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Rank of truncated Hankel operators on model spaces

Caixing Gu[a], **Dong-O Kang**[b,*]

[a]*California Polytechnic State University, United States*
[b]*Chungnam National University, Republic of Korea*

Abstract. We study the rank of truncated Hankel operators on model spaces $K_u^2 := H^2 \ominus uH^2$. Concrete rank formulas will be established for several classes of finite rank truncated Hankel operators. The rank formulas will be derived seperately depending on whether the model space K_u^2 is finite-dimensional, or, infinite-dimensional.

1. Introduction

The study of truncated Toeplitz operators have been quite extensive since the initiating paper of D. Sarason appeared in 2007([16]). Regarding the rank, there also appeared a series of papers on finite rank truncated Toeplitz operators mostly for the last decade([2–5, 9, 11–13]). Meanwhile, the study of truncated Hankel operators has been rather recent compared with that of truncated Toeplitz operators and there seem to be many topics that are still unattempted. A recently published paper [10] which describes various algebraic properties of truncated Hankel operators seems to be offering various tools for further research on truncated Hankel operators. In this paper, we give rank formulas for several classes of truncated Hankel operators in terms of the properties of their symbol functions.

We begin with the introduction of Hardy space. $L^2 \equiv L^2(\mathbb{T})$ denotes the set of all square-integrable functions on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$. $H^2 \equiv H^2(\mathbb{T})$ denotes the closed linear span of all analytic polynomials in L^2 and this space H^2 is called *Hardy space*. The space H^∞ is defined by $H^\infty := H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$. A function $u \in H^\infty$ is called *inner* if $|u(z)| = 1$ almost everywhere on the unit circle $\mathbb{T}$.

For $\phi \in L^\infty$, the Toeplitz operator T_ϕ on H^2 is defined by

$$T_\phi f = P(\phi f),$$

where P is the orthogonal projection of L^2 onto H^2. The function ϕ is called the *symbol* of T_ϕ. The Hankel operator $H_\phi : H^2 \longrightarrow H^2$ with symbol $\phi \in L^\infty$ is defined by

$$H_\phi f = JP_-(\phi f),$$

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* Corresponding author: Dong-O Kang
Email addresses: cgu@calpoly.edu (Caixing Gu), mutjaykang@hanmail.net (Dong-O Kang)
ORCID iDs: https://orcid.org/0000-0001-6289-7755 (Caixing Gu), https://orcid.org/0000-0001-6524-6210 (Dong-O Kang)

where P_- denotes the orthogonal projection of L^2 onto $L^2 \ominus H^2$ and J denotes the unitary operator on L^2 defined by $Jh(z) = \overline{z}h(\overline{z})$ for $h \in L^2$. For the adjoint of Toeplitz operators and Hankel operators, it is well known that

$$T^*_\phi = T_{\overline{\phi}} \text{ and } H^*_\varphi = H_{\hat{\phi}},$$

where $\hat{\phi} := \overline{\phi(\overline{z})}$ for any $\phi \in L^2$. Also well-known are the following basic relations between Toeplitz and Hankel operators:

$$T_{\phi\psi} - T_\phi T_\psi = H^*_{\overline{\phi}} H_\psi, \tag{1}$$

$$T_{\phi\psi} = T_\phi T_\psi \text{ if and only if } \phi \in \overline{H^\infty} \text{ or } \psi \in H^\infty, \tag{2}$$

$$H_\phi T_\psi = H_{\phi\psi} \text{ and } T_{\overline{\psi}} H_\phi = H_{\hat{\psi}\phi}, \text{ if } \psi \in H^\infty, \tag{3}$$

where $\hat{\psi}(z) = \overline{\psi(\overline{z})}$ as mentioned above. The following relations among P, P_- and J are also useful and can be verified easily :

$$JP_- = PJ \text{ and } JP = P_-J. \tag{4}$$

For the unilateral shift $S := T_z$ on H^2, a Hankel operator H_ϕ satisfies $H_\phi S = S^* H_\phi$. From this property we find that $\ker H_\phi$ is invariant for the shift operator S. By Beurling's Theorem, if H_ϕ has a nonzero kernel, then $\ker H_\phi = uH^2$ for some inner function u. Regarding the rank of Hankel operators, the well-known Kronecker's Theorem says H_ϕ is of finite rank if and only if $P_-\phi$ is a rational function and in that case, the rank of H_ϕ equals the number of poles of $P_-\phi$ in the unit disk counting multiplicity.

An inner function u is called a finite Blaschke product if $u = \lambda \prod_{i=1}^n \frac{z-c_i}{1-\overline{c_i}z}$ where λ is a complex number of unit modulus and c_i are complex numbers in the unit disk. For a nonconstant inner function u, define the model space K^2_u by

$$K^2_u := H^2 \ominus uH^2.$$

It is known that the dimension of K^2_u is finite if and only if u is a finite Blaschke product and in that case, the dimension of K^2_u equals the number of zeros of u counting multiplicity. The dimension of K^2_u is also called the *degree* of the inner function u and is denoted by $deg\, u$. If u is not a finite Blaschke product, then K^2_u is an infinite dimensional space and the degree of u is infinite. It is also a well-known fact that if u is a finite Blaschke product, all functions in K^2_u are rational functions. The following set equality is easily verified and is used occasionally in this paper :

$$K^2_u = u\overline{zK^2_u}. \tag{5}$$

From the above equality, we observe that an H^2-function f is an element of K^2_u if and only if $\overline{u}f \in \overline{zH^2}$. It is also easily checked that $f \in K^2_u$ if and only if $\hat{f} \in K^2_{\hat{u}}$. These conditions are also frequently used without mention in this paper. For a function $\phi \in L^2(\mathbb{T})$, the truncated Toeplitz operator A_ϕ on K^2_u is defined by

$$A_\phi f = P_u(\phi f), \text{ for } f \in K^2_u,$$

where P_u denotes the orthogonal projection of L^2 onto K^2_u. For $\phi \in L^2(\mathbb{T})$, the truncated Hankel operator B_ϕ on K^2_u is defined by

$$B_\phi f = P_u(H_\phi f) = P_u[JP_-(\phi f)], \text{ for } f \in K^2_u.$$

To give a familiar examples of truncated Toeplitz operators and truncated Hankel operators,let the inner function u to be z^n. Then $\{1, z, z^2, \cdots, z^{n-1}\}$ forms an orthonormal basis for K^2_u and with respect to this basis, the truncated Toeplitz operator A_ϕ and the truncated Hankel operator B_ϕ are represented as the following forms of matrices, each of which is called a *Toeplitz matrix* and a *Hankel matrix*, respectively :

$$A_\phi = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ddots & \vdots \\ a_2 & a_1 & a_0 & \ddots & a_{-2} \\ \vdots & \ddots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & a_2 & a_1 & a_0 \end{pmatrix},$$

and

$$B_\phi = \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots & a_{-n} \\ a_{-2} & a_{-3} & a_{-4} & \ddots & \vdots \\ a_{-3} & a_{-4} & \ddots & \ddots & a_{-2n-3} \\ \vdots & \ddots & \ddots & a_{-2n-3} & a_{-2n-2} \\ a_{-n} & \cdots & a_{-2n-3} & a_{-2n-2} & a_{-2n-1} \end{pmatrix},$$

where a_i's are the i-th Fourier coefficients of the symbol ϕ. For the adjoints of truncated Toeplitz operators and truncated Hankel operators, the following can be checked easily:

$$A_\phi^* = A_{\overline{\phi}} \text{ and } B_\phi^* = B_{\hat{\phi}}. \tag{6}$$

2. Rank of truncated Hankel operators on infinite-dimensional model spaces

In this section, we focus on the rank of truncated Hankel operators on infinite-dimensional model spaces. Recall that a model space K_u^2 is finite-dimensional if and only if the inner function u is a finite Blaschke product. It is also well-known that an inner function u is a finite Blaschke product if and only if it is a rational function.

Before proceeding to the main results, let's recall some known results about truncated Toeplitz operators, truncated Hankel operators and model spaces. The following two theorems give conditions that a truncated Toeplitz operator or a truncated Hankel operator be the zero operator.

Theorem 2.1. *(Sarason, [16]) A truncated Toeplitz operator A_ϕ is the zero operator if and only if $\phi \in \overline{uH^2} + uH^2$.*

Theorem 2.2. *(Gu-Ma, [10]) A truncated Hankel operator B_ϕ is the zero operator if and only if $\phi \in H^2 + \overline{u\hat{u}H^2}$.*

Note that the above theorems say that the symbol function of a truncated Toeplitz operator, or, a truncated Hankel operator is not uniquely determined. In view of the two theorems above, we may assume that the symbol function ϕ of A_ϕ on K_u^2 lies in $K_u^2 + \overline{K_u^2}$ and the symbol function ψ of B_ψ lies in $\overline{K_{u\hat{u}}^2}$. In this paper, we keep this fact in mind, but, do not always impose this restriction on symbol functions.

The following theorem by Axler, Chang and Sarason which computes the rank of the product of two Hankel operators plays an important role in the proofs of the main results.

Theorem 2.3. *(Axler-Chang-Sarason, [1]) The product $H_f^*H_g$ has finite rank if and only if either H_f or H_g has finite rank. In this case, the rank of $H_f^*H_g$ equals the minimum of the ranks of the two Hankel operators.*

The next lemma was essentially proved by T. Nakazi in [14] and slightly supplemented with the dimension formula by the authors in [9].

Lemma 2.4. *(Gu-Kang-Nakazi,[9, 14]) For finite Blaschke products θ_1 and θ_2, $K^2_{\theta_1} \cap \theta_2 H^2$ is a nontrivial space if and only if $deg\theta_1 > deg\theta_2$ and in this case, $\dim(K^2_{\theta_1} \cap \theta_2 H^2) = deg\theta_1 - deg\theta_2$.*

Let C be a map on $L^2(\mathbb{T})$ defined by $C\phi = u\overline{z\phi}$, then it is known that C is a conjugate linear isometry. Moreover, C maps K^2_u onto itself.

Lemma 2.5. *(Gu-Ma, [10]) A bounded operator B on K^2_u is a truncated Hankel operator if and only if CBC is. Moreover, $CB_\phi C = B_\psi$ with $\psi = \overline{u\hat{u}\phi}$.*

The following identity is frequently used in the paper :

$$P_u = 1 - T_u T_u^* = H^*_{\overline{u}} H_{\overline{u}}, \tag{7}$$

where the second equality comes from (1). Throughout the paper, for a rational function r, we write $\alpha(r)$ to denote the number of poles of r inside the unit circle $\mathbb{T}$ counting multiplicity.

Lemma 2.6. *Let u be an inner function of infinite degree. For a rational function r with no poles on the unit circle, $RankB_r = \alpha(r)$, where, $\alpha(r)$ denotes the number of poles of r inside the unit circle including multiplicity.*

Proof. We first claim

$$\text{rank} H^*_{\overline{u}} H_{\overline{u}} H_r = \text{rank} H_r. \tag{8}$$

Note that every function in RangeH_r is a rational function since r is a rational function. Let $f \in$ RangeH_r be a nonzero function, then $f \notin uH^2 = \ker H^*_{\overline{u}} H_{\overline{u}}$ since f is a nonzero rational function, whereas u is an inner function of infinite degree. Thus Range$H_r \cap \ker H^*_{\overline{u}} H_{\overline{u}} = \{0\}$, which proves the claim. Note that RangeB_r = Range$H^*_{\overline{u}} H_{\overline{u}} H_r H^*_{\overline{u}} H_{\overline{u}}$. Observe

$$\begin{aligned} \text{rank} H_r H^*_{\overline{u}} H_{\overline{u}} &= \text{rank}(H_r H^*_{\overline{u}} H_{\overline{u}})^* \\ &= \text{rank} H^*_{\overline{u}} H_{\overline{u}} H_{\hat{r}} \\ &= \text{rank} H_{\hat{r}} \\ &= \text{rank} H_r, \end{aligned}$$

where the third equality is by the above claim (8) with r replaced by $\hat{r}$ and the fourth equality is obvious since the poles of $\hat{r}$ are exactly the complex conjugate of the poles of r. Thus Range$H_r H^*_{\overline{u}} H_{\overline{u}}$ = RangeH_r and from this we have

$$\begin{aligned} \text{Range} B_r &= \text{Range} H^*_{\overline{u}} H_{\overline{u}} H_r H^*_{\overline{u}} H_{\overline{u}} \\ &= \text{Range} H^*_{\overline{u}} H_{\overline{u}} H_r. \end{aligned}$$

Using the claim (8) again, we have rankB_r = rank$H_r = \alpha(r)$ as desired. □

Lemma 2.7. *Let u be an inner function of infinite degree. For a rational function r with no poles on the unit circle, $RankB_{\overline{u\hat{u}r}} = \alpha(\overline{r})$.*

Proof. Since the map C on L^2 defined by $Ch = u\overline{zh}$ is an isometry of K^2_u onto itself, the rank of $B_{\overline{u\hat{u}r}}$ equals the rank of $CB_{\overline{u\hat{u}r}}C$. Using Lemma 2.5, we have

$$\begin{aligned} \text{rank} B_{\overline{u\hat{u}r}} &= \text{rank} CB_{\overline{u\hat{u}r}} C \\ &= \text{rank} B_{\overline{u\hat{u}\overline{u\hat{u}r}}} \\ &= \text{rank} B_{\overline{r}}. \end{aligned}$$

Since the previous theorem says rank$B_{\overline{r}} = \alpha(\overline{r})$, we get the desired conclusion. □

The following lemma is from [9]. We include the proof just for readers' convenience.

Lemma 2.8. *(Gu-Kang, [9]) For an analytic rational function r and an inner function u, $r - P_u(r) = ur'$ for some analytic rational function r'.*

Proof. Since $\bar{r}$ is also a rational function, we have $\ker H_{u\bar{r}} = \theta H^2$ for some finite Blaschke product θ. Since P_u is the projection of L^2 onto K_u^2, clearly, $r - P_u(r) = uh$ for some $h \in H^2$. So we have $\ker H_{\bar{h}} = \ker H_{u\bar{r} - u\overline{P_u(r)}} = \ker H_{u\bar{r}} = \theta H^2$. Thus Kronecker's theorem says $\bar{h}$ is a rational function, equivalently, h is a rational function. □

The following lemma from [9] plays a crucial role in proofs of the main theorems. The proof was skipped in [9], but, here we include the proof for readers' convenience.

Lemma 2.9. *(Gu-Kang, [9]) For two finite-rank operators A and B in $B(H)$, $rank(A + B) = rank(A) + rank(B)$ if and only if $RangeA \cap RangeB = \{0\}$ and $RangeA^* \cap RangeB^* = \{0\}$.*

Proof. For the proof of sufficiency($\Longleftarrow$), assume $\mathrm{Range}A \cap \mathrm{Range}B = \{0\}$ and $\mathrm{Range}A^* \cap \mathrm{Range}B^* = \{0\}$. First note

$$(\ker A)^\perp \cap (\ker B)^\perp = \mathrm{Range}A^* \cap \mathrm{Range}B^* = \{0\}.$$

We claim that $A + B$ maps $(\ker A)^\perp + (\ker B)^\perp$ onto $\mathrm{Range}A + \mathrm{Range}B$ in 1-1 fashion. Since every operator maps the orthogonal complement of its kernel onto its range in 1-1 fashion, we know

$$\dim(\ker A)^\perp = \mathrm{rank}A \text{ and } \dim(\ker B)^\perp = \mathrm{rank}B.$$

Since it is clear that $\mathrm{Range}(A + B) \subseteq \mathrm{Range}A + \mathrm{Range}B$, proof of the 1-1 part automatically implies to the onto part. Let $a + b \in (\ker A)^\perp + (\ker B)^\perp$ and assume $(A + B)(a + b) = 0$. Observe

$$(A + B)(a + b) = A(a + b) + B(a + b) = 0.$$

Since we assumed $\mathrm{Range}A \cap \mathrm{Range}B = \{0\}$, the equation above implies $A(a + b) = B(a + b) = 0$, that is, $a + b \in \ker A \cap \ker B$. But, recalling

$$a + b \in (\ker A)^\perp + (\ker B)^\perp = (\ker A \cap \ker B)^\perp,$$

we conclude $a + b = 0$, proving the 1-1 part of the claim. As we mentioned above, the onto part of the claim follows automatically. Therefore we conclude $\mathrm{Range}(A + B) = \mathrm{Range}A + \mathrm{Range}B$ which clearly implies $\mathrm{rank}(A + B) = \mathrm{rank}(A) + \mathrm{rank}(B)$.

For the proof of necessity($\Longrightarrow$), we will assume either $\mathrm{Range}A \cap \mathrm{Range}B \neq \{0\}$ or $\mathrm{Range}A^* \cap \mathrm{Range}B^* \neq \{0\}$. First, let $\dim(\mathrm{Range}A \cap \mathrm{Range}B) = m > 0$. Then

$$\begin{aligned} \mathrm{rank}(A + B) &= \dim(\mathrm{Range}(A + B)) \\ &\leq \dim(\mathrm{Range}A + \mathrm{Range}B) \\ &= \dim(\mathrm{Range}A) + \dim(\mathrm{Range}B) - m \\ &< \mathrm{rank}(A) + \mathrm{rank}(B). \end{aligned}$$

To complete the proof, let $\mathrm{Range}A^* \cap \mathrm{Range}B^* = m > 0$, then we have

$$\begin{aligned} \dim((\ker A)^\perp + (\ker B)^\perp) &= \dim(\ker A)^\perp + \dim(\ker B)^\perp - m \\ &= \mathrm{rank}A^* + \mathrm{rank}B^* - m \\ &= \mathrm{rank}A + \mathrm{rank}B - m \\ &< \mathrm{rank}A + \mathrm{rank}B. \end{aligned}$$

Since $\ker A \cap \ker B \subseteq \ker(A + B)$,

$$\mathrm{Range}(A + B) = (A + B)(\ker A \cap \ker B)^\perp = (A + B)((\ker A)^\perp + (\ker B)^\perp).$$

Therefore,

$$\mathrm{rank}(A + B) \leq \dim((\ker A)^\perp + (\ker B)^\perp) < \mathrm{rank}A + \mathrm{rank}B.$$

□

We are now ready for the main theorem of this section. Recall by Theorem 2.2 that general symbol functions of truncated Hankel operators may be considered to lie in $\overline{K^2_{u\hat{u}}}$. Since $K^2_{u\hat{u}} = K^2_u \oplus uK^2_{\hat{u}} = K^2_u \oplus u\hat{u}z\overline{K^2_{\hat{u}}}$, a function ϕ in $\overline{K^2_{u\hat{u}}}$ may be written as $\phi = \overline{f} + \overline{u\hat{u}}g$ with $f \in K^2_u$ and $g \in zK^2_{\hat{u}}$. But, in this section, we don't give such a restriction on f and g. Instead, we give them a restriction that they be rational functions. For a rational function r, it is well-known and easy to show that all functions in the range of H_r are rational functions.

Theorem 2.10. *Let u be an inner function of infinite degree. Then for rational functions r_1 and r_2,*

$$rank B_{r_1+\overline{u\hat{u}}r_2} = \alpha(r_1) + \alpha(\overline{r_2}).$$

Proof. First note $B_{r_1+\overline{u\hat{u}}r_2} = B_{r_1} + B_{\overline{u\hat{u}}r_2}$. To make use of Lemma 2.9, we will prove the two equations :

$$\begin{aligned} &1.\ \text{Range}B_{r_1} \cap \text{Range}B_{\overline{u\hat{u}}r_2} = \{0\} \\ &2.\ \text{Range}B^*_{r_1} \cap \text{Range}B^*_{\overline{u\hat{u}}r_2} = \{0\}. \end{aligned} \tag{9}$$

Note that

$$\begin{aligned} \text{Range}B_{r_1} &= \text{Range}P_uH_{r_1}P_u \\ &= P_uH_{r_1}(K^2_u). \end{aligned}$$

Since every element in RangeH_{r_1} is a rational function, as we mentioned above, every element in RangeB_{r_1} may be written as P_ua for a rational function a. Using Lemma 2.8 we may write

$$P_ua = a - ua_1 \tag{10}$$

for some analytic rational function a_1. To find the forms of functions in Range$B_{\overline{u\hat{u}}r_2}$, recall that C maps K^2_u isometrically onto itself and note $C^2 = I$. Now observe

$$\begin{aligned} \text{Range}B_{\overline{u\hat{u}}r_2} &= \text{Range}B_{\overline{u\hat{u}}r_2}C \\ &= \text{Range}CCB_{\overline{u\hat{u}}r_2}C \\ &= \text{Range}CB_{\overline{r_2}} \\ &= \text{Range}CP_uH_{\overline{r_2}}P_u. \end{aligned} \tag{11}$$

Since the range of $H_{\overline{r_2}}$ consists of rational functions, every element in Range$CP_uH_{\overline{r_2}}P_u$ may be written as CP_ub for some rational function b. Again, using Lemma 2.8 we may write $P_ub = b - ub_1$ for some analytic rational funtion b_1. Thus we have

$$CP_ub = C(b - ub_1) = u\overline{z(b - ub_1)} = u\overline{zb} - \overline{zb_1}. \tag{12}$$

Let $f \in$ Range$B_{r_1} \cap$ Range$B_{\overline{u\hat{u}}r_2}$, then by equations (10) and (12), we have

$$f = a - ua_1 = u\overline{zb} - \overline{zb_1}.$$

Note that the complex conjugate of every rational function defined on the unit circle $\mathbb{T}$ is also rational and therefore both $\overline{zb}$ and $\overline{zb_1}$ are rational functions. Since u is an inner function of infinite degree, it is not a rational function. Therefore, $u(a_1 + \overline{zb}) = a + \overline{zb_1}$ implies $a_1 = -\overline{zb}$ and $a = -\overline{zb_1}$. Note that a and a_1 are analytic functions, whereas $\overline{zb}$ and $\overline{zb_1}$ are coanalytic. Therefore, we find $a = a_1 = b = b_1 = 0$, that is, $f = 0$. Now the proof of the first equation in (9) is complete.

For the second equation in (9), by the properties of the map C that were mentioned above, we find it is sufficient to show

$$\text{Range}(CB^*_{r_1}C) \cap \text{Range}(CB^*_{\overline{u\hat{u}}r_2}C) = \{0\}. \tag{13}$$

By Lemma 2.5, we may observe

$$CB^*_{r_1}C = CB_{\hat{r_1}}C = B_{\overline{u\hat{u}}\cdot\overline{\hat{r_1}}}$$

and

$$\begin{aligned} CB^*_{\overline{u\hat{u}}r_2}C &= CB_{\overline{u\hat{u}}\hat{r_2}}C \\ &= B_{\overline{u\hat{u}u\hat{u}\hat{r_2}}} \\ &= B_{\overline{\hat{r_2}}}. \end{aligned}$$

Since $\overline{\hat{r_1}}$ and $\overline{\hat{r_2}}$ are both rational functions, the same argument as the proof of the first equation of (9) will prove (13). Therefore the proof of the two equations in (9) is complete. Now the application of Lemma 2.9 together with Lemma 2.6 and Lemma 2.7 directly leads us to the desired conclusion. The proof is complete. □

In [10], all finite rank truncated Hankel operators were characterized. It says B_ϕ is of finite rank if and only if $\phi = r_1 + \overline{u\hat{u}}r_2 + \sum_{i=1}^{l} c_i s_i(z)$ where r_1 and r_2 are rational functions in H^2, c_i are constant complex numbers and $s_i(z)$ are kernel functions and their angular derivatives in the sense of Carathéodory at points on $\mathbb{T}$. In this section we derived the rank formula for truncated Hankel operators on infinite dimensional model spaces when the symbol functions don't contain the terms $\sum_{i=1}^{l} c_i s_i(z)$.

3. Rank of truncated Hankel operators on finite dimensional model spaces

In this section, we focus on rank of truncated Hankel operators on finite-dimensional model spaces. Recall again that a model space K_u^2 is finite-dimensional if and only if the inner function u is a finite Blaschke product, or, equivalently, if and only if u is a rational function. For this section, we use Theorem 2.2 and assume that the symbol functions of truncated Hankel operators lie in $\overline{K^2_{u\hat{u}}}$. Since $K^2_{u\hat{u}} = K^2_u \oplus uK^2_{\hat{u}} = K^2_u \oplus u\hat{u}\overline{zK^2_{\hat{u}}}$, a function ϕ in $\overline{K^2_{u\hat{u}}}$ may be written as $\phi = \overline{f} + \overline{u\hat{u}}g$ with $f \in K^2_u$ and $g \in zK^2_{\hat{u}}$.

Lemma 3.1. *Let u be a finite Blaschke product and let $\varphi = \overline{f}$, where $f \in K^2_u$. Then $rankB_\varphi = rankH_{\overline{f}} = \alpha(\overline{f})$.*

Proof. Since the range of B_φ equals the range of $B_\varphi P_u$, by an abuse of notation,

$$\begin{aligned} B_\varphi &= P_u H_\varphi P_u \\ &= H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}}H^*_{\overline{u}}H_{\overline{u}} \\ &= H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}}(1 - T_u T_{\overline{u}}) \\ &= H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}} - H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}}T_u T_{\overline{u}} \\ &= H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}} - H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}\cdot u}T_{\overline{u}} \\ &= H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}}. \end{aligned} \tag{14}$$

For the third equality we used (1) and for the fifth equality, we used (3). Since $f \in K^2_u$, we find $\overline{f}u$ is an analytic function and therefore $H_{\overline{f}\cdot u} = 0$, which gives the last equality. We claim that

$$\ker H^*_{\overline{u}}H_{\overline{u}} \cap \text{range}H_{\overline{f}} = \{0\}. \tag{15}$$

Observe $T_{\overline{\hat{u}}}H_{\overline{f}} = H_{u\cdot\overline{f}} = 0$ since $u\overline{f}$ is an analytic symbol of a Hankel operator. Therefore we find

$$\text{range}H_{\overline{f}} \subseteq \ker T_{\overline{\hat{u}}} = K^2_{\hat{u}}. \tag{16}$$

We may verify the above inclusion also by observing

$$H_{\overline{f}}h = J(I-P)(\overline{f}h) = PJ(\overline{f}h) = P(\overline{z}\hat{\overline{f}}\hat{h}),$$

where the last one is easily seen to lie in $K^2_{\hat{u}}$ for any $h \in H^2$. Indeed, we may write $P(\overline{z}\hat{f}\hat{\overline{h}}) = \overline{z}\hat{f}\hat{\overline{h}} - \overline{zh_1}$ for some $h_1 \in H^2$ and we observe $\overline{\hat{u}}P(\overline{z}\hat{f}\hat{\overline{h}}) = \overline{\hat{u}}\hat{f}\overline{z}\hat{\overline{h}} - \overline{z\hat{u}h_1} \in \overline{zH^2}$ since $\overline{\hat{u}}\hat{f} \in \overline{zH^2}$, which proves $P(\overline{z}\hat{f}\hat{\overline{h}}) \in K^2_{\hat{u}}$.

Note that $\ker H^*_{\overline{u}}H_{\overline{u}} = uH^2$. Since $\deg \hat{u} = \deg u$, using Lemma2.4, we have $uH^2 \cap K^2_{\hat{u}} = \{0\}$ and this together with (16) gives $\ker H^*_{\overline{u}}H_{\overline{u}} \cap \mathrm{range}H_{\overline{f}} = \{0\}$. Therefore, we conclude that $\mathrm{rank}H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}} = \mathrm{rank}H_{\overline{f}} = \alpha(\overline{f})$ as we wanted. □

Lemma 3.2. *Let u be a finite Blaschke product and let $\varphi = \overline{u\hat{u}}g$ with $g \in K^2_{\hat{u}}$. Then $\mathrm{rank}B_\varphi = \mathrm{rank}H_{\overline{\hat{g}}} = \alpha(\overline{\hat{g}})$.*

Proof. We first claim

$$P_uH_{\overline{u\hat{u}}g}P_u = H^*_{\overline{u}}H_{\overline{\hat{g}}}H_{\overline{u}}. \tag{17}$$

For the proof of the claim, let $h = h_1 + uh_2 \in H^2$ be an arbitrary element in H^2, where, $h_1 = P_uh$ and $uh_2 = h - h_1$. The following observation proves the claim.

$$\begin{aligned} P_uH_{\overline{u\hat{u}}g}P_uh &= P_uJ(I-P)(\overline{u\hat{u}}gh_1) \\ &= P_uJ(\overline{u\hat{u}}gh_1) \\ &= P_u(\overline{z}\overline{u(\overline{z})\hat{u}(\overline{z})}g(\overline{z})h_1(\overline{z})) \\ &= P_u(\hat{u}uz\overline{\hat{g}\hat{h}_1}) \\ &= P_uT_{u\overline{\hat{g}}}(\hat{u}z\overline{\hat{h}_1}) \\ &= H^*_{\overline{u}}H_{\overline{u}}T_{u\overline{\hat{g}}}J(I-P)(\overline{u}h) \\ &= H^*_{\overline{u}}H_{\overline{u}\cdot u\overline{\hat{g}}}H_{\overline{u}}h \\ &= H^*_{\overline{u}}H_{\overline{\hat{g}}}H_{\overline{u}}h. \end{aligned} \tag{18}$$

Now we claim

$$\mathrm{rank}H^*_{\overline{u}}H_{\overline{\hat{g}}}H_{\overline{u}} = \mathrm{rank}H_{\overline{\hat{g}}}H_{\overline{u}}. \tag{19}$$

For its proof, let a be a nonzero element in $\mathrm{Range}H_{\overline{\hat{g}}}H_{\overline{u}}$. Then $a = H_{\overline{\hat{g}}}H_{\overline{u}}h$ for some $h \in H^2$. Since a is nonzero, $h \notin \ker H_{\overline{\hat{g}}}H_{\overline{u}}$. Using the relation (3), we have

$$T_{\overline{\hat{u}}}H_{\overline{\hat{g}}}H_{\overline{u}}h = H_{u\cdot\overline{\hat{g}}}H_{\overline{u}}h = 0$$

since $u \cdot \overline{\hat{g}}$ is an analytic symbol of a Hankel operator. Thus we find $a = H_{\overline{\hat{g}}}H_{\overline{u}}h \in \ker T_{\overline{\hat{u}}} = K^2_{\hat{u}}$. Since $\ker H^*_{\overline{u}} = \ker H_{\overline{\hat{u}}} = \hat{u}H^2$, we find that $a \notin \ker H^*_{\overline{u}}$. Since a is an arbitrary nonzero element in $\mathrm{range}H_{\overline{\hat{g}}}H_{\overline{u}}$, the proof of the claim(i.e., equation (19)) is complete. Since Axler-Chang-Sarason Theorem says that the rank of two Hankel operators is the minimum of the ranks of individual Hankel operators, we conclude

$$\mathrm{rank}H_{\overline{\hat{g}}}H_{\overline{u}} = \min\{\mathrm{rank}H_{\overline{\hat{g}}}, \mathrm{rank}H_{\overline{u}}\} = \mathrm{rank}H_{\overline{\hat{g}}}. \tag{20}$$

Combining equations (17),(19) and (20), we get the desired formula $\mathrm{rank}B_{\overline{u\hat{u}}g} = \mathrm{rank}H_{\overline{\hat{g}}} = \alpha(\overline{\hat{g}})$, where the last equality is by Kronecker's Theorem. □

We here note that $\alpha(\overline{\hat{g}}) = \alpha(\overline{g})$. Indeed, it is easy to verify that a complex number c is a pole of $\overline{g}$ if and only if $\overline{c}$ is the pole of $\overline{\hat{g}}$. Therefore, the conclusion of the above lemma may be rewritten as $\mathrm{rank}B_{\overline{u\hat{u}}g} = \mathrm{rank}H_{\overline{g}} = \alpha(\overline{g})$. The following is the main result of this section.

Theorem 3.3. *Let* $deg\, u = n$ *and let* $\varphi = \overline{f} + \overline{u\hat{u}g}$, *with* $f \in K_u^2$ *and* $g \in zK_{\hat{u}}^2$. *If* $rankB_{\overline{f}} + rankB_{\overline{u\hat{u}g}} \leq n$, *then* $rankB_\varphi = rankB_{\overline{f}} + rankB_{\overline{u\hat{u}g}} = \alpha(\overline{f}) + \alpha(\overline{g})$.

Proof. Assume $\text{rank}B_{\overline{f}} + \text{rank}B_{\overline{u\hat{u}g}} \leq n$ and let θ_1 and θ_2 be inner functions defined by $\ker H_{\overline{f}} = \theta_1 H^2$ and $\ker H_{\overline{\hat{g}}} = \theta_2 H^2$. Note that since $f \in K_u^2$ and $g \in K_{\hat{u}}^2$, both $\overline{f}u$ and $\overline{\hat{g}}u$ are analytic functions and therefore u belongs to both $\ker H_{\overline{f}}$ and $\ker H_{\overline{\hat{g}}}$, which implies both θ_1 and θ_2 divides u. Since we showed $\text{rank}B_{\overline{f}} = \text{rank}H_{\overline{f}} = \text{deg}\theta_1$ and $\text{rank}B_{\overline{u\hat{u}g}} = \text{rank}H_{\overline{u\hat{u}g}} = \text{deg}\theta_2$, we know that $\text{deg}\theta_1 + \text{deg}\theta_2 \leq n$. To make use of Lemma 2.9, we first claim

$$\text{Range}B_{\overline{f}} \cap \text{Range}B_{\overline{u\hat{u}g}} = \{0\}. \tag{21}$$

By Lemma 3.1,

$$\begin{aligned}\text{Range}B_{\overline{f}} &= \text{Range}H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}}\\ &= H^2 \ominus \ker (H^*_{\overline{u}}H_{\overline{u}}H_{\overline{f}})^*\\ &= H^2 \ominus \ker(H^*_{\overline{f}}H^*_{\overline{u}}H_{\overline{u}})\\ &= H^2 \ominus \ker(H_{\overline{\hat{f}}}P_u).\end{aligned}$$

Let $h = h_1 + uh_2 \in \ker(H_{\overline{\hat{f}}}P_u)$, where $h_1 \in K_u^2$ and $h_2 \in H^2$, then $H_{\overline{\hat{f}}}P_u h = H_{\overline{\hat{f}}}h_1 = 0$. Since $H_{\overline{\hat{f}}} = \hat{\theta}_1 H^2$, $h_1 \in \hat{\theta}_1 H^2$. Thus $h_1 \in K_u^2 \cap \hat{\theta}_1 H^2$, which implies $\ker H_{\overline{\hat{f}}}P_u = (K_u^2 \cap \hat{\theta}_1 H^2) \oplus uH^2$. Therefore,

$$\begin{aligned}\text{Range}B_{\overline{f}} &= H^2 \ominus \ker H_{\overline{\hat{f}}}P_u\\ &= H^2 \ominus [(K_u^2 \cap \hat{\theta}_1 H^2) \oplus uH^2]\\ &= K_u^2 \ominus (K_u^2 \cap \hat{\theta}_1 H^2)\\ &= K_u^2 \cap [(K_u^2)^\perp + (\hat{\theta}_1 H^2)^\perp]\\ &= K_u^2 \cap (uH^2 + K^2_{\hat{\theta}_1}).\end{aligned} \tag{22}$$

To compute $\text{Range}B_{\overline{u\hat{u}g}}$, observe the following :

$$\begin{aligned}\text{Range}B_{\overline{u\hat{u}g}} &= \text{Range}H^*_{\overline{u}}H_{\overline{\hat{g}}}H_{\overline{u}}\\ &= [\ker (H^*_{\overline{u}}H_{\overline{g}}H_{\overline{u}})^*]^\perp\\ &= [\ker (H^*_{\overline{u}}H_{\overline{g}}H_{\overline{u}})]^\perp\\ &= [\ker (H_{\overline{g}}H_{\overline{u}})]^\perp.\end{aligned} \tag{23}$$

Here we used (17) for the first equality. Let $h' = h_1' + uh_2' \in \ker (H_{\overline{g}}H_{\overline{u}})$ with $h_1' \in K_u^2$ and $h_2' \in H^2$, then

$$H_{\overline{g}}H_{\overline{u}}h' = H_{\overline{g}}J(\overline{u}h_1') = H_{\overline{g}}\hat{u}z\overline{\hat{h}_1'} = 0.$$

Since $\ker H_{\overline{g}} = \hat{\theta}_2 H^2$, we have $\hat{u}z\overline{\hat{h}_1'} = \hat{\theta}_2 h_3'$ for some $h_3' \in H^2$. Recall that θ_2 divides u and set an inner function $\theta_2' := \overline{\theta_2}u$. Thus we have $h_1' = u\overline{\theta_2}z\overline{\hat{h}_3'} = \theta_2' z\overline{\hat{h}_3'}$ for some h_3', or, equivalently, $h_1' \in K^2_{\theta_2'}$. Now we have $\ker (H_{\overline{g}}H_{\overline{u}}) = K^2_{\theta_2'} \oplus uH^2$ and therefore,

$$\text{Range}B_{\overline{u\hat{u}g}} = K_u^2 \ominus K^2_{\theta_2'} = \theta_2' K^2_{\theta_2}. \tag{24}$$

Let $a \in \text{Range}B_{\overline{f}} \cap \text{Range}B_{\overline{u\hat{u}g}}$, then by (22),

$$a = ub + c \text{ with } b \in H^2 \text{ and } c \in K^2_{\hat{\theta}_1}. \tag{25}$$

On the other hand, referring to equation (24), we may also write

$$a = \theta_2' a' \text{ with } a' \in K^2_{\theta_2}. \tag{26}$$

From (25) and (26) we have $ub + c = \theta_2' a'$. Since $u = \theta_2\theta_2'$, we have $c = \theta_2'(a' - \theta_2 b)$, which implies

$$c \in \theta_2' H^2. \tag{27}$$

Recall that we assumed $\deg\theta_1 + \deg\theta_2 \leq n$. Note $\deg\hat{\theta}_1 = \deg\theta_1$ and $\deg\theta_2' = n - \deg\theta_2$, which, together with our assumption, implies $\deg\hat{\theta}_1 \leq \deg\theta_2'$. By Lemma 2.4,

$$K^2_{\hat{\theta}_1} \cap \theta_2' H^2 = \{0\}. \tag{28}$$

Since $c \in K^2_{\hat{\theta}_1} \cap \theta_2' H^2$ by (25) and (27), equation (28) gives $c = 0$. Now (25) reduces to $a = ub$. But, since $a \in K^2_u$ by (24), $a = ub$ implies $a = b = 0$. Therefore we conclude $\text{Range}B_{\overline{f}} \cap \text{Range}B_{\overline{u\hat{u}g}} = \{0\}$.

To reach our goal, we still need to show

$$\text{Range}B^*_{\overline{f}} \cap \text{Range}B^*_{\overline{u\hat{u}g}} = \text{Range}B_{\overline{\hat{f}}} \cap \text{Range}B_{\overline{u\hat{u}\hat{g}}} = \{0\}. \tag{29}$$

Since a map C on $L^2(\mathbb{T})$ defined by $C\phi = u\overline{z\phi}$ is an isometry of K^2_u onto itself, it is sufficient to show that

$$\text{Range}(CB_{\overline{\hat{f}}}C) \cap \text{Range}(CB_{\overline{u\hat{u}\hat{g}}}C) = \{0\}. \tag{30}$$

By Lemma 2.5, we need to show

$$\text{Range}(CB_{\overline{\hat{f}}}C) \cap \text{Range}(CB_{\overline{u\hat{u}\hat{g}}}C) = \text{Range}B_{\overline{u\hat{u}f}} \cap \text{Range}B_{\overline{g}} = \{0\}. \tag{31}$$

Note $rankA = rankA^* = rankCA^*C$ for any operator A on K^2_u. Thus

$$\begin{aligned} \text{rank}B_{\overline{u\hat{u}f}} + \text{rank}B_{\overline{\hat{g}}} &= \text{rank}(CB_{\overline{\hat{f}}}C) + \text{rank}(CB_{\overline{u\hat{u}\hat{g}}}C) \\ &= \text{rank}(CB^*_{\overline{f}}C) + \text{rank}(CB^*_{\overline{u\hat{u}g}}C) \\ &= \text{rank}B_{\overline{f}} + \text{rank}B_{\overline{u\hat{u}g}} \\ &\leq n, \end{aligned} \tag{32}$$

where the last inequality is our assumption. By the same argument as we derived (21), we can show

$$\text{Range}B_{\overline{u\hat{u}f}} \cap \text{Range}B_{\overline{\hat{g}}} = \{0\},$$

which proves (31). Thus

$$\text{Range}B^*_{\overline{f}} \cap \text{Range}B^*_{\overline{u\hat{u}g}} = \{0\} \tag{33}$$

as explained above. Now by Lemma 2.9, equation (21) and equation (33), we conclude

$$\text{rank}B_{\overline{f}+\overline{u\hat{u}g}} = \text{rank}B_{\overline{f}} + \text{rank}B_{\overline{u\hat{u}g}}.$$

Since we showed $\text{rank}B_{\overline{f}} = \alpha(\overline{f})$ and $\text{rank}B_{\overline{u\hat{u}g}} = \alpha(\overline{\hat{g}}) = \alpha(\overline{g})$ in previous two lemmas, we finally have

$$\text{rank}B_{\overline{f}+\overline{u\hat{u}g}} = \text{rank}B_{\overline{f}} + \text{rank}B_{\overline{u\hat{u}g}} = \alpha(\overline{f}) + \alpha(\overline{g})$$

as desired. □

In the case rank$B_{\overline{f}}$ + rank$B_{\overline{u\hat{u}g}} > n$, it seems difficult to find a nice relation among rank$B_{\overline{f}}$, rank$B_{\overline{u\hat{u}g}}$ and rank$B_{\overline{f}+\overline{u\hat{u}g}}$. We conclude the paper with an example showing such difficulty.

Example 3.4. *Let $u = z^4$ and let $f(z) = g(z) = z + z^3$ so that the condition $f \in K_u^2$ and $g \in zK_{\hat{u}}^2$ is satisfied. Then the matrix representation of $B_{\overline{f}}$ and $B_{\overline{u\hat{u}g}}$ are*

$$B_{\overline{f}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B_{\overline{u\hat{u}g}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

with respect to the standard ordered basis of K_u^2. Although we verify that rank$B_{\overline{f}}$ = rank$B_{\overline{u\hat{u}g}} = 3$, we also see that the rank of $B_{\overline{f}+\overline{u\hat{u}g}}$ collapses to 2 by observing the matrix representation

$$B_{\overline{f}+\overline{u\hat{u}g}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

On the other hand, it is easy to see that

$$\text{rank}B_{\overline{f}+2\overline{u\hat{u}g}} = \text{rank}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} = 4,$$

while rank$B_{2\overline{u\hat{u}g}}$ = rank$B_{\overline{u\hat{u}g}} = 3$.

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