\sigma : [a,b] \to (0,\infty)$ if the function v satisfies (5).*

$$|v(t) - u(t)| \leq \varepsilon\frac{(1-0)}{(1-C)(1-CL)}\psi(t)\frac{\sigma(t)}{\sigma(0)}. \tag{44}$$

Proof. Let υ fulfil (5) for each $\varepsilon > 0$. Based on Remark (6.1), we have that there exists a continuous function $\xi(t)$ on J such that $|\xi(t)| \leq \varepsilon\sigma(t)$ and $\upsilon'(t) = \rho(t) + \xi(t)$. It may be inferred from this that the integral equation is satisfied by the function υ.

$$\upsilon(t) = \upsilon_0 + \int_0^t \rho(s)d(s) + \int_0^t \xi(s)d(s), \tag{45}$$

where

$$\begin{aligned}\int_0^t \rho(s)ds = \int_0^t \Big[\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) \\ f_2(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1 \int_0^s k_1(s,\tau) \\ f_1(\tau, u(\tau))d\tau + \lambda_2 \int_0^1 k_2(s,\tau) f_2(\tau, u(\tau))d\tau\Big]ds\Big]ds.\end{aligned}$$

Similar to the proof of Theorem (4.1), we also reconsider the sequence $(u_n)_n \geq 0$ defined as in (24) with $u_0(t) = \upsilon(t), \forall t \in J$. Now, by (24), the hypothesis (H3) and (45), for n=1 one has

$$|u_1(t) - u_0(t)| = \Big|\upsilon_0 + \int_0^t \rho_0(s)ds - \upsilon(t)\Big| \leq \varepsilon \int_0^t \psi(s)d(s)\frac{\sigma(t)}{\sigma(0)} \leq \varepsilon C\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J.$$

For $n = 1, 2, \cdots$ and from the hypothesis (H1), one has

$$|u_{n+1}(t) - u_n(t)| \leq L \int_0^t \Big(\int_0^s |u_n(s) - u_{n-1}(s)|ds + \int_0^1 |u_n(r) - u_{n-1}(r)|dr\Big)ds,$$

where $L = \max\{L_1, L_2\}$. In particular for n=1 one has

$$\begin{aligned}|u_2(t) - u_1(t)| &\leq \varepsilon LC \int_0^t \int_0^s \psi(s)dsds\frac{\sigma(t)}{\sigma(0)} + \varepsilon LC \int_0^t \int_0^1 \psi(r)drds\frac{\sigma(t)}{\sigma(0)} \\ &= \varepsilon L(C^2 + C^3)\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J, \\ &\leq 2\varepsilon L(C^2 + C^3)\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J\end{aligned}$$

and so, for n=2, we also obtain

$$\begin{aligned}|u_3(t) - u_2(t)| &\leq L \int_0^t \int_0^s |u_2(s) - u_1(s)|dsds + L \int_0^t \int_0^1 |u_2(r) - u_1(r)|drds \\ &\leq 3\varepsilon L^2(C^3 + C^4 + C^5)\psi(t)\frac{\sigma(t)}{\sigma(0)}.\end{aligned}$$

and for $n \geq 4$ we have

$$|u_n(t) - u_{n-1}(t)| \leq n\varepsilon(C^n + C^{n+1} + \cdots + C^{2n} + C^{2n+1})L^{n-1}\psi(t)\frac{\sigma(t)}{\sigma(0)}. \tag{46}$$

Then, by the hypothesis (H3), the estimation (26) is rewritten by:

$$\begin{aligned}|u_n(t) - u_{n-1}(t)| &\leq \varepsilon(1-0)(CL)^{n-1}(1 + C^1 + \cdots + C^{n+1})\psi(t)\frac{\sigma(t)}{\sigma(0)} \\ &\leq \varepsilon(1)\Big(\frac{1 - C^{n+1}}{1 - C}\Big)(CL)^{n-1}\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J.\end{aligned}$$

In addition, if the assumption

$$|u_n(t) - u_{n-1}(t)| \le \varepsilon(1)\Big(\frac{1 - C^{n+1}}{1 - C}\Big)(CL)^{n-1}\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J, \tag{47}$$

is satisfied, then by using the mathematical induction we also get

$$|u_{n+1}(t) - u_n(t) \le \varepsilon(1)\Big(\frac{1 - C^{n+2}}{1 - C}\Big)(CL)^{n}\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J.$$

This yields that

$$\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \le \varepsilon(1)\Big(\frac{1}{1 - C}\Big)\sum_{n=0}^{\infty}(CL)^{n}\psi(t)\frac{\sigma(t)}{\sigma(0)}. \tag{48}$$

By the hypothesis (H3), we observe that $\sum_{n=0}^{\infty}(CL)^n \to \frac{1}{1-CL}$ as $n \to \infty$. Hence for every $\varepsilon > 0$ we infer that the series $u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t) - u_n(t)]$ is uniformly convergent on J and

$$\sum_{n=0}^{\infty} |u_{n+1}(t) - u_n(t)| \le \varepsilon\frac{1}{(1 - C)(1 - CL}\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J. \tag{49}$$

With the same manner as in the proof of theorem (4.1), we also can show that u(t) is a solution of (2) which has form

$$u(t) = v_0 + \int_0^t \rho(s)d(s), \forall t \in J,$$

where

$$\begin{aligned}\rho(t) :=& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[g(s) + \lambda_1\int_0^s k_1(s,\tau)f_1(\tau, u(\tau))d\tau + \lambda_2\int_0^1 k_2(s,\tau)\\ & f_2(\tau, u(\tau))d\tau\Big]ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Big[g(s) + \lambda_1\int_0^s k_1(s,\tau)\\ & f_1(\tau, u(\tau))d\tau + \lambda_2\int_0^1 k_2(s,\tau)f_2(\tau, u(\tau))d\tau\Big]ds.\end{aligned}$$

In addition, the following estimate is also satisfied

$$|u(t) - v(t)| \le \varepsilon\frac{1}{(1 - C)(1 - CL)}\psi(t)\frac{\sigma(t)}{\sigma(0)}, \forall t \in J.$$

This completes the proof. □

7. Illustrative example

In this section, some examples are presented to illustrate our results.

Example 7.1. *Consider the following problem*

$$\begin{cases} {}^cD^{0.5}u(t) = & 1 + t^2 + \frac{1}{6}\int_0^t ts\sqrt{s + [u(s)]^2}ds - \\ & \frac{1}{4}\int_0^1 (t-s)\sqrt{1 + [u(s)]^2}ds, \\ 4u(0) + 2u(1) = 1. \end{cases} \tag{50}$$

We see that $v(t) = 1,\ \forall t \in [0,1]$ *and* $\alpha = 0.5,\ \lambda_1 = \frac{1}{6},\ \lambda_2 = \frac{-1}{4}$, *a=4, b=2,* $g(t) = 1+t^2,\ f_1(t,u(t)) = \sqrt{t+[u(t)]^2}$, $f_2((t,u(t)) = \sqrt{1+[u(t)]^2},\ k_1(s,t) = ts$ *and* $k_2(s,t) = t-s$ *complies with the following inequality*

$$|D^{0.5}u(t) - (1+t^2) - \frac{1}{6}\int_0^t ts\sqrt{s+[u(s)]^2}ds + \frac{1}{4}\int_0^1 (t-s)\sqrt{1+[u(s)]^2}ds| \le 10.$$

Now, we can choose $v_0(t) = u(0) = 1$. *By using the successive approximation method as in Theorem (3.2) , we obtain the following successive solution to (50) as*

$$v_0(t) = 1,$$

$$\begin{aligned} u_1(t) &= v(0) + \int_o^t \Big[1+t^2+\frac{1}{6}\int_0^s s\tau\sqrt{\tau+[u(\tau)]^2}d\tau - \frac{1}{4}\int_0^1 (t-s)\sqrt{1+[u(r)]^2}dr\Big]ds \\ &= 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!} \end{aligned}$$

Then it is no difficult to see that $u(t) = 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}$ *forms a solution (50) and one gets the estimate*

$$|v(t) - u(t)| = |1 - (1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!})| \le \frac{8}{5}.$$

Next, we define the function $u^*(t) = 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\cdots$ *is also a solution of (50) and we also have*

$$|v(t) - u^*(t)| = |1 - (1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!})| \le \frac{38}{42}.$$

Therefore, it shows the function $u^*(t)$ *is better approximate solution than the function u(t).*

Example 7.2. *Consider the following problem*

$$\begin{cases} {}^cD^1u(t) & = 1+\int_0^t \Big[1+\int_0^s \frac{(t-s)}{2}ds - \\ & \int_0^1 \frac{s}{4}ds\Big]\sigma(s)ds, \forall t \in [0,1], \\ 3u(0)+5u(1) = 1 & \end{cases} \tag{51}$$

where, $\rho(t) = 1+\int_0^s \frac{(t-s)}{2}ds - \int_0^1 \frac{s}{4}ds$ *is continuous and integrable for* $t \in [0,1]$ *and* $\sigma(s) \in C[0,1]$.
Now, we can choose $v_0(t) = u(0) = 1$. *By using the successive approximation method as in Theorem (4.1) , we obtain the following successive solution to (51) as*

$$u_1(t) = v(0) + \int_0^t \Big[\int_0^s \frac{(t-s)}{2} - \int_0^1 \frac{s}{4}\Big]\sigma(s)ds = 1 + [\frac{t^2}{4} - \frac{t^2}{8}] = 1 + \frac{t^2}{8}.$$

There is no difficult to see that $u(t) = 1+\frac{t^2}{8}$ *forms a solution (51) and one gets the estimate*

$$|v(t) - u(t)| = |1 - (1+\frac{t^2}{8})| \le \frac{10}{13}.$$

Next we define the function $u^*(t) = 1+\frac{t^2}{8}+\frac{9t^3}{30}+\cdots$ *is also a solution of (51) and we also have*

$$|v(t) - u^*(t)| = |1 - (1+\frac{t^2}{8}+\frac{9t^3}{30})| \le \frac{33}{45}.$$

Therefore , it shows the function $u^*(t)$ *is better approximate solution than the function u(t).*

Example 7.3. *Consider the following problem*

$$\begin{cases} {}^cD^\alpha u(t) & = t^2 + \frac{1}{2}\int_0^t (s^2 + [u(s)]^2)ds + \\ & \frac{1}{3}\int_0^1 t(s + [u(s)])ds, \\ u(0) + 2u(1) = 2 \end{cases} \tag{52}$$

where $v(t) = 2,\ \ \forall t \in [0,2]$ *and* $\alpha = 1,\ \lambda_1 = \frac{1}{2},\ \lambda_2 = \frac{1}{3},\ a=1,\ b=2,\ g(t) = t^2,\ f_1(t,u(t)) = t^2 + [u(t)]^2,$ $f_2((t,u(t)) = t + [u(t)],\ k_1(s,t) = 1$ *and* $k_2(s,t) = t$ *complies with the following inequality*

$$\left|{}^cD^\alpha u(t) - t^2 - \frac{1}{2}\int_0^t (s^2 + [u(s)]^2 ds - \frac{1}{3}\int_0^1 t(s + [u(s)])ds\right| \leq 3e^t.$$

Now, we can $v_0(t) = u(0) = 2$. *By using the successive approximation method as in Theorem (5.2) , we obtain the following successive solution to (52) as*

$$v_0(t) = 2,$$

$$u_1(t) = v(0) + \int_o^t \Big[t^2 + \frac{1}{2}\int_0^s (\tau^2 + (u[\tau])^2 d\tau + \frac{1}{3}\int_0^1 t(s + u[s])ds\Big]ds = 2 + t^3 + \frac{t^2}{2!} + \frac{t^4}{3!}$$

Then $u(t) = 2 + t^3 + \frac{t^2}{2!} + \frac{t^4}{3!}$ *is a solution (52) and we gets the*

$$|v(t) - u(t)| = \Big|2 - \Big(2 + t^3 + \frac{t^2}{2!} + \frac{t^4}{3!}\Big)\Big| \leq \frac{3}{7}e^t.$$

We define the function $u^*(t) = 2 + t^3 + \frac{t^2}{2!} + \frac{t^4}{3!} + \frac{t^6}{4!} + \cdots$ *is also a solution of (52) and we have*

$$|v(t) - u^*(t)| = \Big|2 - \Big(2 + t^3 + \frac{t^2}{2!} + \frac{t^4}{3!} + \frac{t^6}{4!}\Big)\Big| \leq \frac{31}{25}e^t.$$

Therefore, the function $u^*(t)$ *is better approximate solution than the function u(t).*

Example 7.4. *Consider the following problem*

$$\begin{cases} {}^cD^\alpha u(t) & = 3 + \int_0^t \Big[\int_0^s \frac{(t-s)}{3}ds - \\ & \int_0^1 \frac{(1-s)}{4}ds\Big]\sigma(s)ds, \forall t \in [0,3], \\ 2u(0) + 3u(1) = 3 \end{cases} \tag{53}$$

where, $\rho(t) = \int_0^s \frac{(t-s)}{3}ds - \frac{(1-s)}{4}ds$ *is continuous and integrable for* $t \in [0,3]$ *and* $\sigma(s) \in C[0,3]$.
Now, we can $v_0(t) = u(0) = 3$. *By using the successive approximation method as in Theorem (6.2) , we obtain the following successive solution to (53) as*

$$u_1(t) = v(0) + \int_0^t \Big[\int_0^s \frac{(t-s)}{3} - \int_0^1 \frac{(1-s)}{4}\Big]\sigma(s)ds = 3 + \Big[\frac{t^2}{6} + \frac{t^2}{8}\Big] = 3 + \frac{7t^2}{24}.$$

where $u(t) = 3 + \frac{7t^2}{24}$ *is a solution of (53) and we get*

$$|v(t) - u(t)| = \Big|3 - \Big(3 + \frac{7t^2}{24}\Big)\Big| \leq \frac{7}{8}e^{2t}.$$

We define the function $u^*(t) = 3 + \frac{7t^2}{24} + \frac{7t^3}{36} + \cdots$ *is also a solution of (53)*

$$|v(t) - u^*(t)| = \Big|3 - \Big(3 + \frac{7t^2}{24} + \frac{7t^3}{36}\Big)\Big| \leq \frac{29}{38}e^{2t}.$$

Therefore, the function $u^*(t)$ *is better approximate solution than the function u(t).*

8. Conclusion

This research focuses on solving and analysing a nonlinear Volterra-Fredhlom integro-differential equations of fractional order with boundary conditions. A new successive approximation approach is used to solve the Hyers-Ulam stability problem. The results also demonstrate that the Ulam stability study field finds the successive approximation to be more efficient and practical. The results demonstrate that there is only one solution to the nonlinear Volterra-Fredhlom integro-differential equation of fractional order with boundary conditions and that it is possible to properly restrict the approximate solutions. The investigation's findings validate relevant cases. The Volterra-Fredholm integro-fractional differential problems are promising as a kind of highly integrated boundary value problem in integrated fractional operators; however, our work will continue to focus on improving numerical solutions and gearbox efficiency.

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