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Hyers–Ulam stability for Hammerstein integral equations with impulses and delay

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Abstract. This paper mainly focuses on solving the Hyers–Ulam and Hyers–Ulam–Rassias stability problem of the Hammerstein integral equation with impulses and delay using fixed point technique. The consequent different cases of a finite interval and an infinite interval are studied. Finally, a concrete example is given at the end to justify the efficiency of the obtained theoretical results.

1. Introduction

Over the past sixty years, a great deal of research has been done on the concept of stability for different kinds of equations. Because of their many applications in elasticity, electronic devices, heat conduction, fluid flow, scattering theory, chemical reactions, and population dynamics, among other fields, special attention has been paid to the Hyers-Ulam stability and the Hyers-Ulam–Rassias stability of a wide range of functional equations, differential equations, and integral equations (see [1, 3, 4, 5, 6, 7, 9, 10]).

The first results on this type of stability for functional equations came from a famous question of S. M. Ulam [18] from 1940 regarding the requirement of some form of similarity between the solution of a given equation and a solution of an equation that is slightly different from it. With regard to Banach spaces, D.H. Hyers provided a partial solution to S. M. Ulam's query in the case of the additive Cauchy equation [13]. The term Hyers-Ulam-Rassias stability refers to the additional directions in which Th. M. Rassias, see [22], gave a stability analysis in 1978. Guo [10] provided some applications to initial value problems for the first-order impulsive differential equations in Banach spaces, as well as some existence theorems for external solutions to nonlinear, impulsive Volterra equations on a finite interval in Banach spaces, with a finite number of moments of impulse effect. Since many applied mathematics problems involve systems of differential or integral equations, the existence of solutions for the system of nonlinear impulsive Volterra integral equations on the infinite interval with an infinite number of moments of impulse effect in Banach spaces is studied.

Several important results in Banach spaces for fractional impulsive integro-differential equations were established by K. Balachandran et al. [5]. The Hyers-Ulam stability of integral equations has recently

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received attention. L. Hua et al. [23] investigated the Hyers–Ulam stability of a subset of Fredholm integral equations in 2015. Furthermore, the Stability of the Hyers–Ulam and Hyers–Ulam–Rassias classes of Hammerstein Integral Equations was explored in 2017 by L. P. Castro and A. M. Simoes [24]. Since it can be difficult to find exact answers for such physical issues, the concept of Hyers–Ulam stability is extremely helpful in many fields (such as economics, biology, numerical analysis, and optimization). It is extremely useful in the study of integral and differential equations see [1, 8, 11, 12, 15, 16, 19, 28–30], for a few recent works.

In 2009 and 2010, Castro and Ramos ([9,10]) dealt with the Hyers–Ulam stability and Hyers–Ulam–Rassias stabilities in the following non-linear Volterra IEs without delay and with delay. In 2013, Castro and Guerra [8] investigated the Hyers–Ulam–Rassias stability of the nonlinear Volterra IE with variable delay using the Banach FPT and the Bielecki metric.

In this work, we will be devoted to analyze the Hyers–Ulam and Hyers–Ulam–Rassias stability for the following class of the impulsive Hammerstein delay integral equation:

$$y(x) = p(x) + f(x, y(x), y(\tau(x))) \int_a^x g(x,\tau)h(\tau, y(\tau), y(\vartheta(\tau)))d\tau + \sum_{a<x_k<b} I_k(y(x_k^-)), \tag{1}$$

where, for starting, a and b are fixed real numbers, $p : [a, b] \to \mathbb{C}$ is a continuous function, $\tau, \vartheta : [a, b] \to [a, b]$ be continuous delay functions with $\tau(x) \le x$ and $\vartheta(\tau) \le \tau$, $f : [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $h : [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous functions, and the kernel $g : [a, b] \times [a, b] \to \mathbb{C}$ is also a continuous function. For all $x \in [a, b]$, $I_k : \mathbb{C} \to \mathbb{C}, k = 1, 2, \ldots, m$ and $y(x_k^-)$ represents the left limit of $y(x)$ at $x = x_k$.

2. Preliminary Results

This segment briefly discusses some basic concepts from the literature. For a non-empty set X, we introduce the definition of a generalized metric in X as follows:

Definition 2.1. *([2]) A mapping $d : X \times X \to [0, \infty]$ is called a generalized metric on set X if and only if d holds:*
(C_1) *$d(x, y) = 0$ if and only if $x = y$;*
(C_2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(C_3) *$d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

Next, we recall the main result of J. B. Diaz and B. Margolis [9], with the help of which we will be able to prove our main results.

Theorem 2.2. *[9] Let (X, d) be a complete generalized metric space. Assume that $T : X \to X$ is a strictly contractive operator with $L < 1$, where L is a Lipschitz constant. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following are true:*
(a) The sequence $T^n x$ converges to a fixed point x^* of T;
(b) x^* is the unique fixed point of T in

$$X^* = \left\{ y \in X \mid d(T^k x, y) < \infty \right\};$$

(c) If $y \in X^*$, then

$$d(y, x^*) \le \frac{1}{1-L} d(Ty, y). \tag{2}$$

Now, we give the definitions of Hyers–Ulam–Rassias and Hyers–Ulam stability of the impulsive Hammerstein delay integral equation (1).

Definition 2.3. *If for each continuously differentiable function $y(x)$ satisfying*

$$\left| y(x) - p(x) - f(x, y(x), y(\tau(x))) \int_a^x g(x,\tau)h(\tau, y(\tau), y(\vartheta(\tau)))d\tau - \sum_{a<x_k<x} I_k(y(x_k^-)) \right| \le \sigma(x),$$

for some $\sigma : I \to (0, \infty)$, *there exists a solution* $y_0(x)$ *of the delay impulsive Hammerstein integral equation* (1) *and a constant* $K > 0$ *with*

$$\left| y(x) - y_0(x) \right| \leq K\sigma(x),$$

for all $x \in I$, *where K is independent of* $y(x)$ *and* $y_0(x)$, *then we say that the impulsive Hammerstein integral equation* (1) *has the Hyers–Ulam–Rassias stability. If* $\sigma(x)$ *is a constant function in the above inequalities, we say that the impulsive Hammerstein delay integral equation* (1) *has Hyers–Ulam stability.*

In the current work, using the idea of C*ădariu* and Radu [22], we shall study the Ulam-type stability results of the impulsive Hammerstein delay integral equation (1).

3. Main Results

In this section, we study the Hyers–Ulam–Rassias and Hyers–Ulam stability for Eq. (1).

3.1. Hyers-Ulam-Rassias stability in the finite Interval Case

This section presents the necessary conditions for the Hyers-Ulam-Rassias stability of the impulsive Hammerstein delay integral equation (1), where $x \in [a, b]$, for some fixed real numbers a and b.
We will use the space $C([a, b])$ of continuous functions on [a,b], provided with the metric ([25])

$$d(u, v) = \sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{\sigma(x)}, \tag{3}$$

where σ is a non-decreasing continuous function $\sigma : [a, b] \to (0, \infty)$. Recall that $(C([a, b]), d)$ is a complete metric space (cf., e.g.,([26]), ([27])).

Theorem 3.1. *Let us consider a continuous given function* μ: $[a, b] \to [0, \infty)$. *Moreover, suppose that* $p : [a, b] \to \mathbb{C}$ *is a continuous function,* $f : [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *is a continuous function such that there exists* $M > 0$ *so that*

$$M = \sup_{a<x_k<b} |f(x, y)|, \quad y \in \mathbb{C},$$

and the kernal $g : [a, b] \times [a, b] \to \mathbb{C}$ *is also continuous. In addition, suppose that there is* $\beta \in [0, 1)$ *such that*

$$\int_a^x |g(x, \tau)|\mu(\tau)\sigma(\tau)d\tau \leq \beta\sigma(x)$$

and $h : [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *is a continuous function which fulfills the condition*

$$|h(x, u(x), u(\vartheta(x))) - h(x, v(x), v(\vartheta(x)))| \leq \mu(x)|u(x) - v(x)|$$

for all $x \in [a, b]$ *and* $u, v \in C([a, b])$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ *and there exists a constant* $L > 0$ *such that*

$$|I_k(u(x)) - I_k(v(x))| \leq L|u(x) - v(x)|$$

for all $u, v \in C([a, b])$.

If $y \in C([a, b])$ *is such that*

$$\left| y(x) - p(x) - f(x, y(x), y(\tau(x))) \int_a^x g(x, \tau)h(\tau, y(\tau), y(\vartheta(\tau)))d\tau - \sum_{a<x_k<x} I_k(y(x_k^-)) \right| \leq \sigma(x), \tag{4}$$

for all $x \in [a,b]$ and $M\beta + L < 1$, then there exists a unique function $y_0 \in C([a,b])$ such that

$$y_o(x) = p(x) + f(x, y_o(x), y_o(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_o(\tau), y_o(\vartheta(\tau)))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-))$$

and

$$|y(x) - y_o(x)| \le \frac{\sigma(x)}{1-(M\beta+L)} \tag{5}$$

for all $x \in [a,b]$.

This means that under the above conditions, of the impulsive Hammerstein delay integral equation (1) exhibits Hyers-Ulam-Rassias stability.

Proof. *We will consider the operator $\Omega : C([a,b]) \to C([a,b])$, defined by*

$$(\Omega(u))(x) = p(x) + f(x, u(x), u(\tau(x))) \int_a^x g(x,\tau)h(\tau, u(\tau), u(\vartheta(\tau)))d\tau + \sum_{a<x_k<x} I_k(u(x_k^-)),$$

for all $x \in [a,b]$ and $u \in C([a,b])$. Note that for any continuous function u, $\Omega(u)$ is also continuous. Indeed,

$$\begin{aligned}
|(\Omega(u))(x) - (\Omega(u))(x_o)| &= \Big| p(x) + f(x,u(x),u(\tau(x))) \int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
&\quad + \sum_{a<x_k<x} I_k(u(x_k^-)) - p(x_o) \\
&\quad - f(x_o,u(x_o),u(\tau(x_0))) \int_a^{x_o} g(x_o,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
&\quad - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-)) \Big| \\
&\le \Big| p(x) - p(x_0) \Big| + \Big| f(x,u(x),u(\tau(x))) \int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
&\quad + \sum_{a<x_k<x} I_k(u(x_k^-)) - f(x_o,u(x_o),u(\tau(x_o))) \int_a^{x_o} g(x_o,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
&\quad - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-)) \Big| \\
&\le \Big| p(x) - p(x_o) \Big| + \Big| f(x,u(x),u(\tau(x)) \int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
&\quad - f(x_o,u(x_o),u(\tau(x_o)) \int_a^{x_o} g(x_o,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \Big| \\
&\quad + \Big| \sum_{a<x_k<x} I_k(u(x_k^-)) - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-)) \Big|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \Big|p(x)-p(x_o)\Big| + M\Big|\int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
& - \int_a^{x_0} g(x_0,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau\Big| \\
& + \Big|\sum_{a<x_k<x} I_k(u(x_k^-)) - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-))\Big| \\
\leq\ & \Big|p(x)-p(x_o)\Big| + M\Big|\int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
& - \int_a^x g(x_o,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau + \int_a^x g(x_0,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau \\
& - \int_a^{x_o} g(x_o,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau\Big| + \Big|\sum_{a<x_k<x} I_k(u(x_k^-)) - \sum_{a<x_0k<x_0} I_k(u(x_{0_k}^-))\Big| \\
\leq\ & |p(x)-p(x_o)| + M\Big(\int_a^x |g(x,\tau)-g(x_o,\tau)||h(\tau,u(\tau),u(\vartheta(\tau)))|d\tau \\
& + \Big|\int_{x_o}^x g(x_o,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau\Big|\Big) + \Big|\sum_{a<x_k<x} I_k(u(x_k^-)) \\
& - \sum_{a<x_{0_k}<x_0} I_k(u(x_{0_k}^-))\Big| \to 0,
\end{aligned}
$$

when $x \to x_o$.

Based on the present conditions, we can conclude that the operator Ω is strictly contractive (for the given metric). For all $u, v \in C([a,b])$, we have,

$$
\begin{aligned}
& d(\Omega(u),\Omega(v)) = \sup_{x\in[a,b]} \frac{|(\Omega(u))(x)-(\Omega(v))(x)|}{\sigma(x)} \\
=\ & \sup_{x\in[a,b]} \frac{|f(x,u(x))\int_a^x g(x,\tau)h(\tau,u(\tau),u(\vartheta(\tau)))d\tau - f(x,v(x))\int_a^x g(x,\tau)h(\tau,v(\tau))d\tau| + \sum_{a<x_k<x} I_k[u(x_k^-)-v(x_k^-)]}{\sigma(x)} \\
\leq\ & M\sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)||h(\tau,u(\tau),u(\vartheta(\tau)))-h(\tau,v(\tau),v(\vartheta(\tau)))|d\tau}{\sigma(x)} \\
+\ & \sup_{x\in[a,b]} \frac{\sum_{a<x_k<x} I_k[u(x_k^-)-v(x_k^-)]}{\sigma(x)} \\
=\ & M\sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)\frac{|u(\tau)-v(\tau)|}{\sigma(\tau)}d\tau}{\sigma(x)} + L\sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} \\
\leq\ & M\sup_{\tau\in[a,b]} \frac{|u(\tau)-v(\tau)|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)d\tau}{\sigma(x)} + L\sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} \\
\leq\ & Md(u,v)\beta + Ld(u,v) \\
\leq\ & (M\beta + L)d(u,v).
\end{aligned}
$$

Due to the fact that $M\beta + L < 1$ it follows that Ω is strictly contractive. Using the above mentioned Banach fixed point theorem, the impulsive Hammerstein delay integral equation achieves Hyers-Ulam-Rassias stability. Additionally, (5) *follows from* (4) *and* (2).

3.2. Hyers-Ulam-Rassias stability in the infinite Interval Case.

This subsection focuses on the Hyers-Ulam-Rassias stability of the impulsive Hammerstein delay integral equation across infinite intervals. This means that instead of considering, as before, a finite interval [a, b] (with $a, b \in \mathbb{R}$), we will now consider e.g., corresponding intervals $[a, \infty)$, for some fixed $a \in \mathbb{R}$.

Thus, we will now be dealing with the integral equation

$$y(x) = p(x) + f((x, y(x)), y(\tau(x))) \int_a^x g(x,\tau)h(\tau, y(\tau), y(\vartheta(\tau)))d\tau + \sum_{a<x_k<x} I_k(y(x_k^-)), \tag{6}$$

for all $x \in [a, \infty)$, where a is a fixed real number, $p : [a, \infty) \to \mathbb{C}$ is a bounded continuous function, $\tau, \vartheta : [a, \infty) \to [a, \infty)$ be bounded continuous delay functions with $\tau(x) \leq x$ and $\vartheta(\tau) \leq \tau$ and $f : [a, \infty)\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is bounded continuous function, $h : [a, \infty)\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ and the kernel $g : [a, \infty)\times[a, \infty) \to \mathbb{C}$ are also continuous functions. Moreover, $I_k : \mathbb{C} \to \mathbb{C}, k = 1, 2, \ldots, m$ and $y(x_k^-)$ represents the left limit of $y(x)$ at $x = x_k$. Our technique relies on a recurrence procedure, as the result for the finite interval situation is already available.

Let us consider a fixed non-decreasing continuous function $\phi : [a, \infty) \to (\epsilon, \eta)$, for some $\epsilon, \eta > 0$, and the space $C_b([a, \infty))$ of bounded continuous functions endowed with the weighted metric ([25])

$$d_b(u, v) = \sup_{x\in[a,\infty)} \frac{|u(x) - v(x)|}{\phi(x)}.$$

Theorem 3.2. *Let us consider a continuous function $\mu : [a, \infty) \to [0, \infty)$. Further, assume that $p : [a, \infty) \to \mathbb{C}$ is a bounded continuous function, $f : [a, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a bounded continuous function such that there exists $M > 0$ so that*

$$M = \sup_{x\in[a,\infty)} |f(x, y(x), y(\tau(x)))|, \quad y \in \mathbb{C},$$

and the kernel $g : [a, \infty)\times[a, \infty) \to \mathbb{C}$ and $h : [a, \infty)\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ are continuous functions so that $\int_a^x g(x, \tau)h(\tau, z(\tau))d\tau$ is a bounded continuous function for any bounded continuous function z.

In addition, suppose that there is $\beta \in [0, 1)$ such that

$$\int_a^x |g(x, \tau)|\mu(\tau)\varphi(\tau)d\tau \leq \beta\varphi(x)$$

and $h : [a, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function which fulfills the condition

$$|h(x, u(x), u(\vartheta(x))) - h(x, v(x), v(\vartheta(x)))| \leq \mu(x)|u(x) - v(x)|$$

for all $x \in [a, \infty)$ and $u, v \in C_b([a, \infty))$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ and there exists a constant $L > 0$ such that

$$|I_k(u(x)) - I_k(v(x))| \leq L|u(x) - v(x)|$$

for all $u, v \in C_b([a, \infty))$.

If $y \in C_b([a, \infty))$ is such that

$$\left|y(x) - p(x) - f(x, y(x), y(\tau(x)) \int_a^x g(x, \tau)h(\tau, y(\tau), y(\vartheta(\tau)))d\tau - \sum_{a<x_k<x} I_k(y(x_k^-))\right| \leq \varphi(x),$$

for all $x \in [a, \infty)$ and $M\beta + L < 1$, then there exists a unique function $y_o \in C_b([a, \infty))$ such that

$$y_o(x) = p(x) + f(x, y_0(x), y_0(\tau(x)) \int_a^x g(x, \tau)h(\tau, y_o(\tau), y_0(\tau(x))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-)) \tag{7}$$

and

$$|y(x) - y_o(x)| \leq \frac{\varphi(x)}{1 - (M\beta + L)} \tag{8}$$

for all $x \in [a, \infty)$.

Under the given conditions, the impulsive Hammerstein delay integral equation (1) *exhibits Hyers-Ulam-Rassias stability.*

Proof. For any $n \in \mathbb{N}$, we will define $I_n = [a, a+n]$. By Theorem (3.1), there exists a unique bounded continuous function $y_{o,n} : I_n \to \mathbb{C}$ such that

$$y_{o,n}(x) = p(x) + f(x, y_{o,n}(x), y_{o,n}(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_{o,n}(\tau), y_{o,n}(\vartheta(\tau)))d\tau + \sum_{a<x_k<x} I_k(y_{o,n}(x_k^-)) \tag{9}$$

and

$$|y(x) - y_{o,n}(x)| \leq \frac{\varphi(x)}{1 - (M\beta + L)} \tag{10}$$

for all $x \in I_n$. The uniqueness of $y_{o,n}$ implies that if $x \in I_n$ then

$$y_{o,n}(x) = y_{o,n+1}(x) = y_{o,n+2}(x) = \cdots . \tag{11}$$

For any $x \in [a, \infty)$, let us define $n(x) \in \mathbb{N}$ as $n(x) = min\{n \in \mathbb{N} \mid x \in I_n\}$. We also define a function $y_o : [a, \infty) \to \mathbb{C}$ by

$$y_o(x) = y_{o,n(x)}(x). \tag{12}$$

For any $x_1 \in [a, \infty)$, let $n_1 = n(x_1)$. Then $x_1 \in IntI_{n_1+1}$ and there exists an $\epsilon > 0$ such that $y_0(x) = y_{0,n_1+1}(x)$ for all $x \in (x_1 - \epsilon, x_1 + \epsilon)$. By Theorem (3.1), y_{0,n_1+1} is continuous at x_1, and so it is y_0.

Now, we will prove y_o satisfies

$$y_o(x) = p(x) + f(x, y_o(x), y_o(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_o(\tau), y_o(\vartheta(\tau)))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-))$$

and

$$|y(x) - y_o(x)| \leq \frac{\varphi(x)}{1 - (M\beta + L)} \tag{13}$$

for all $x \in [a, \infty)$. For an arbitrary $x \in [a, \infty)$ we choose $n(x)$ such that $x \in I_{n(x)}$. By (9) and (12), we have

$$\begin{aligned} y_o(x) &= y_{o,n(x)}(x) = p(x) + f(x, y_{o,n(x)}(x), y_{o,n(x)}(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_{o,n(x)}(\tau), y_{o,n(x)}(\vartheta(\tau)))d\tau \\ &+ \sum_{a<x_k<x} I_k(y_{o,n(x)}(x_k^-)) \\ &= p(x) + f(x, y_o(x), y_o(\tau(x))) \int_a^x g(x,\tau)h(\tau, y_o(\tau), y_o(\vartheta(\tau))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-)). \end{aligned} \tag{14}$$

Note that $n(\zeta) \leq n(x)$, for any $\zeta \in I_{n(x)}$, and it follows from (11) that $y_o(\zeta) = y_{o,n(\zeta)}(\zeta) = y_{o,n(x)}(\zeta)$, so, the last equality in (14) holds.

To prove (13), by (12) and (10), we have for all $x \in [a, \infty)$,

$$|y(x) - y_o(x)| = |y(x) - y_{o,n(x)}(x)| \leq \frac{\varphi(x)}{1 - (M\beta + L)}.$$

Finally, we will prove the uniqueness of y_o. Let us consider another bounded continuous function y_1 which satisfies (7) and (8), for all $x \in [a, \infty)$. By the uniqueness of the solution on $I_{n(x)}$ for any $n(x) \in \mathbb{N}$ we have that $y_{o|I_{n(x)}} = y_{o,n(x)}$ and $y_{1|I_{n(x)}}$ satisfies (7) and (8) for all $x \in I_{n(x)}$, so

$$y_o(x) = y_{o|I_{n(x)}}(x) = y_{1|I_{n(x)}}(x) = y_1(x).$$

3.3. Hyers–Ulam stability in the finite interval case

This section provides adequate requirements for the Hyers-Ulam stability of the impulsive Hammerstein delay integral equation (1). For a given non-decreasing continuous function $\sigma : [a, b] \to (0, \infty)$, we shall utilize the same metric (3).

Theorem 3.3. *Let us consider a continuous function $\mu : [a, b] \to [o, \infty)$. Moreover, assume that $p : [a, b] \to \mathbb{C}$ is a continuous function, $f : [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function such that there exists $M > 0$ so that*

$$M = \sup_{x \in [a,b]} |f(x, y)|, \quad y \in \mathbb{C},$$

and the kernel $g : [a, b] \times [a, b] \to \mathbb{C}$ is also continuous. In addition, suppose that there is $\beta \in [0, 1)$ such that

$$\int_a^x |g(x, \tau)|\mu(\tau)\sigma(\tau)d\tau \leq \beta\sigma(x)$$

and $h : [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function which fulfils the condition

$$|h(x, u(x)) - h(x, v(x))| \leq \mu(x)|u(x) - v(x)|$$

for all $x \in [a, b]$ and $u, v \in C([a, b])$.
Moreover, $I_k : \mathbb{C} \to \mathbb{C}$ and there exists a constant $L > 0$ such that

$$|I_k(u(x)) - I_k(v(x))| \leq L|u(x) - v(x)|$$

for all $u, v \in C([a, b])$.

If $y \in C([a, b])$ is such that

$$\left| y(x) - p(x) - f(x, y(x), y(\tau(x))) \int_a^x g(x, \tau)h(\tau, y(\tau), y(\vartheta(\tau)))d\tau - \sum_{a<x_k<x} I_k(y(x_k^-)) \right| \leq \theta, \tag{15}$$

$x \in [a, b]$, where $\theta \geq 0$ and $M\beta + L < 1$, then there is a unique function $y_o \in C([a, b])$ such that

$$y_o(x) = p(x) + f(x, y_o(x), y_o(\tau(x))) \int_a^x g(x, \tau)h(\tau, y_o(\tau), y_o(\vartheta(\tau)))d\tau + \sum_{a<x_k<x} I_k(y_o(x_k^-))$$

and

$$|y(x) - y_o(x)| \leq \frac{\theta}{1 - (M\beta + L)} \tag{16}$$

for all $x \in [a, b]$.

This means that under the above conditions, the impulsive Hammerstein delay integral equation (1) has the Hyers-Ulam stability.

Proof. We will consider the operator $T : C([a,b]) \to C([a,b])$, defined by

$$(Tu)(x) = p(x) + f(x, u(x), u(\tau(x))) \int_a^x g(x,\tau)h(\tau, u(\tau), u(\vartheta(\tau)))d\tau + \sum_{a<x_k<x} I_k(u(x_k^-))$$

for all $x \in [a,b]$ and $u \in C([a,b])$.

T is strictly contractive (with respect to the metric under consideration). Indeed, for all $u, v \in C([a,b])$, we have,

$$\begin{aligned}
d(Tu,Tv) &= \sup_{x\in[a,b]} \frac{|(Tu)(x) - (Tv)(x)|}{\sigma(x)} \\
&\leq M \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)||h(\tau,u(\tau),u(\vartheta(\tau))) - h(\tau,v(\tau),v(\vartheta(\tau)))|d\tau}{\sigma(x)} \\
&+ \sup_{x\in[a,b]} \frac{\sum_{a<x_k<x} I_k|u(x_k^-) - v(x_k^-)|}{\sigma(x)} \\
&= M \sup_{x\in[a,b]} \int_a^x \frac{|g(x,\tau)|\mu(\tau)\sigma(\tau)\frac{|u(\tau)-v(\tau)|}{\sigma(\tau)}d\tau}{\sigma(x)} \\
&+ L \sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{\sigma(x)} \\
&\leq M \sup_{\tau\in[a,b]} \frac{|u(\tau) - v(\tau)|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)d\tau}{\sigma(x)} \\
&+ L \sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{\sigma(x)} \\
&\leq Md(u,v)\beta + Ld(u,v) \\
&\leq (M\beta + L)d(u,v).
\end{aligned}$$

Due to the fact that $M\beta + L < 1$ it follows that T is strictly contractive. Thus, again we can apply the Banach fixed point theorem, which ensures that we have the Hyers-Ulam stability for the impulsive Hammerstein delay integral equation with (16) which is obtained by using (2) and (15).

3.4. Example

In this section, we present an application of the results derived.

Example 3.4. *Consider the following impulsive Hammerstein delay integral equation:*

$$\begin{aligned}
y(x) =& \frac{1}{2000}\exp(-3x) + x\sqrt{x^2+4} + \frac{1}{1000}\sin\left(\frac{1}{x}\right) \\
&+ \frac{\sin(y(x))}{2000(3+\exp(x^2))}\int_0^x \exp[-4(x-\tau)]\left[\frac{3y(\tau)+\sin(y(\tau))}{2000+\tau^3+3\tau^2+\tau}\right]d\tau \\
&+ \frac{\left|y\left(\frac{5}{9}^-\right)\right|}{40+\left|y\left(\frac{5}{9}^-\right)\right|}.
\end{aligned} \quad (17)$$

We note that Equation (17) *is in the form of Equation* (1) *with the data as follows:*

$[a,b] = [0,1], x, \tau \in [0,1],$

$$p(x) = \frac{1}{2000}\exp(-3x) + x\sqrt{x^2+4} + \frac{1}{1000}\sin\Big(\frac{1}{x}\Big),$$

$$f(x, y(x), y(\tau(x))) = \frac{\sin(y(x))}{2000(3+\exp(x^2))}, \tau(x) = 0,$$

$$g(x,\tau) = \exp[-4(x-\tau)],$$

$$h(\tau, y(\tau), y(\vartheta(\tau))) = \frac{3y(\tau)+\sin(y(\tau))}{2000+\tau^3+3\tau^2+\tau}, \vartheta(\tau) = 0.$$

$$\triangle y|_{x=\frac{5}{9}} = \frac{\left|y\left(\frac{5}{9}^-\right)\right|}{40+\left|y\left(\frac{5}{9}^-\right)\right|}.$$

We now check the conditions of Theorem 3.1. To show that the conditions of Theorem 3.1 hold, we let $M = 2000^{-1}$, $\beta = 3^{-1}$, $\mu(x) = x^{-1}$, $\sigma(x) = x$, $L = 40^{-1}$, *and calculate:*

$$\max_{x\in[a,b]}|f(x, y(x), y(\tau(x)))| = \frac{1}{2000}\max_{x\in[0,1]}\left|\frac{\sin(y(x))}{3+\exp(x^2)}\right| \le \frac{1}{2000},$$

$$\forall\, x\in[0,1], \forall\, y\in\mathbb{C},$$

$$\begin{aligned}\int_a^x |g(x,\tau)|\mu(\tau)\sigma(\tau)d\tau &= \int_0^x \exp[-4(x-\tau)]\frac{1}{\tau}\tau d\tau\\ &= \int_0^x \exp[-4(x-\tau)]d\tau \le \frac{1}{3}x, \forall\, x,\tau\in[0,1],\end{aligned}$$

$$\begin{aligned}&|h(x,u(x),u(\vartheta(x))) - h(x,v(x),v(\vartheta(x)))|\\ &\qquad= \left|\frac{3u(x)+\sin(u(x))}{2000+x^3+3x^2+x} - \frac{3v(x)+\sin(v(x))}{2000+x^3+3x^2+x}\right|\\ &\qquad\le \frac{1}{x}|u(x)-v(x)|, \forall\, x\in[0,1], \forall\, u,v\in\mathbb{C},\end{aligned}$$

$$\begin{aligned}|I_k(u(x)) - I_k(v(x))| &= \left|\frac{u(x)}{40+u(x)} - \frac{v(x)}{40+v(x)}\right|\\ &= \frac{40|u(x)-v(x)|}{(40+u(x))(40+v(x))}\\ &\le \frac{1}{40}|u(x)-v(x)|, \forall\, x\in[0,1], \forall\, u,v\in\mathbb{C}.\end{aligned}$$

It is also clear that $M\beta + L = \frac{1}{2000} \times \frac{1}{3} + \frac{1}{40} = 0.02516666667 < 1$. *Hence, using this value, we have*

$$|y(x) - y_o(x)| \leq \frac{x}{1 - \left[\frac{1}{2000} \times \frac{1}{3} + \frac{1}{40}\right]} = \frac{\sigma(x)}{1 - (M\beta + L)},\ x \in [0, 1].$$

Hence, equation (17) *admits the Hyers–Ulam–Rassias stability in the finite interval case. Since* $x \in [0, 1]$, *obviously, we also have*

$$|y(x) - y_o(x)| \leq \frac{x}{1 - \left[\frac{1}{2000} \times \frac{1}{3} + \frac{1}{40}\right]} = \frac{1}{1 - (M\beta + L)} = \frac{\theta}{1 - (M\beta + L)},\ \theta = 1.$$

Thus, equation (17) *is stable in the Hyers–Ulam sense.*

4. Conclusion

In this study, the Hammerstein integral equation for impulses and delay was considered. We developed new sufficient criteria for the impulsive Hammerstein delay integral equation and obtained the results of Hyers–Ulam–Rassias stability on finite and infinite intervals, as well as Hyers–Ulam stability on finite intervals. An example was provided to demonstrate the numerical application of the results.

Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

The authors declare that they have no competing interests.

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