Filomat 39:7 (2025), 2405–2416
https://doi.org/10.2298/FIL2507405S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Hyers–Ulam stability of a class of impulsive Hammerstein integral equations

Rahim Shah[a,*], Haleema Bibi[a], Natasha Irshad[a], Hajra Imtiaz Abbasi[a]

[a]*Department of Mathematics, Kohsar University Murree, Murree, Pakistan*

Abstract. The objective of this study is to examine different types of stability results for a class of impulsive Hammerstein integral equations. We provide adequate conditions for achieving Hyers–Ulam and Hyers–Ulam–Rassias stability for impulsive Hammerstein integral equations. The consequent different cases of a finite interval and an infinite interval are studied. Finally, a concrete example is given at the end of this study for illustrations.

1. Introduction and Preliminaries

The idea of stability for various types of equations has been extensively investigated during the last six decades. Particular attention has been given to the Hyers–Ulam stability and Hyers–Ulam–Rassias stability of a wide range of functional equations, differential equations, and integral equations because of their numerous applications, including those in elasticity, electronic devices, heat conduction, flow of fluids, scattering theory, chemical reactions, and population dynamics, among other fields (see [1, 3, 4, 5, 6, 7, 9, 10]).

The initial results on this type of stability for functional equations stemmed from a well-known question posed by S. M. Ulam [18] in 1940 regarding the necessity of some form of closeness between the solution of a given equation and the solution of an equation that differs slightly from it. D. H. Hyers [13] provided a partial answer to Ulam's question for Banach spaces in the case of the additive Cauchy equation [13]. In 1978, T. M. Rassias [22] introduced new directions to stability analysis, leading to what is now known as Hyers–Ulam–Rassias stability.

Accordingly, the Hammerstein integral equation is stable in the Hyers–Ulam sense if, for every function that approximately satisfies the Hammerstein integral equation, there exists a solution of the equation that is close to it. This implies that the stability of the Hammerstein integral equations concerns how the solutions of the inequality deviate from those of the considered Hammerstein integral equation. Despite extensive research on several types of equations, there are relatively few studies on these types of stability

2020 *Mathematics Subject Classification*. Primary 47H10, 34A40, 35B35; Secondary 26D10.
Keywords. Hyers–Ulam stability; Hyers–Ulam–Rassias stability; Impulsive Hammerstein integral equations; Fixed point method; Lipschitz condition.
Received: 11 November 2024; Accepted: 22 November 2024
Communicated by Miodrag Spalević
* Corresponding author
Email addresses: rahimshah@kum.edu.pk, shahraheem1987@gmail.com (Rahim Shah), haleemasatti682@gmail.com (Haleema Bibi), natashairshad24@gmail.com (Natasha Irshad), hajraabbasi168@gmail.com (Hajra Imtiaz Abbasi)
ORCID iDs: https://orcid.org/0009-0001-9044-5470 (Rahim Shah), https://orcid.org/0009-0007-8218-0963 (Haleema Bibi), https://orcid.org/0009-0008-8166-6520 (Natasha Irshad)

for integral equations; in particular, only a limited number address the stability of Hammerstein integral equations.

Currently, based on the existing literature, we will summarize a few relevant results related to the content of this study in the following lines. As a starting point in the relevant literature, the first work on Ulam stability was introduced in the context of functional equations by S. M. Ulam; see [18]. In 2007, Jung [14] used the fixed point method to study the uniqueness of the solution and the Hyers–Ulam–Rassias stability of the Volterra integral equation. Indeed, Jung's work [14] served as a starting point in the literature for subsequent studies on Ulam stabilities of integral equations.

In 1993, Guo [10] proved several existence theorems for external solutions to nonlinear impulsive Volterra integral equations on a finite interval in Banach spaces, considering a finite number of moments of impulsive effects. He also provided applications to initial value problems for first-order impulsive differential equations in Banach spaces. Furthermore, the existence of solutions for systems of nonlinear impulsive Volterra integral equations on an infinite interval, with an infinite number of moments of impulsive effects in Banach spaces, was studied. These investigations are significant, as many problems in applied mathematics involve systems of differential or integral equations.

K. Balachandran et al. [5] established several important results in Banach spaces for fractional impulsive integro-differential equations. Recently, the Hyers–Ulam stability of integral equations has gained significant attention. In 2015, L. Hua et al. [23] studied the Hyers–Ulam stability of certain types of Fredholm integral equations. Similarly, in 2017, L. P. Castro and A. M. Simoes [24] investigated the Hyers–Ulam and Hyers–Ulam–Rassias stability of a class of Hammerstein integral equations.

The concept of Hyers–Ulam stability is highly valuable in various fields, including economics, biology, numerical analysis, and optimization, as it is often challenging to determine exact solutions for such physical problems. This concept has been extensively applied in the study of differential and integral equations. For some recent works, see [1, 8, 11, 12, 15, 16, 19–21] and the references therein.

The stability of impulsive Hammerstein integral equations is a topic of significant interest within the mathematical community due to its connections with nonlinear dynamics, functional analysis, and the study of systems that exhibit sudden changes or "impulses." Impulsive effects frequently arise in various real-world scenarios, such as biological systems, control theory, and mechanical systems experiencing discontinuities or shocks. When these impulsive behaviors are integrated into the framework of the Hammerstein integral equation, the problem becomes considerably more complex. The interplay between the nonlinear integral operator and the impulsive dynamics necessitates advanced mathematical techniques for stability analysis.

Investigating stability involves analyzing how solutions respond to small perturbations or initial disturbances and determining whether they converge toward equilibrium or exhibit unbounded growth. Addressing these challenges often requires tools from fixed-point theory, functional analysis, and Lyapunov's direct method to establish stability criteria for impulsive systems (see [28], [29]).

Furthermore, several interesting results related to Ulam stability, Lyapunov stability, and other qualitative behaviors of solutions for various classes of integrodifferential equations, integral equations, and related problems have been discussed by Chauhan et al. [30], Deep et al. [31], Gãvruţa [32], Graef and Tunç [33], Hammami and Hnia [34], Murail et al. [35], Alinejad et al. [36], Abdollahpour et al. [37], [38], Aczel et al. [39], Czerwik [40], and in the books or articles cited in these sources.

In this work, we will focus on analyzing the Hyers–Ulam and Hyers–Ulam–Rassias stability for the following class of impulsive Hammerstein integral equations:

$$y(x) = p(x) + f(x, y(x)) \int_a^x g(x,\tau)h(\tau, y(\tau))\, d\tau + \sum_{a<x_k<b} I_k(y(x_k^-)), \tag{1}$$

where a and b are fixed real numbers, and $p : [a, b] \to \mathbb{C}$, $f : [a, b] \times \mathbb{C} \to \mathbb{C}$, and $h : [a, b] \times \mathbb{C} \to \mathbb{C}$ are continuous functions. The kernel $g : [a, b] \times [a, b] \to \mathbb{C}$ is also a continuous function. For all $x \in [a, b]$, $I_k : \mathbb{C} \to \mathbb{C}$ for $k = 1, 2, \ldots, m$, and $y(x_k^-)$ represents the left-hand limit of $y(x)$ at $x = x_k$. This segment briefly discusses some basic concepts from the literature. For a nonempty set X, we introduce the definition of generalized metric on X as follows:

Definition 1.1 ([2]). *A mapping $d : X \times X \to [0, \infty]$ is called a generalized metric on set X if and only if d holds:*
(C_1) $d(x, y) = 0$ *if and only if* $x = y$;
(C_2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
(C_3) $d(x, z) \le d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Next, we recall the main result of J. B. Diaz and B. Margolis [9], with the help of which we will be able to prove our main results.

Theorem 1.2 ([9]). *Let (X, d) be a generalized complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with $L < 1$, where L is a Lipschitz constant. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following are true:*
(a) *The sequence $T^n x$ converges to a fixed point x^* of T;*
(b) *x^* is the unique fixed point of T in*

$$X^* = \left\{ y \in X \mid d(T^k x, y) < \infty \right\};$$

(c) *If $y \in X^*$, then*

$$d(y, x^*) \le \frac{1}{1 - L}\, d(Ty, y). \tag{2}$$

Now, we give the definitions of Hyers–Ulam–Rassias and Hyers–Ulam stability of impulsive Hammerstein integral equation (1).

Definition 1.3. *If for each continuously differentiable function $y(x)$ satisfying*

$$\left| y(x) - p(x) - f(x, y(x)) \int_a^x g(x, \tau) h(\tau, y(\tau)) d\tau - \sum_{a < x_k < x} I_k(y(x_k^-)) \right| \le \sigma(x),$$

for some $\sigma : I \to (0, \infty)$, there exists a solution $y_0(x)$ of the impulsive Hammerstein integral equation (1) *and a constant $K > 0$ with*

$$\left| y(x) - y_0(x) \right| \le K\sigma(x),$$

for all $x \in I$, where K is independent of $y(x)$ and $y_0(x)$, then we say that the impulsive Hammerstein integral equation (1) *has the Hyers–Ulam–Rassias stability. If $\sigma(x)$ is a constant function in the above inequalities, we say that the impulsive Hammerstein integral equation* (1) *has the Hyers–Ulam stability.*

In the current work, using the idea of Cădariu and Radu [22], we will study the Ulam–type stability results of the impulsive Hammerstein integral equation (1).

2. Hyers–Ulam–Rassias stability in the finite Interval Case

This section presents the necessary conditions for the Hyers–Ulam–Rassias stability of the impulsive Hammerstein integral equation (1), where $x \in [a, b]$, for some fixed real numbers a and b.

We will use the space C($[a, b]$) of continuous functions on [a,b], provided with the metric ([25])

$$d(u, v) = \sup_{x \in [a,b]} \frac{|u(x) - v(x)|}{\sigma(x)}, \tag{3}$$

where σ is a non-decreasing continuous function $\sigma : [a, b] \to (0, \infty)$. Recall that $(C([a, b]), d)$ is a complete metric space (cf., e.g.,([26]), ([27])).

Theorem 2.1. *Let us consider a continuous given function* μ: $[a,b] \to [0,\infty)$. *Moreover, suppose that* $p : [a,b] \to \mathbb{C}$ *is a continuous function,* $f : [a,b] \times \mathbb{C} \to \mathbb{C}$ *is a continuous function such that there exists* $M > 0$ *so that*

$$M = \sup_{x\in[a,b]} |f(x,y)|, \quad y \in \mathbb{C},$$

and the kernal $g : [a,b] \times [a,b] \to \mathbb{C}$ *is also continuous. In addition, suppose that there is* $\beta \in [0,1)$ *such that*

$$\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)d\tau \leq \beta\sigma(x)$$

and $h : [a,b] \times \mathbb{C} \to \mathbb{C}$ *is a continuous function which fulfils the condition*

$$|h(x,u(x)) - h(x,v(x))| \leq \mu(x)|u(x) - v(x)|$$

for all $x \in [a,b]$ *and* $u, v \in C([a,b])$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ *and there exists a constant* $L > 0$ *such that*

$$|I_k(u(x)) - I_k(v(x))| \leq L|u(x) - v(x)|$$

for all $u, v \in C([a,b])$.
If $y \in C([a,b])$ *is such that*

$$\left| y(x) - p(x) - f(x,y(x)) \int_a^x g(x,\tau)h(\tau,y(\tau))d\tau - \sum_{a<x_k<x} I_k(y(x_k^-)) \right| \leq \sigma(x), \tag{4}$$

for all $x \in [a,b]$ *and* $M\beta + L < 1$, *then there exists a unique function* $y_0 \in C([a,b])$ *such that*

$$y_o(x) = p(x) + f(x,y_o(x)) \int_a^x g(x,\tau)h(\tau,y_o(\tau))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-))$$

and

$$|y(x) - y_o(x)| \leq \frac{\sigma(x)}{1 - (M\beta + L)} \tag{5}$$

for all $x \in [a,b]$.

This means that under the above conditions, the impulsive Hammerstein integral equation (1) *exhibits Hyers–Ulam–Rassias stability.*

Proof. We will consider the operator $\Omega : C([a,b]) \to C([a,b])$, defined by

$$(\Omega u)(x) = p(x) + f(x,u(x)) \int_a^x g(x,\tau)h(\tau,u(\tau))d\tau + \sum_{a<x_k<x} I_k(u(x_k^-)),$$

for all $x \in [a,b]$ and $u \in C([a,b])$. Note that for any continuous function u, $\Omega(u)$ is also continuous. Indeed,

$$\begin{aligned} |(\Omega u)(x) - (\Omega u)(x_o)| &= \Big| p(x) + f(x,u(x)) \int_a^x g(x,\tau)h(\tau,u(\tau))d\tau + \sum_{a<x_k<x} I_k(u(x_k^-)) - p(x_o) \\ &\quad - f(x_o,u(x_o)) \int_a^{x_o} g(x_o,\tau)h(\tau,u(\tau))d\tau - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-)) \Big| \end{aligned}$$

$$
\begin{aligned}
\leq\ & \Big|p(x)-p(x_0)\Big| + \Big|f(x,u(x))\int_a^x g(x,\tau)h(\tau,u(\tau))d\tau + \sum_{a<x_k<x} I_k(u(x_k^-)) \\
& -f(x_o,u(x_o))\int_a^{x_o} g(x_o,\tau)h(\tau,u(\tau))d\tau - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-))\Big| \\
\leq\ & \Big|p(x)-p(x_o)\Big| + \Big|f(x,u(x))\int_a^x g(x,\tau)h(\tau,u(\tau))d\tau \\
& -f(x_o,u(x_o))\int_a^{x_o} g(x_o,\tau)h(\tau,u(\tau))d\tau\Big| \\
& +\Big|\sum_{a<x_k<x} I_k(u(x_k^-)) - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-))\Big| \\
\leq\ & \Big|p(x)-p(x_o)\Big| + M\Big|\int_a^x g(x,\tau)h(\tau,u(\tau))d\tau - \int_a^{x_0} g(x_0,\tau)h(\tau,u(\tau))d\tau\Big| \\
& +\Big|\sum_{a<x_k<x} I_k(u(x_k^-)) - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-))\Big| \\
\leq\ & \Big|p(x)-p(x_o)\Big| + M\Big|\int_a^x g(x,\tau)h(\tau,u(\tau))d\tau - \int_a^x g(x_o,\tau)h(\tau,u(\tau))d\tau \\
& +\int_a^x g(x_0,\tau)h(\tau,u(\tau))d\tau - \int_a^{x_o} g(x_o,\tau)h(\tau,u(\tau))d\tau\Big| \\
& +\Big|\sum_{a<x_k<x} I_k(u(x_k^-)) - \sum_{a<x_0k<x_0} I_k(u(x_{0_k}^-))\Big| \\
\leq\ & |p(x)-p(x_o)| + M\Big(\int_a^x |g(x,\tau)-g(x_o,\tau)||h(\tau,u(\tau))|d\tau \\
& +\Big|\int_{x_o}^x g(x_o,\tau)h(\tau,u(\tau))d\tau\Big|\Big) + \Big|\sum_{a<x_k<x} I_k(u(x_k^-)) - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-))\Big| \to 0
\end{aligned}
$$

when $x \to x_o$.

Based on the present conditions, we can conclude that the operator Ω is strictly contractive (for the given metric). For all $u, v \in C([a,b])$, we have,

$$
d(\Omega(u),\Omega(v)) = \sup_{x\in[a,b]} \frac{|(\Omega(u))(x)-(\Omega(v))(x)|}{\sigma(x)}
$$

$$
\begin{aligned}
\leq\ & M\sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)||h(\tau,u(\tau))-h(\tau,v(\tau))|d\tau}{\sigma(x)} + \sup_{x\in[a,b]} \frac{\sum_{a<x_k<x} I_k[u(x_k^-)-v(x_k^-)]}{\sigma(x)} \\
=\ & M\sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)\frac{|u(\tau)-v(\tau)|}{\sigma(\tau)}d\tau}{\sigma(x)} + L\sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} \\
\leq\ & M\sup_{\tau\in[a,b]} \frac{|u(\tau)-v(\tau)|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)d\tau}{\sigma(x)} + L\sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} \\
\leq\ & (M\beta+L)d(u,v).
\end{aligned}
$$

Due to the fact that $M\beta + L < 1$ it follows that Ω is strictly contractive. Using the above mentioned Banach fixed point theorem, the impulsive Hammerstein integral equation achieves Hyers–Ulam–Rassias stability. Additionally, (5) follows from (4) and (2). □

3. Hyers–Ulam–Rassias stability in the infinite Interval Case

This subsection focuses on the Hyers–Ulam–Rassias stability of the impulsive Hammerstein integral equation across infinite intervals. This means that instead of considering, as before, a finite interval [a, b] (with $a, b \in \mathbb{R}$), we will now consider e.g. corresponding intervals $[a, \infty)$, for some fixed $a \in \mathbb{R}$.

Thus, we will now be dealing with the integral equation

$$y(x) = p(x) + f(x, y(x)) \int_a^x g(x, \tau) h(\tau, y(\tau)) d\tau + \sum_{a<x_k<x} I_k(y(x_k^-)), \tag{6}$$

for all $x \in [a, \infty)$ where a is a fixed real number, $p : [a, \infty) \to \mathbb{C}$ and $f : [a, \infty) \times \mathbb{C} \to \mathbb{C}$ are bounded continuous functions, and $h : [a, \infty) \times \mathbb{C} \to \mathbb{C}$ and the kernel $g : [a, \infty) \times [a, \infty) \to \mathbb{C}$ are continuous functions. Moreover, $I_k : \mathbb{C} \to \mathbb{C}, k = 1, 2, \ldots, m$ and $y(x_k^-)$ represents the left limit of $y(x)$ at $x = x_k$. Our technique relies on a recurrence procedure, as the result for the finite interval situation is already available.

Let us consider a fixed non-decreasing continuous function $\phi : [a, \infty) \to (\epsilon, \eta)$, for some $\epsilon, \eta > 0$, and the space $C_b([a, \infty))$ of bounded continuous functions endowed with the weighted metric ([25])

$$d_b(u, v) = \sup_{x \in [a,\infty)} \frac{|u(x) - v(x)|}{\phi(x)}.$$

Theorem 3.1. *Let us consider a continuous given function $\mu : [a, \infty) \to [0, \infty)$. Further, assume that $p : [a, \infty) \to \mathbb{C}$ is a bounded continuous function, $f : [a, \infty) \times \mathbb{C} \to \mathbb{C}$ is a continuous function such that there exists $M > 0$ so that*

$$M = \sup_{x \in [a,\infty)} |f(x, y)|, \quad y \in \mathbb{C},$$

and the kernel $g : [a, \infty) \times [a, \infty) \to \mathbb{C}$ and $h : [a, \infty) \times \mathbb{C} \to \mathbb{C}$ are continuous functions so that $\int_a^x g(x, \tau) h(\tau, z(\tau)) d\tau$ is a bounded continuous function for any bounded continuous function z.

In addition, suppose that there is $\beta \in [0, 1)$ such that

$$\int_a^x |g(x, \tau)| \mu(\tau) \varphi(\tau) d\tau \leq \beta \varphi(x)$$

and $h : [a, \infty) \times \mathbb{C} \to \mathbb{C}$ is a continuous function which fulfills the condition

$$|h(x, u(x)) - h(x, v(x))| \leq \mu(x) |u(x) - v(x)|$$

for all $x \in [a, \infty)$ and $u, v \in C_b([a, \infty))$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ and there exists a constant $L > 0$ such that

$$|I_k(u(x)) - I_k(v(x))| \leq L |u(x) - v(x)|$$

for all $u, v \in C_b([a, \infty))$.

If $y \in C_b([a, \infty))$ is such that

$$\left| y(x) - p(x) - f(x, y(x)) \int_a^x g(x, \tau) h(\tau, y(\tau)) d\tau - \sum_{a<x_k<x} I_k(y(x_k^-)) \right| \leq \varphi(x),$$

for all $x \in [a, \infty)$ and $M\beta + L < 1$, then there exists a unique function $y_o \in C_b([a, \infty))$ such that

$$y_o(x) = p(x) + f(x, y_o(x)) \int_a^x g(x, \tau) h(\tau, y_o(\tau)) d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-)) \tag{7}$$

and

$$|y(x) - y_o(x)| \leq \frac{\varphi(x)}{1-(M\beta + L)} \tag{8}$$

for all $x \in [a, \infty)$.

Under the given conditions, the Impulsive Hammerstein integral equation (1) *exhibits Hyers–Ulam–Rassias stability.*

Proof. For any $n \in \mathbb{N}$, we will define $I_n = [a, a+n]$. By Theorem (2.1), there exists a unique bounded continuous function $y_{o,n} : I_n \to \mathbb{C}$ such that

$$y_{o,n}(x) = p(x) + f(x, y_{o,n}(x)) \int_a^x g(x,\tau)h(\tau, y_{o,n}(\tau))d\tau + \sum_{a<x_k<x} I_k(y_{o,n}(x_k^-)) \tag{9}$$

and

$$|y(x) - y_{o,n}(x)| \leq \frac{\varphi(x)}{1-(M\beta + L)} \tag{10}$$

for all $x \in I_n$. The uniqueness of $y_{o,n}$ implies that if $x \in I_n$ then

$$y_{o,n}(x) = y_{o,n+1}(x) = y_{o,n+2}(x) = \cdots . \tag{11}$$

For any $x \in [a, \infty)$, let us define $n(x) \in \mathbb{N}$ as $n(x) = min\{n \in \mathbb{N} \mid x \in I_n\}$. We also define a function $y_o : [a, \infty) \to \mathbb{C}$ by

$$y_o(x) = y_{o,n(x)}(x). \tag{12}$$

For any $x_1 \in [a, \infty)$, let $n_1 = n(x_1)$. Then $x_1 \in Int I_{n_1+1}$ and there exists an $\epsilon > 0$ such that $y_0(x) = y_{0,n_1+1}(x)$ for all $x \in (x_1 - \epsilon, x_1 + \epsilon)$. By Theorem (2.1), y_{0,n_1+1} is continuous at x_1, and so it is y_0.

Now, we will prove y_o satisfies

$$y_o(x) = p(x) + f(x, y_o(x)) \int_a^x g(x,\tau)h(\tau, y_o(\tau))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-))$$

and

$$|y(x) - y_o(x)| \leq \frac{\varphi(x)}{1-(M\beta + L)} \tag{13}$$

for all $x \in [a, \infty)$. For an arbitrary $x \in [a, \infty)$ we choose $n(x)$ such that $x \in I_{n(x)}$. By (9) and (12), we have

$$\begin{aligned} y_o(x) &= y_{o,n(x)}(x) \\ &= p(x) + f(x, y_{o,n(x)}(x)) \int_a^x g(x,\tau)h(\tau, y_{o,n(x)}(\tau))d\tau + \sum_{a<x_k<x} I_k(y_{o,n(x)}(x_k^-)) \\ &= p(x) + f(x, y_o(x)) \int_a^x g(x,\tau)h(\tau, y_o(\tau))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-)). \end{aligned} \tag{14}$$

Note that $n(\tau) \leq n(x)$, for any $\tau \in I_{n(x)}$, and it follows from (11) that $y_o(\tau) = y_{o,n(\tau)}(\tau) = y_{o,n(x)}(\tau)$, so, the last equality in (14) holds.

To prove (13), by (12) and (10), we have for all $x \in [a, \infty)$,

$$|y(x) - y_o(x)| = |y(x) - y_{o,n(x)}(x)| \leq \frac{\varphi(x)}{1-(M\beta + L)}.$$

Finally, we will prove the uniqueness of y_o. Let us consider another bounded continuous function y_1 which satisfies (7) and (8), for all $x \in [a,\infty)$. By the uniqueness of the solution on $I_{n(x)}$ for any $n(x) \in \mathbb{N}$ we have that $y_{o|I_{n(x)}} = y_{o,n(x)}$ and $y_{1|I_{n(x)}}$ satisfies (7) and (8) for all $x \in I_{n(x)}$, so

$$y_o(x) = y_{o|I_{n(x)}}(x) = y_{1|I_{n(x)}}(x) = y_1(x).$$

□

4. Hyers–Ulam stability in the finite interval case

This section provides adequate requirements for the Hyers–Ulam stability of the impulsive Hammerstein integral equation (1). For a given non-decreasing continuous function $\sigma : [a,b] \to (0,\infty)$, we shall utilize the same metric (3).

Theorem 4.1. *Let us consider a continuous given function $\mu : [a,b] \to [o,\infty)$. Moreover, assume that $p : [a,b] \to \mathbb{C}$ is a continuous function, $f : [a,b] \times \mathbb{C} \to \mathbb{C}$ is a continuous function such that there exists $M > 0$ so that*

$$M = \sup_{x\in[a,b]} |f(x,y)|, \quad y \in \mathbb{C},$$

and the kernel $g : [a,b] \times [a,b] \to \mathbb{C}$ is also continuous. In addition, suppose that there is $\beta \in [0,1)$ such that

$$\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)d\tau \leq \beta\sigma(x)$$

and $h : [a,b] \times \mathbb{C} \to \mathbb{C}$ is a continuous function which fulfills the condition

$$|h(x,u(x)) - h(x,v(x))| \leq \mu(x)|u(x) - v(x)|$$

for all $x \in [a,b]$ and $u,v \in C([a,b])$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ and there exists a constant $L > 0$ such that

$$|I_k(u(x)) - I_k(v(x))| \leq L|u(x) - v(x)|$$

for all $u,v \in C([a,b])$.

If $y \in C([a,b])$ is such that

$$\left|y(x) - p(x) - f(x,y(x))\int_a^x g(x,\tau)h(\tau,y(\tau))d\tau - \sum_{a<x_k<x} I_k(y(x_k^-))\right| \leq \theta, \quad x \in [a,b], \tag{15}$$

where $\theta \geq 0$ and $M\beta + L < 1$, then there is a unique function $y_o \in C([a,b])$ such that

$$y_o(x) = p(x) + f(x,y_o(x))\int_a^x g(x,\tau)h(\tau,y_o(\tau))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-))$$

and

$$|y(x) - y_o(x)| \leq \frac{\theta}{1 - (M\beta + L)} \tag{16}$$

for all $x \in [a,b]$.

This means that under the above conditions, the impulsive Hammerstein integral equation (1) has the Hyers–Ulam stability.

Proof. We will consider the operator $T : C([a,b]) \to C([a,b])$, defined by

$$(Tu)(x) = p(x) + f(x,u(x)) \int_a^x g(x,\tau)h(\tau,u(\tau))d\tau + \sum_{a<x_k<x} I_k(u(x_k^-))$$

for all $x \in [a,b]$ and $u \in C([a,b])$.

T is strictly contractive (with respect to the metric under consideration). Indeed, for all $u, v \in C([a,b])$, we have,

$$\begin{aligned} d(Tu,Tv) &= \sup_{x\in[a,b]} \frac{|(Tu)(x)-(Tv)(x)|}{\sigma(x)} \\ &\leq M \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)||h(\tau,u(\tau)) - h(\tau,v(\tau))|d\tau}{\sigma(x)} + \sup_{x\in[a,b]} \frac{\sum_{a<x_k<x} I_k|u(x_k^-) - v(x_k^-)|}{\sigma(x)} \\ &= M \sup_{x\in[a,b]} \int_a^x \frac{|g(x,\tau)|\mu(\tau)\sigma(\tau)\frac{|u(\tau)-v(\tau)|}{\sigma(\tau)}d\tau}{\sigma(x)} + L \sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} \\ &\leq M \sup_{\tau\in[a,b]} \frac{|u(\tau)-v(\tau)|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)d\tau}{\sigma(x)} + L \sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} \\ &\leq (M\beta + L)d(u,v). \end{aligned}$$

Due to the fact that $M\beta + L < 1$ it follows that T is strictly contractive. Thus, again we can apply the Banach fixed point theorem, which ensures that we have the Hyers–Ulam stability for the impulsive Hammerstein integral equation with (16) being obtained by using (2) and (15). □

5. Illustrative example

To demonstrate that the conditions described above are attainable, we will provide an example. First, we will make some considerations about the solution of our equation.

Example 5.1. *Let us consider the impulsive integral equation*

$$y(x) = x - \frac{x}{2}\ln\Big(\frac{x+1}{2}\Big) + \frac{xy(x)}{2}\int_1^x \frac{1}{x+\tau}\frac{y(\tau)}{\tau^2}d\tau + \sum_{1<\frac{5}{4}<x} \frac{|y(\frac{5^-}{4})|}{3+|y(\frac{5^-}{4})|}, \tag{17}$$

for any $x \in [1,2]$ and the continuous given function $\mu : [1,2] \to [0,\infty)$ such that $\mu(x) = \frac{1}{x}$. We know that the exact solution of this equation is $y_o(x) = x$. We can see that all the conditions of Theorem (2.1) are satisfied. In fact, $p : [1,2] \to \mathbb{C}$ such that

$$p(x) = x - \frac{x}{2}\ln\Big(\frac{x+1}{2}\Big)$$

is a continuous function; $f : [1,2]\times\mathbb{C} \to \mathbb{C}$ such that $f(x,y(x)) = \frac{xy(x)}{2}$ is a continuous function where $\sup_{x\in[1,2]} |f(x,y(x))| \leq 2 = M$; the kernel $g : [1,2]\times[1,2] \to \mathbb{C}$, given by $g(x,\tau) = \frac{1}{x+\tau}$, is a continuous; there exists $\beta \in [0,1)$ such that

$$\int_1^x |g(x,\tau)|\mu(\tau)\sigma(\tau)d\tau = \int_1^x \Big|\frac{1}{x+\tau}\Big|\frac{1}{\tau}\tau d\tau = \ln\frac{2x}{x+1} \leq \ln\frac{4}{3}x = \beta\sigma(x),$$

where $\sigma : [1,2] \to (0,\infty)$ *is a non-decreasing continuous function* $\sigma(x) = x$, *and* $h : [1,2] \times \mathbb{C} \to \mathbb{C}$ *such that* $h(\tau, y(\tau)) = \frac{y(\tau)}{\tau^2}$ *is a continuous function which fulfills the condition*

$$\begin{aligned} |h(x,u(x)) - h(x,v(x))| &= \left|\frac{u(x)}{x^2} - \frac{v(x)}{x^2}\right| \\ &= \frac{1}{x^2}|u(x) - v(x)| \\ &\leq \frac{1}{x}|u(x) - v(x)| \\ &= \mu(x)|u(x) - v(x)|, \end{aligned}$$

for all $x \in [1,2]$.
Moreover,

$$I_k(y(x_k^-)) = \triangle y \mid_{x=x_k},$$

so that,

$$\Delta y \mid_{x=\frac{5}{4}} = I_k(y(\frac{5}{4}^-)) = \frac{|y(\frac{5}{4}^-)|}{3 + |y(\frac{5}{4}^-)|}.$$

Clearly,

$$\begin{aligned} |I_k(u) - I_k(v)| &= \left|\frac{u}{3+u} - \frac{v}{3+v}\right| \\ &= \left|\frac{(u)(3+v) - (v)(3+u)}{(3+u)(3+v)}\right| \\ &= \left|\frac{3u + uv - 3v - uv}{(3+u)(3+v)}\right| \\ &= \frac{3|u-v|}{(3+u)(3+v)} \leq \frac{1}{3}|u-v|. \end{aligned}$$

Note that $L = \frac{1}{3}$. *If we choose* $y(x) = \frac{x}{0.3}$, *it follows,*

$$\begin{aligned} &\left|y(x) - p(x) - f(x,y(x)) \int_a^x g(x,\tau)h(\tau,y(\tau))d\tau - \sum_{a<x_k<x} I_k[y(x_k^-)]\right| \\ =& \left|\frac{17}{6} - \frac{50}{9}\ln\frac{x+1}{2} - \frac{1}{0.9+x}\right|x \\ \leq& \left|\frac{17}{6} - \frac{50}{9}ln\frac{3}{2} - \frac{10}{29}\right|x \leq x = \sigma(x), \quad x \in [1,2]. \end{aligned}$$

This exhibits the Hyers–Ulam–Rassias stability of the impulsive Hammerstein integral equation (17). *In addition, having in mind the exact solution* $y_o(x) = x$ *and* $M\beta + L = 2 \times \ln\frac{4}{3} + \frac{1}{3} \approx 0.090869 < 1$, *we have*

$$|y(x) - y_o(x)| = \left|\frac{x}{0.3} - x\right| \leq \frac{\sigma(x)}{1 - (M\beta + L)} = \frac{x}{1 - (2 \times \ln\frac{4}{3} + \frac{1}{3})}, \quad x \in [1,2].$$

6. Conclusion

In this study, the impulsive Hammerstein integral equation was considered. We developed new sufficient criteria for the impulsive Hammerstein integral equation's for Hyers–Ulam–Rassias stability on finite and infinite intervals, as well as its Hyers–Ulam stability on finite intervals. An example was provided to demonstrate the numerical application of the results.

References

[1] M. Akkouchi, *Hyers–Ulam–Rassias stability of nonlinear Volterra integral equations via a fixed point approach*, Acta Univ. Apulensis Math. Inform. **26** (2011), 257–266.

[2] W. A. J. Luxemburg, *On the convergence of successive approximations in the theory of ordinary differential equations. II*, Nederl. Akad. Wetensch. Proc. Ser. **20** (1958), 540–546.

[3] M. Benchohra, B. A. Slimani, *Existence and uniqueness of solutions to impulsive fractional differential equations*, Electron. J. Diff. Equ. **(2009)(10)** (2009), 1–11.

[4] M. Benchohra, D. Seba, *Impulsive fractional differential equations in Banach spaces*, Electron. J. Qual. Theo. Diff. Equ. Spec. Edit. **(2009)(8)** (2009), 1–14.

[5] K. Balachandran, S. Kiruthika, J. J. Trujillo, *Existence results for fractional impulsive integrodifferential equations in Banach spaces*, Commun. Nonlinear Sci. Numer. Simulat. **16** (2011), 1970–1977.

[6] M. Benchohra, A. Ouahab, *Impulsive neutral functional differential inclusions with variable times*, Electron. J. Diff. Equ. **2003** (2003), 1–12.

[7] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[8] L. P. Castro, A. Ramos, *Hyers–Ulam–Rassias stability for a class of nonlinear Volterra integral equations*, Banach J. Math. Anal. **3** (2009), 36–43.

[9] J. B. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[10] D. Guo, *Nonlinear impulsive Volterra integral equations in Banach spaces and applications*, J. Appl. Math. Stoch. Anal. **6(1)** (1993), 35–48.

[11] M. Gachpazan, O. Baghani, *Hyers–Ulam stability of Volterra integral equations*, J. Nonlinear Anal. Appl. **1** (2010), 19–25.

[12] M. Gachpazan, O. Baghani, *Hyers–Ulam stability of nonlinear integral equations*, Fixed Point Theory Appl. **2010** (2010), 6 pages.

[13] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.

[14] S. M. Jung, *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory Appl. **2007** (2007), 9 pages.

[15] S. M. Jung, *A fixed point approach to the stability of differential equations $y' = F(x, y)$*, Bull. Malays. Math. Sci. Soc. **33** (2010), 47–56.

[16] S. M. Jung, S. Sevgin, H. Sevli, *On the perturbation of Volterra integro-differential equations*, Appl. Math. Lett. **26** (2013), 665–669.

[17] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[18] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York, 1964.

[19] A. Zada, O. Shah, R. Shah, *Hyers–Ulam stability of non-autonomous systems in terms of boundedness of Cauchy problems*, Appl. Math. Comp. **271** (2015), 512–518.

[20] R. Shah, A. Zada, *A fixed point approach to the stability of a nonlinear Volterra integro-differential equation with delay*, Hacet. J. Math. Stat. **47**(3) (2018), 615–623.

[21] R. Shah, A. Zada, *Hyers–Ulam–Rassias stability of impulsive Volterra integral equations via a fixed point approach*, J. Linear Topological Algebra **8**(4) (2019), 219–227.

[22] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Iteration Theory (ECIT'02), **346** (2004), 43–52.

[23] L. Hua, J. Huang, Y. Li, *Hyers–Ulam stability of some Fredholm integral equations*, Int. J. Pure Appl. Math. **104**(1) (2015), 107–117.

[24] Z. Gu, J. Huang, *Hyers–Ulam stability of Fredholm integral equations*, Math. Aeterna **5**(2) (2015), 257–261.

[25] L. P. Castro, A. M. Simões, *Hyers–Ulam and Hyers–Ulam–Rassias stability of a class of Hammerstein integral equations*, AIP Conf. Proc. **1798** (2017), 020036.

[26] L. Cădariu, L. Găvruţa, P. Găvruça, *Weighted space method for the stability of some nonlinear equations*, Appl. Anal. Discrete Math. **6**(1) (2012), 126–139.

[27] C. C. Tisdell, A. Zaidi, *Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling*, Nonlinear Anal. **68**(1) (2008), 3504–3524.

[28] V. Lakshmikantham, S. Leela, *Theory of Impulsive Differential Equations. World Scientific*, (2009).

[29] M. M. Rao, *Nonlinear Integral Equations and Stability Analysis*, Wiley, (2012).

[30] H. V. S. Chauhan, B. Singh, C. Tunç, O. Tunç, *On the existence of solutions of non-linear 2D Volterra integral equations in a Banach space*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, **116** (2022), 101.

[31] A. Deep, Deepmala, C. Tunç, *On the existence of solutions of some non-linear functional integral equations in Banach algebra with applications*, Arab. J. Basic Appl. Sci. **27** (2020), 279–286.

[32] P. Găvrută, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[33] J. R. Graef, O. Tunç, *Asymptotic behavior of solutions of Volterra integro–differential equations with and without retardation*, J. Integral Equations Appl. **33** (2021), 289–300.

[34] M. A. Hammami, N. Hnia, *On the stability of perturbed Volterra integro–differential equations*, J. Integral Equations Appl. **32** (2020), 325–339.

[35] R. Murali et al., *Aboodh transform and Ulam stability of second–order linear differential equations*, In: Mathematical Analysis, Differential Equations and Applications, World Scientific Publ. Co., 587–616.

[36] A. Alinejad et al., *A functional equation related to inner product spaces in Serstnev probabilistic normed spaces*, In: Mathematical Analysis, Differential Equations and Applications, World Scientific Publ. Co., 77–96.

[37] M. R. Abdollahpour et al., *Hyers–Ulam stability of hypergeometric differential equations*, Aequationes Mathematicae **93**(4) (2019), 691–698.

[38] M. R. Abdollahpour et al., *Hyers–Ulam stability of associated Laguerre differential equations in a subclass of analytic functions*, J. Math. Anal. Appl. **437** (2016), 605–612.

[39] J. Aczel et al., *Functional Equations in Several Variables*, Cambridge University Press, Vol. **31**, 1989.
[40] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.