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Ulam–Hyers stability of higher dimensional weakly singular Volterra integral equations

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Abstract. Using the Gronwall lemma method, we prove the Ulam–Hyers and Ulam–Hyers–Rassias stability of higher-dimensional weakly singular Volterra integral equations. To highlight the practical use of the theoretical results, two examples are presented.

1. Introduction And Preliminaries

A functional equation is considered stable if an exact solution can be found that is near each approximate solution. The notion of stability was first presented by S. M. Ulam [7] in 1940 and subsequently developed into a central concept in mathematical analysis. The purpose was to determine whether or not there exists a functional equation with an approximate solution that is as close to the exact solution as is practical. This difficulty was partially addressed for Banach spaces in 1945 by D. H. Hyers (see [9]), specifically for the additive Cauchy equation $f(\chi + \zeta) = f(\chi) + f(\zeta)$, giving rise to the Ulam–Hyers stability. Th. M. Rassias [3] built upon Hyers's work in the 1970s, adding new concepts and establishing the Ulam–Hyers–Rassias stability. The contributions made by Rassias [4] significantly expanded the field of stability research, leading to a number of generalizations and applications.

The stability principle developed by Ulam–Hyers has been extended and used to a wide range of mathematical disciplines, covering differential equations, differentiation equations, and numerous branches of analysis. These ideas are widely used in domains such as control theory and mathematical physics, where they contribute to the stability of physical models (see [2],[10],[11], [13], [15], [16], [17]). It provides a valuable foundation for understanding how slight changes in input or function may be handled without affecting the system's stability. Numerous strategies and techniques are used while trying to find exact and approximate solutions to mathematical problems, particularly those involving differential and integral equations (see for example [2], [5], [10], [12]). In this paper, we shall employ the Gronwall lemma. It demonstrates that an approximate solution converges to a single exact solution. This is crucial in many stability issues, notably those involving integral or differential equations (see [1], [2], and [8]).

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Motivated by the stability theories (see [1], [2], [8]) this research examine the subsequent higher-dimensional weakly singular Volterra integral equation.

$$\begin{aligned}\Im(\chi_1,\chi_2...\chi_n) \quad = \quad & \alpha + \int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\Im(\zeta_1,\chi_2,...\chi_n)d\zeta_1 \\ & + \int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\Im(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2 + ... \\ & + \int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\Im(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n,\end{aligned} \tag{1}$$

where

$\alpha > 0\ ,\ b_i > 0\ ,\ i = 1, 2...n\ ,\ D = \prod_{i=1}^{n}[0, b_i]$ and $K_i \in C(D)$.

and $V_i(\chi_i - \zeta_i) \in C(D), \forall \chi_i \neq \zeta_i$ is the weakly singular part and $P_{K_i} > 0$ is so that:

$$|K_i(x)| \leq P_{K_i}, \forall \chi \in D, i = 1, 2, ...n.$$

We also define an operator $O : C(D) \to C(D)$ so that

$$\begin{aligned}O(\Im)(\chi_1,\chi_2,...\chi_n) \quad := \quad & \alpha + \int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\Im(\zeta_1,\chi_2,...\chi_n)d\zeta_1 \\ & + \int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\Im(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2 + ... \\ & + \int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\Im(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n.\end{aligned}$$

Weakly singular Volterra integral equations are useful in many scientific and engineering domains because they can be used in any scenario in which the current state of a system is determined by a range of previous states or historical impacts [11]. First, we cover the fundamentals of Picard operators theory. Our results are connected to resent paper (see Lungu [1]). The Gronwall lemmas (Lemma 1.3, Lemma 1.4) will be applied to determine Ulam–Hyers and Ulam–Hyers–Rassias stability. For each situation, we will present examples that demonstrate the uses of Ulam–Hyers and Ulam–Hyers–Rassias stability.

Our work is structured as follows: initially, we will go over some formal definitions, basic findings, and lemmas that aids in the illustration of our findings. We shall demonstrate (1)'s Ulam–Hyers stability in section 2. For equation (1), the Ulam–Hyers–Rassias stability will be presented in section 3. Section 4 will include a few illustrative cases that will align with our findings. The conclusion will be given in the end.

We will now present the formal definitions of the stabilities discussed earlier for the higher-dimensional weakly singular Volterra integral equation. To confirm the Ulam-Hyers and Ulam-Hyers-Rassias stability of this equation, the following definitions, theorems, and lemmas are crucial.

Definition 1.1. *In terms of ϵ, equation* (1) *is considered Ulam–Hyers stable if a constant $G_K > 0$ exists with the condition that, for every solution $\Im \in C(D)$ of the inequality*

$$\begin{aligned}&\Big|\Im(\chi_1,\chi_2,...\chi_n) - \alpha - \int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\Im(\zeta_1,\chi_2,...\chi_n)d\zeta_1 \\ &- \int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\Im(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2 - ... \\ &- \int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\Im(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n\Big| \leq \epsilon\end{aligned}$$

For equation (1), *there is a solution* $\Im^* \in C(D)$ *such that*

$$|\Im(\chi_1, \chi_2, ...\chi_n) - \Im^*(\chi_1, \chi_2, ...\chi_n)| \leq G_K.\epsilon, \forall \chi_1, \chi_2, ...\chi_n \in D.$$

Definition 1.2. *In terms of* $\psi(\chi_1, \chi_2, ...\chi_n)$, *equation* (1) *is considered Ulam–Hyers–Rassias stable if a constant* $G_k > 0$ *exists with the condition that, for every solution* $\Im \in C(D)$ *of the inequality*

$$\Big|\Im(\chi_1, \chi_2...\chi_n) - \alpha - \int_0^{\chi_1} V_1(\chi_1 - \zeta_1)K_1(\zeta_1, \chi_2, ...\chi_n)\Im(\zeta_1, \chi_2, ...\chi_n)d\zeta_1$$
$$- \int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2 - \zeta_2)K_2(\zeta_1, \zeta_2, \chi_3...\chi_n)\Im(\zeta_1, \zeta_2, \chi_3...\chi_n)d\zeta_1, d\zeta_2 - ...$$
$$- \int_0^{\chi_1}\int_0^{\chi_2} ... \int_0^{\chi_n} V_n(\chi_n - \zeta_n)K_n(\zeta_1, \zeta_2, \zeta_3...\zeta_n)\Im(\zeta_1, \zeta_2, \zeta_3...\zeta_n)d\zeta_1, d\zeta_2...d\zeta_n\Big| \leq \psi(\chi_1, \chi_2, ...\chi_n)$$

For equation (1), *there is a solution* $\Im^* \in C(D)$ *such that:*

$$\Big|\Im(\chi_1, \chi_2, ...\chi_n) - \Im^*(\chi_1, \chi_2, ...\chi_n)\Big| \leq G_K.\psi(\chi_1, \chi_2, ...\chi_n), \forall \chi_1, \chi_2, ...\chi_n \in D.$$

Lemma 1.3. *([18])). Let* $(Y, \to, \leq)$ *be an ordered L-space and O from Y to itself be an operator. Assume that:*
(i) O is picard operator $(F_O = \{\chi_O^*\})$;
(ii) O is an increasing operator.
Consequently, we obtain:
$(a)\chi \in Y, \chi \leq O(\chi) \Rightarrow \chi \leq \chi_O^*$;
$(b)\chi \in Y, \chi \geq O(\chi) \Rightarrow \chi \geq \chi_O^*$.

Lemma 1.4. ([19])) *Let* $(Y, \to, \leq)$ *be an ordered L-space and* O, O_1 *from Y to itself be two operators. Assume that:*
(i) O and O_1 *are picard operators;*
(ii) O is an increasing operator;
(iii) $O \leq O_1$.
Consequently, we obtain:
$\chi \leq O(\chi) \Rightarrow \chi \leq \chi_O^* \leq \chi_{O_1}^*$;

Lemma 1.5. ([19]) *Let* $(Y, \to, \leq)$ *be an ordered L-space and* O, O_1 *from Y to itself be two operators. Assume that:*
(i) O and O_1 *are picard operators;*
(ii) O is an increasing operator;
(iii) $\chi = O(\chi) \Rightarrow \chi \leq O_1(\chi)$
Consequently, we obtain:
$\chi \leq O(\chi) \Rightarrow \chi \leq \chi_{O_1}^*$.

Theorem 1.6. ([1]) *Let* $\alpha > 0$, $K_i \in C(D, \mathbb{R}+)$, $i = 1, 2, ...n$. *subsequently,*
(a) $\Im^*(\chi_1, \chi_2, ...\chi_n) > 0, \forall\ \chi_1, \chi_2, ...\chi_n \in D$;
(b) If $K_i(\chi_1, \chi_2, ...\chi_n)$ *increases with regard to* $\chi_{i+1}, ..., \chi_n$ *then,* $\Im^*$ *is increasing, where* $\Im^*$ *is the only solution for the equation* (1).

2. Ulam–Hyers stability

This section aims to establish the Ulam-Hyers stability of the higher dimensional weakly singular Volterra integral equation through the use of the Gronwall lemma. We will also include an example for a particular case to showcase how Theorem 2.1 and Theorem 3.1 can be applied.

Theorem 2.1. *Assume that*
(i) $K_i \in (C(D), \mathbb{R}+)$, $i = 1, 2, ..n$ *and let* $P_{K_i} > 0$, *so that*

$$|K_i(\chi)| \leq P_{K_i}, \forall \chi \in D, i = 1, 2, ...n.$$

Subsequently:

(a) The unique solution $\Im^*$ *for equation* (1.1) *exists within* $C(D)$;

(b) For each $\epsilon > 0$, *if* $\Im \in C(D)$ *is the solution of inequation*

$$\Big|\Im(\chi_1,\chi_2...\chi_n) - \alpha - \int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\Im(\zeta_1,\chi_2,...\chi_n)d\zeta_1$$
$$-\int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\Im(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2$$
$$-...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\Im(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n\Big|, \leq \epsilon, \forall\chi \in D.$$

Then:

$$|\Im(\chi) - \Im^*(\chi)| \leq G_K.\epsilon,$$

for all $\chi \in D$, *where,*

$$G_k = \exp(P_{K_1}.c_1.b_1 + P_{K_2}.c_2.b_1b_2 + ... + P_{K_n}.c_nb_1b_2...b_n).$$

i.e., the equation is Ulam–Hyers stable.

Proof. (*a*) This result is well established (see [1],[2]).

(*b*) Let us consider:

$|\Im(\chi_1,\chi_2...\chi_n) - \Im^*(\chi_1,\chi_2...\chi_n)|$

$$|\Im(\chi_1,\chi_2...\chi_n) - \Im^*(\chi_1,\chi_2...\chi_n)| = \Big|\Im(\chi_1,\chi_2...\chi_n) - \alpha - \int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\Im^*(\zeta_1,\chi_2,...\chi_n)d\zeta_1$$
$$-\int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\Im^*(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2$$
$$-...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\Im^*(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n\Big|$$

From Lemma 1.4 and Lemma 1.5

$$\leq \Big|\Im(\chi_1,\chi_2...\chi_n) - \alpha - \int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\Im(\zeta_1,\chi_2,...\chi_n)d\zeta_1$$
$$-\int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\Im(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2$$
$$-...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\Im(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n\Big|$$
$$+\Big|\int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\Im(\zeta_1,\chi_2,...\chi_n)d\zeta_1$$
$$+\int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\Im(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2$$
$$+...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\Im(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n$$
$$-\int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\Im^*(\zeta_1,\chi_2,...\chi_n)d\zeta_1$$

$$-\int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\mathfrak{I}^*(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2$$
$$-...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\mathfrak{I}^*(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n\Big|.$$

which implies that,

$$\begin{aligned}
\leq\ & \epsilon+\int_0^{\chi_1} K_1(\zeta_1,\chi_2,...\chi_n)V_1(\chi_1-\zeta_1)|\mathfrak{I}(\zeta_1,\chi_2,...\chi_n)-\mathfrak{I}^*(\zeta_1,\chi_2,...\chi_n)|d\zeta_1\\
&+\int_0^{\chi_1}\int_0^{\chi_2} K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)V_2(\chi_2-\zeta_2)|\mathfrak{I}(\zeta_1,\zeta_2,\chi_3...\chi_n)-\mathfrak{I}^*(\zeta_1,\zeta_2,\chi_3...\chi_n)|d\zeta_1d\zeta_2\\
&+...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)V_n(\chi_n-\zeta_n)|\mathfrak{I}(\zeta_1,\zeta_2,\zeta_3...\zeta_n)-\mathfrak{I}^*(\zeta_1,\zeta_2,\zeta_3...\zeta_n)|d\zeta_1d\zeta_2...d\zeta_n
\end{aligned}$$

using Lemma 1.5 Theorem 1.6 and Gronwall Lemma we get,

$$\begin{aligned}
\leq\ & \epsilon.\exp\Big(\int_0^{\chi_1} K_1(\zeta_1,\chi_2,...\chi_n)V_1(\chi_1-\zeta_1)|\mathfrak{I}(\zeta_1,\chi_2,...\chi_n)-\mathfrak{I}^*(\zeta_1,\chi_2,...\chi_n)|d\zeta_1\\
&+\int_0^{\chi_1}\int_0^{\chi_2} K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)V_2(\chi_2-\zeta_2)|\mathfrak{I}(\zeta_1,\zeta_2,\chi_3...\chi_n)-\mathfrak{I}^*(\zeta_1,\zeta_2,\chi_3...\chi_n)|d\zeta_1d\zeta_2\\
&+...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)V_n(\chi_n-\zeta_n)|\mathfrak{I}(\zeta_1,\zeta_2,\zeta_3...\zeta_n)-\mathfrak{I}^*(\zeta_1,\zeta_2,\zeta_3...\zeta_n)|d\zeta_1d\zeta_2...d\zeta_n\Big)\\
\leq\ & \epsilon.\exp\left(P_{K_1}.c_1.b_1+P_{K_2}.c_2.b_1b_2+...P_{K_n}.c_n.b_1b_2...b_n\right)\\
\leq\ & \epsilon.G_K.
\end{aligned}$$

where,

$$G_K=\exp\left(P_{K_1}.c_1.b_1+P_{K_2}.c_2.b_1b_2+...+P_{K_n}.c_n.b_1b_2...b_n\right).$$

Thus, equation (1) demonstrates Ulam–Hyers stability. □

3. Ulam–Hyers–Rassias stability

This section will focus on proving the Ulam–Hyers–Rassias stability of the higher dimensional weakly singular Volterra integral equation through the application of the Gronwall lemma.

Theorem 3.1. *Assume that:*
(i)$K_i\in(C(D),\mathbb{R}+),\ i=1,2,..n$ *there exists* $P_{K_i}>0$ *so that :*
$|K_i(\chi)|\leq P_{K_i},\ \forall\chi\in D,\ i=1,2,...n$ *and* $\psi\in(C(D),\mathbb{R}+)$;
(ii) ψ *is a non-decreasing function*
Then:
(a) The unique solution $\mathfrak{I}^*$ *for equation* (1.1) *exists within* $C(D)$*;*
(b) If $\mathfrak{I}\in C(D)$ *is with the condition that:*

$$\Big|\mathfrak{I}(\chi_1,\chi_2...\chi_n)-\alpha-\int_0^{\chi_1} V_1(\chi_1-\zeta_1)K_1(\zeta_1,\chi_2,...\chi_n)\mathfrak{I}(\zeta_1,\chi_2,...\chi_n)d\zeta_1$$
$$-\int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2-\zeta_2)K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)\mathfrak{I}(\zeta_1,\zeta_2,\chi_3...\chi_n)d\zeta_1,d\zeta_2$$
$$-...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n-\zeta_n)K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)\mathfrak{I}(\zeta_1,\zeta_2,\zeta_3...\zeta_n)d\zeta_1,d\zeta_2...d\zeta_n\Big|\leq\psi(\chi_1,\chi_2,...\chi_n),\forall\chi\in D$$

then,

$$|\mathfrak{I}(\chi) - \mathfrak{I}^*(\chi)| \leq G_K.\psi(\chi_1, \chi_2, ...\chi_n), \forall \chi \in D.$$

Where,

$$G_K = \exp(P_{K_1}.c_1.b_1 + P_{K_2}.c_2.b_1b_2 + ...P_{K_n}.c_n.b_1b_2...b_n)$$

$\mathfrak{I}^*(\chi_1, \chi_2...\chi_n)$ *is the only fixed point of the equation* (1); *thus,* (1) *is regarded as Ulam–Hyers–Rassias stable.*

Proof. (a) This result is well established (see [1],[2]).
(b)Let us consider:

$$\begin{aligned}|\mathfrak{I}(\chi_1, \chi_2...\chi_n) - \mathfrak{I}^*(\chi_1, \chi_2...\chi_n)| \quad = \quad & \Big|\mathfrak{I}(\chi_1, \chi_2...\chi_n) - \alpha - \int_0^{\chi_1} V_1(\chi_1 - \zeta_1)K_1(\zeta_1, \chi_2, ...\chi_n)\mathfrak{I}^*(\zeta_1, \chi_2, ...\chi_n)d\zeta_1 \\ & - \int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2 - \zeta_2)K_2(\zeta_1, \zeta_2, \chi_3...\chi_n)\mathfrak{I}^*(\zeta_1, \zeta_2, \chi_3...\chi_n)d\zeta_1, d\zeta_2 \\ & -...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n - \zeta_n)K_n(\zeta_1, \zeta_2, \zeta_3...\zeta_n)\mathfrak{I}^*(\zeta_1, \zeta_2, \zeta_3...\zeta_n)d\zeta_1, d\zeta_2...d\zeta_n\Big|\end{aligned}$$

From Lemma 1.4, Lemma 1.5 and Theorem 1.6 we have;

$$\begin{aligned}\leq \quad & \Big|\mathfrak{I}(\chi_1, \chi_2...\chi_n) - \alpha - \int_0^{\chi_1} V_1(\chi_1 - \zeta_1)K_1(\zeta_1, \chi_2, ...\chi_n)\mathfrak{I}(\zeta_1, \chi_2, ...\chi_n)d\zeta_1 \\ & - \int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2 - \zeta_2)K_2(\zeta_1, \zeta_2, \chi_3...\chi_n)\mathfrak{I}(\zeta_1, \zeta_2, \chi_3...\chi_n)d\zeta_1, d\zeta_2 \\ & -...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n - \zeta_n)K_n(\zeta_1, \zeta_2, \zeta_3...\zeta_n)\mathfrak{I}(\zeta_1, \zeta_2, \zeta_3...\zeta_n)d\zeta_1, d\zeta_2...d\zeta_n\Big| \\ & +\Big|\int_0^{\chi_1} V_1(\chi_1 - \zeta_1)K_1(\zeta_1, \chi_2, ...\chi_n)\mathfrak{I}(\zeta_1, \chi_2, ...\chi_n)d\zeta_1 \\ & + \int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2 - \zeta_2)K_2(\zeta_1, \zeta_2, \chi_3...\chi_n)\mathfrak{I}(\zeta_1, \zeta_2, \chi_3...\chi_n)d\zeta_1, d\zeta_2 \\ & +...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n - \zeta_n)K_n(\zeta_1, \zeta_2, \zeta_3...\zeta_n)\mathfrak{I}(\zeta_1, \zeta_2, \zeta_3...\zeta_n)d\zeta_1, d\zeta_2...d\zeta_n \\ & - \int_0^{\chi_1} V_1(\chi_1 - \zeta_1)K_1(\zeta_1, \chi_2, ...\chi_n)\mathfrak{I}^*(\zeta_1, \chi_2, ...\chi_n)d\zeta_1 \\ & - \int_0^{\chi_1}\int_0^{\chi_2} V_2(\chi_2 - \zeta_2)K_2(\zeta_1, \zeta_2, \chi_3...\chi_n)\mathfrak{I}^*(\zeta_1, \zeta_2, \chi_3...\chi_n)d\zeta_1, d\zeta_2 \\ & -...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} V_n(\chi_n - \zeta_n)K_n(\zeta_1, \zeta_2, \zeta_3...\zeta_n)\mathfrak{I}^*(\zeta_1, \zeta_2, \zeta_3...\zeta_n)d\zeta_1, d\zeta_2...d\zeta_n\Big|\end{aligned}$$

which means that,

$$\begin{aligned}\leq \quad & \psi(\chi_1, \chi_2, ...\chi_n) + \int_0^{\chi_1} K_1(\zeta_1, \chi_2, ...\chi_n)V_1(\chi_1 - \zeta_1).|\mathfrak{I}(\zeta_1, \chi_2, ...\chi_n) - \mathfrak{I}^*(\zeta_1, \chi_2, ...\chi_n)|d\zeta_1 \\ & + \int_0^{\chi_1}\int_0^{\chi_2} K_2(\zeta_1, \zeta_2, \chi_3...\chi_n)V_2(\chi_2 - \zeta_2).|\mathfrak{I}(\zeta_1, \zeta_2, \chi_3...\chi_n) - \mathfrak{I}^*(\zeta_1, \zeta_2, \chi_3...\chi_n)|d\zeta_1 d\zeta_2 \\ & +...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} K_n(\zeta_1, \zeta_2, \zeta_3...\zeta_n)V_n(\chi_n - \zeta_n).|\mathfrak{I}(\zeta_1, \zeta_2, \zeta_3...\zeta_n) - \mathfrak{I}^*(\zeta_1, \zeta_2, \zeta_3...\zeta_n)|d\zeta_1 d\zeta_2...d\zeta_n\end{aligned}$$

By applying Lemma 1.4 and the Gronwall Lemma, we obtain,

$$\begin{aligned}
&\leq \psi(\chi_1,\chi_2,...\chi_n).\exp\Big(\int_0^{\chi_1} K_1(\zeta_1,\chi_2,...\chi_n)V_1(\chi_1-\zeta_1).|\mathfrak{I}(\zeta_1,\chi_2,...\chi_n)-\mathfrak{I}^*(\zeta_1,\chi_2,...\chi_n)|d\zeta_1\\
&\quad+\int_0^{\chi_1}\int_0^{\chi_2} K_2(\zeta_1,\zeta_2,\chi_3...\chi_n)V_2(\chi_2-\zeta_2).|\mathfrak{I}(\zeta_1,\zeta_2,\chi_3...\chi_n)-\mathfrak{I}^*(\zeta_1,\zeta_2,\chi_3...\chi_n)|d\zeta_1 d\zeta_2\\
&\quad+...\int_0^{\chi_1}\int_0^{\chi_2}...\int_0^{\chi_n} K_n(\zeta_1,\zeta_2,\zeta_3...\zeta_n)V_n(\chi_n-\zeta_n).|\mathfrak{I}(\zeta_1,\zeta_2,\zeta_3...\zeta_n)-\mathfrak{I}^*(\zeta_1,\zeta_2,\zeta_3...\zeta_n)|d\zeta_1 d\zeta_2...d\zeta_n)\\
&\leq \psi(\chi_1,\chi_2,...\chi_n).\exp\,(P_{K_1}.c_1.b_1+P_{K_2}.c_2.b_1b_2+...+P_{K_n}c_n.b_1b_2...b_n)\\
&\leq \psi(\chi_1,\chi_2,...\chi_n).G_k.
\end{aligned}$$

where,

$$G_K=\exp\,(P_{K_1}.c_1.b_1+P_{K_2}.c_2.b_1b_2+...+P_{K_n}.c_n.b_1b_2...b_n).$$

This indicates that equation (1) is Ulam–Hyers–Rassias stable. □

4. Illustrative Examples

In this section, we will present several illustrative examples that comply with Theorems 2.1 and 3.1.

Example 4.1. *Consider the integral equation presented below. It's easy to confirm for n=1. Next, let's examine the case when n=2,*

$$\begin{aligned}
\mathfrak{I}(\chi_1,\chi_2) &= \frac{3}{2}+\int_0^{\chi_1}\frac{1}{\chi_1-\zeta_1}\Big(\frac{3}{32}\zeta_1+\frac{2}{21}\chi_2\Big),\frac{1}{5}\mathfrak{I}(\zeta_1,\chi_2)d\zeta_1\\
&\quad+\int_0^{\chi_1}\int_0^{\chi_2}\frac{1}{\chi_2-\zeta_2}\Big(\frac{3}{31}\zeta_1.\frac{3}{52}\zeta_2\Big),\frac{1}{6}\mathfrak{I}(\zeta_1,\zeta_2)d\zeta_2 d\zeta_1,\chi_i\neq\zeta_i\forall\ i=1,2
\end{aligned}\tag{2}$$

We have observed that equation (2) *resembles equation* (1)*. By comparing these two equations, we obtain the following information:*

$$\begin{aligned}
&\alpha=\frac{3}{2},\ V_1(\chi_1-\zeta_1)=\frac{1}{\chi_1-\zeta_1},\ V_2(\chi_2-\zeta_2)=\frac{1}{\chi_2-\zeta_2}\\
&K_1(\zeta_1,\chi_2),\mathfrak{I}(\zeta_1,\chi_2)=\Big(\frac{3}{32}\zeta_1+\frac{2}{21}\chi_2\Big),\frac{1}{5}\mathfrak{I}(\zeta_1,\chi_2)\\
&K_2(\zeta_1,\zeta_2),\mathfrak{I}(\zeta_1,\zeta_2)=\Big(\frac{3}{31}\zeta_1.\frac{3}{52}\zeta_2\Big),\frac{1}{6}\mathfrak{I}(\zeta_1,\zeta_2)
\end{aligned}$$

We get,

$$\begin{aligned}
&|K_1(\zeta_1,\chi_2),\mathfrak{I}_1(\zeta_1,\chi_2)-K_1(\zeta_1,\chi_2),\mathfrak{I}_2(\zeta_1,\chi_2)|+|K_2(\zeta_1,\zeta_2),\mathfrak{I}_1(\zeta_1,\zeta_2)-K_2(\zeta_1,\zeta_2),\mathfrak{I}_2(\zeta_1,\zeta_2)|\\
&=\Big|\Big(\frac{3}{32}\zeta_1+\frac{2}{21}\chi_2\Big),\frac{1}{5}\mathfrak{I}_1(\zeta_1,\chi_2)-\Big(\frac{3}{32}\zeta_1+\frac{2}{21}\chi_2\Big),\frac{1}{5}\mathfrak{I}_2(\zeta_1,\chi_2)\Big|\\
&\quad+\Big|\Big(\frac{3}{31}\zeta_1\frac{3}{52}\zeta_2\Big),\frac{1}{6}\mathfrak{I}_1(\zeta_1,\zeta_2)-\Big(\frac{3}{31}\zeta_1\frac{3}{52}\zeta_2\Big),\frac{1}{6}\mathfrak{I}_2(\zeta_1,\zeta_2)\Big|\\
&\leq\frac{1}{5}|\mathfrak{I}_1(\zeta_1,\chi_2)-\mathfrak{I}_2(\zeta_1,\chi_2)|+\frac{1}{6}|\mathfrak{I}_1(\zeta_1,\zeta_2)-\mathfrak{I}_2(\zeta_1,\zeta_2)|\\
&\leq P_{K_1}|\mathfrak{I}_1(\zeta_1,\chi_2)-\mathfrak{I}_2(\zeta_1,\chi_2)|+P_{K_2}|\mathfrak{I}_1(\zeta_1,\zeta_2)-\mathfrak{I}_2(\zeta_1,\zeta_2)|.
\end{aligned}$$

Moreover,

$$\Big|\Im(\chi_1,\chi_2)-\frac{3}{2}-\Big(\int_0^{\chi_1}\frac{1}{\chi_1-\zeta_1}\Big(\frac{3}{32}\zeta_1+\frac{2}{21}\chi_2\Big),\frac{1}{5}\Im(\zeta_1,\chi_2)d\zeta_1 + \int_0^{\chi_1}\int_0^{\chi_2}\frac{1}{\chi_2-\zeta_2}\Big(\frac{3}{31}\zeta_1.\frac{3}{52}\zeta_2\Big),\frac{1}{6}\Im(\zeta_1,\zeta_2)d\zeta_2 d\zeta_1\Big)\Big|\leq\epsilon$$

Now, if we choose $\Im(\zeta_1,\chi_2)=3\zeta_1+6\chi_2$ *and* $\Im(\zeta_1,\zeta_2)=\zeta_1-2\zeta_2$ *Clearly we have;*

$$\begin{aligned}
&\Big|\Im(\chi_1,\chi_2)-\frac{3}{2}-\Big(\int_0^{\chi_1}\frac{1}{\chi_1-\zeta_1}\Big(\frac{3}{32}\zeta_1+\frac{2}{21}\chi_2\Big),\frac{1}{5}(3\zeta_1+6\chi_2)d\zeta_1\\
&+\int_0^{\chi_1}\int_0^{\chi_2}\frac{1}{\chi_2-\zeta_2}\Big(\frac{3}{31}\zeta_1.\frac{3}{52}\zeta_2\Big),\frac{1}{6}(\zeta_1-2\zeta_2)d\zeta_2 d\zeta_1\Big)\Big|\leq\epsilon\\
= \;&\Big|\Im(\chi_1,\chi_2)-\frac{3}{2}-\Big[\Big(\frac{3}{32}(\chi_1\ln\chi_1-\chi_1)+\frac{2}{21}(\chi_1\ln\chi_1)\Big)\\
&+\frac{1}{1242}\Big(\frac{\chi_1^3\chi_2\ln\chi_2}{3}-\frac{\chi_1^3\chi_2}{3}-\frac{1}{2}\chi_1^2\chi_2^2\ln\chi_2+\frac{1}{2}\chi_1^2\chi_2^2\Big)\Big]\Big|\leq\epsilon=\frac{1}{3}.
\end{aligned}$$

Hence, based on these assumptions, all requirements of Theorem 2.1 are satisfied. Therefore, we can state that equation (1) *possesses Ulam–Hyers stability. Also for any* $\epsilon>0$ *i.e.* $\epsilon=\frac{1}{3}$ *and by considering exact and approximate solutions, we have:*

$$|\Im(\chi_1,\chi_2)-\Im^*(\chi_1,\chi_2)|=\frac{1}{3}.G_k,$$

where, G_k *is also a constant.*

Remark 4.2. *If we take* $\psi(\chi_1,\chi_2)=exp\,(3\chi_1+2\chi_2)$, *the conditions of Theorem 3.1 hold true. Therefore, equation* (1) *is classified as Ulam–Hyers–Rassias stable, which means the difference between the approximate and exact solutions does not exceed a constant multiplied by a specific function.*

5. Conclusion

Higher-dimensional weakly singular Volterra integral equations are frequently employed in mathematical modeling of physical phenomena, such as diffusion processes in heterogeneous materials, where the singularity represents the spatial variability of the material properties. In this paper, we studied two types of stabilities for higher-dimensional weakly singular Volterra integral equations with supporting examples.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Availability of data and materials

No data were used to support this study.

Competing interests

The authors declare that they have no competing interests.

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