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Quantum Weddle's type inequalities for convex functions with their computational analysis and applications

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Abstract. In this paper, quantum Weddle's type inequalities for convex functions have been derived, involving the extension of the classical results in the framework of q-calculus. We present new conditions that describe the behaviour of convex functions using the quantum calculus, which undermines the systematical theory of the related phenomena in theoretical and applied mathematics. We also investigate some aspects of these inequalities, such as scaling and translation, and demonstrate how some of the inequalities are connected to other existing inequalities in the intersection of convex analysis and optimization. Furthermore, numerical and graphical solutions to the inequalities applied to real-life problems are given, along with an illustration of the computation and the connection to the relevant inequalities. Therefore, the findings of the present work are also useful in extending the theory of convex functions.

1. Introduction

The literature on convexity dates back ages, with mathematical techniques that are believed to have originated in Egypt and Babylon. Even as early as the prehistoric era, people have been drawing circles and triangles, while convexity, though may not be as old as numbers, is quite old. The German mathematician Karl Hermann Amandus Schwarz was the one who introduced convex functions at the end of the 19th century, which in his time caused significant changes in the development of the theory of mathematics and its numerous applications. These functions are used in numerous optimization problems in different fields, as will be shown later. Business, civil, and software, helping with issues such as asset allocation and problem-solving with computation. For other works that focus on the background and uses of convexity, readers are encouraged to consult [9, 10, 16, 24].

In the case of inequalities, convexity is defined as the characteristic for which of every pair of points on the graph of a function, the line segment joining them lies above and on the same side of the graph as the function. Hermite-Hadamard inequality was postulated by two French mathematicians namely

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Charles Hermite and Jacques Salomon Hadamard. C. Hermite and J. S. Hadamard did significant work in Number theory, Complex analysis, and much more. To know about their works, see [20, 23]. If a function $F : [\alpha, \beta] \to \mathbb{R}$ is convex, then

$$\frac{F(\alpha) + F(\beta)}{2} \geq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} F(\varkappa) d\varkappa \geq F\left(\frac{\alpha + \beta}{2}\right). \tag{1}$$

In mathematics, there is a correlation between integrals, functions, and convexity. This correlation is called Jensen's Inequality.

The Latin name "quanta" means how much it started in the 17th century. Thus, the meaning of the word quantum can be defined as the general size or amount associated with something. In physics, it is used frequently to denote the smallest quantity of something. In addition to that, from the mathematical perspective, the term quantum means the minimal and discrete amount used to describe a quantum system. It is noteworthy that all the equations relevant to the description of quantum mechanics can be considered mathematical equations. q-calculus emerged as a translation between mathematics and physics.

At the beginning of the 18th century, q-calculus is defined by Euler (1707-1783) and C. G. Jacobi [18]. The q-calculus, more popularly referred to as non-standard calculus, is to the standard calculus as calculus using infinitesimal quantities is to calculus using limits. Thus, q-calculus is a type of calculus that is developed from differentiation and integration operations without using limits. The Jackson derivative or q-derivative, whose introduction is credited to Frank Hilton Jackson, was discussed in [17]. It is the opposite operation of Jackson's q-integration. It is the inverse of Jackson's q integration. Other types of q-derivative can be found in [7]. Jackson integral, or q-integral, was also introduced by F. H. Jackson. q-calculus is used in quantum mechanics, quantum field theory, number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other areas of theoretical physics to study and describe quantum systems and their properties. For more detailed applications of q-calculus, one can see [3, 6, 12, 13, 15, 27, 28]. q-Hermite-Hadamard inequality stated as:

Theorem 1.1. *[4] Let $F : [\alpha, \beta] \to \mathbb{R}$ be a convex function and $q \in (0, 1)$. Then, we have*

$$\frac{qF(\alpha) + F(\beta)}{1 + q} \geq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} F(\varkappa)\ {}_{\alpha}d^{q}\varkappa \geq F\left(\frac{q\alpha + \beta}{1 + q}\right), \tag{2}$$

also

$$\frac{F(\alpha) + qF(\beta)}{1 + q} \geq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} F(\varkappa)\ {}^{\beta}d^{q}\varkappa \geq F\left(\frac{\alpha + q\beta}{1 + q}\right), \tag{3}$$

and for limit $q \to 1^-$ in last two inequalities, we get (1).

This type of inequality is known under the name of q-calculus; The following literature helps solve the problems presented in this paper. The first article connected with q integral inequalities in the frame of q-calculus was published in 2004 by H. Gauchman [14]. H. Budak proves the q-midpoint and trapezoidal type inequalities in [5]. Quantum Simpson-type inequalities for convex functions: A novel investigation was proved by Sabah *et al.* in [19]. Quantum Simpson-like type inequalities for q-differentiable convex functions were established by Meftah et al. in [22]. Afterwards, some other q-analogues of the classical inequalities have been proved, some of which are listed here [1, 2, 4, 11, 25].

Inspired by the above literature, new Weddle's type inequalities in the framework of q-calculus are established by using the newly established identity. To check the validity of newly proved inequalities, 2D and 3D plots were constructed using Mathematica 13.3.1.

The remainder of the paper is structured as follows: Section 2 explains the origins of q-calculus and the basic ideas of convexity. They provide the background information that allows the reader to follow the remaining discussions and arguments of the paper. Section 3 is devoted to the quantum Weddle's type inequalities and their proof, along with an illustration of their importance and use in quantum information theory. In Section 4, we present numerical examples and their graphical analysis. Section 5 presents applications to the quadrature formula and special means of real numbers. In turn, the last section of the paper is Section 6, with the conclusion that encompasses the authors' final statements as well as the major outcomes and implications of the study carried out; additionally, possible directions for further research in the intriguing area of mathematics and quantum processes have been outlined.

2. Basics of q-calculus and convexity

Definition 2.1 (Convex Set). *[8] A set $\Omega \subset \mathbb{R}^n$ is convex, if for any two points $\alpha, \beta \in \Omega$ the entire segment joining α and β lies in Ω. The points in the segment are of the form:*

$$\varpi\alpha + (1-\varpi)\beta, \ \forall\, \varpi \in [0,1]. \tag{4}$$

Definition 2.2 (Convex Function). *[8] Let Ω be a convex subset of a real vector space. Function $F : \Omega \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if*

$$\varpi F(\varkappa) + (1-\varpi)F(y) \geq F(\varpi\varkappa + (1-\varpi)y) \tag{5}$$

for all $\varpi \in [0,1]$ and $\varkappa, y \in \Omega$.

Proposition 2.3 (Jensen's Inequality). *[21] Let F be a convex function defined on the real interval $\Omega \subset \mathbb{R}$. If $\varkappa_1, \varkappa_2, \varkappa_3, \ldots, \varkappa_n \in \Omega$ and $\mu_1, \mu_2, \mu_3, \ldots, \mu_n \geq 0$, then*

$$\sum_{i=1}^{n} \mu_i F(\varkappa_i) \geq F\left(\sum_{i=1}^{n} \mu_i \varkappa_i\right), \tag{6}$$

where $\sum_{i=1}^{n} \mu_i = 1$.

Definition 2.4 (Quantum Number). *Let $q \in (0,1)$. A q-natural number $[n]_q$ is defined as:*

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}.$$

Definition 2.5. *[17] For a real function F, the q-derivative of $F(\varkappa)$ is defined as:*

$$D^q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa)}{\varkappa - q\varkappa}, \tag{7}$$

with $q \in (0,1)$. The q derivative is also known as the Jackson derivative.

Definition 2.6 (Classical Jackson Integral). *[17] The following series expansion defines the classic Jackson integral of a real function:*

$$\int_0^{\varkappa} F(\varpi) d^q \varpi = (1-q)\varkappa \sum_{k=0}^{\infty} q^k F(q^k \varkappa). \tag{8}$$

Definition 2.7. *[26] For a continuous function $F : [\alpha, \beta] \longrightarrow \mathbb{R}$ and $q \in (0,1)$ the q_α-derivative of F at $\varkappa \in [\alpha, \beta]$ is defined as:*

$${}_{\alpha}d^q F(\varkappa) = \frac{F(\alpha + q(\varkappa - \alpha)) - F(\varkappa)}{(q-1)(\varkappa - \alpha)}, \tag{9}$$

for $\varkappa = \alpha$ *we define* ${}_{\alpha}d^{q}F(\alpha) = \lim_{\varkappa\to\alpha}({}_{\alpha}d^{q}F(\varkappa))$ *if it exists and it is finite. Similarly, let* q^{β}*-derivative of* F *at* $\varkappa \in [\alpha,\beta]$ *is defined as [4]:*

$$ {}^{\beta}d^{q}F(\varkappa) = \frac{F(\beta + q(\varkappa-\beta)) - F(\varkappa)}{(q-1)(\varkappa-\beta)}, \tag{10}$$

for $\varkappa = \beta$*, we define* ${}^{\beta}d^{q}F(\beta) = \lim_{\varkappa\to\beta}({}^{\beta}d^{q}F(\varkappa))$*, if it exists and it is finite.*

Definition 2.8. *[26] For a continuous function* $F : [\alpha,\beta] \longrightarrow \mathbb{R}$ *and* $q \in (0,1)$*, then* q_{α}*-definite integral of* F *at* $\varkappa \in [\alpha,\beta]$ *is defined as:*

$$\int_{\alpha}^{\varkappa} F(\varpi)\, {}_{\alpha}d^{q}\varpi = (q-1)(\alpha-\varkappa)\sum_{k=0}^{\infty} q^{k}F(\alpha + (\varkappa-\alpha)\,q^{k}),\ \varkappa \in [\alpha,\beta]. \tag{11}$$

Similarly, the q^{β}*-definite integral of* F *at* $\varkappa \in [\alpha,\beta]$ *is defined as [4]:*

$$\int_{\varkappa}^{\beta} F(\varpi)\, {}^{\beta}d^{q}\varpi = (q-1)(\varkappa-\beta)\sum_{k=0}^{\infty} q^{k}F(\beta + (\varkappa-\beta)q^{k}),\ \varkappa \in [\alpha,\beta]. \tag{12}$$

Remark 2.9. *If we choose* $\alpha = 0$ *in (11), then we get the classical Jackson* q*-integral, defined in (8).*

$$\int_{0}^{1} F(\varpi)\, {}_{0}d^{q}\varpi = (1-q)\sum_{k=0}^{\infty} q^{k}F(q^{k}).$$

Similarly, if $\beta = 1$ *and* $\varkappa = \alpha = 0$ *in (12), then*

$$\int_{0}^{1} F(\varpi)\, {}^{1}d^{q}\varpi = (1-q)\sum_{k=0}^{\infty} q^{k}F(1-q^{k}).$$

3. Main Results

In this section, new identity for q-differentiable functions are established. By using newly established identity Weddle's type inequalities for differentiable convex functions are established via q-calculus.

Lemma 3.1. *Assume that* $F : [\alpha,\beta] \to \mathbb{R}$ *be a* q*-differentiable function and* $F' \in L[\alpha,\beta]$*, then*

$$Q(\alpha,\beta) - \mathbf{I}_{q} = \frac{\beta-\alpha}{36}\sum_{i=1}^{6} L_{i}, \tag{13}$$

where

$$\begin{aligned}
L_{1} &:= \int_{0}^{1}\left(q\varpi - \frac{6}{20}\right) {}_{\alpha}D_{q}F\left(\frac{6-\varpi}{6}\alpha + \frac{\varpi}{6}\beta\right) d_{q}\varpi,\\
L_{2} &:= \int_{0}^{1}\left(q\varpi - \frac{16}{20}\right) {}_{\alpha}D_{q}F\left(\frac{5-\varpi}{6}\alpha + \frac{1+\varpi}{6}\beta\right) d_{q}\varpi,\\
L_{3} &:= \int_{0}^{1}\left(q\varpi - \frac{2}{20}\right) {}_{\alpha}D_{q}F\left(\frac{4-\varpi}{6}\alpha + \frac{2+\varpi}{6}\beta\right) d_{q}\varpi,\\
L_{4} &:= \int_{0}^{1}\left(q\varpi - \frac{18}{20}\right) {}_{\alpha}D_{q}F\left(\frac{3-\varpi}{6}\alpha + \frac{3+\varpi}{6}\beta\right) d_{q}\varpi,
\end{aligned}$$

$$L_5 := \int_0^1 \left(q\varpi - \frac{4}{20}\right) {}_\alpha D_q F\left(\frac{2-\varpi}{6}\alpha + \frac{4+\varpi}{6}\beta\right) d_q\varpi,$$
$$L_6 := \int_0^1 \left(q\varpi - \frac{14}{20}\right) {}_\alpha D_q F\left(\frac{1-\varpi}{6}\alpha + \frac{5+\varpi}{6}\beta\right) d_q\varpi,$$

$$\mathbf{I}_q := \frac{1}{\beta-\alpha}\left[\int_\alpha^{\frac{5\alpha+\beta}{6}} F(\xi)_\alpha\, d_q\xi + \int_{\frac{5\alpha+\beta}{6}}^{\frac{2\alpha+\beta}{3}} F(\xi)_{\frac{5\alpha+\beta}{6}}\, d_q\xi + \int_{\frac{2\alpha+\beta}{3}}^{\frac{\alpha+\beta}{2}} F(\xi)_{\frac{2\alpha+\beta}{3}}\, d_q\xi \right.$$
$$\left. + \int_{\frac{\alpha+\beta}{2}}^{\frac{\alpha+2\beta}{3}} F(\xi)_{\frac{\alpha+\beta}{2}}\, d_q\xi + \int_{\frac{\alpha+2\beta}{3}}^{\frac{\alpha+5\beta}{6}} F(\xi)_{\frac{\alpha+2\beta}{3}}\, d_q\xi + \int_{\frac{\alpha+5\beta}{6}}^{\beta} F(\xi)_{\frac{\alpha+5\beta}{6}}\, d_q\xi\right],$$

and

$$Q(\alpha,\beta) := \frac{1}{20}\left[F(\alpha) + 5F\left(\frac{5\alpha+\beta}{6}\right) + F\left(\frac{2\alpha+\beta}{3}\right) + 6F\left(\frac{\alpha+\beta}{2}\right) + F\left(\frac{\alpha+2\beta}{3}\right) + 5F\left(\frac{\alpha+5\beta}{6}\right) + F(\beta)\right].$$

Proof. Taking into account RHS of Eq. (13)

$$\frac{\beta-\alpha}{36}\sum_{i=1}^{6} L_i.$$

Now, integrate all integrals in the framework of quantum calculus, we have

$$\begin{aligned}
L_1 &= \int_0^1 \left(q\varpi - \frac{6}{20}\right) {}_\alpha D_q F\left(\frac{6-\varpi}{6}\alpha + \frac{\varpi}{6}\beta\right) d_q\varpi \\
&= \frac{6}{\beta-\alpha}\left(q\varpi - \frac{6}{20}\right) F\left(\frac{6-\varpi}{6}\alpha + \frac{\varpi}{6}\beta\right)\Big|_{\varpi=0}^{1} - \frac{6q}{\beta-\alpha}\int_0^1 F\left(\frac{6-q\varpi}{6}\alpha + \frac{q\varpi}{6}\beta\right) {}_\alpha d_q\varpi \\
&= \frac{6}{\beta-\alpha}\left[\left(q - \frac{6}{20}\right) F\left(\frac{5\alpha+\beta}{6}\right) + \frac{6}{20}F(\alpha)\right] - \frac{6q}{\beta-\alpha}\int_0^1 F\left((1-q\varpi)\alpha + q\varpi\left(\frac{5\alpha+\beta}{6}\right)\right) {}_\alpha d_q\varpi \\
&= \frac{6}{\beta-\alpha}\left[\left(q - \frac{6}{20}\right) F\left(\frac{5\alpha+\beta}{6}\right) + \frac{6}{20}F(\alpha)\right] - \frac{6(1-q)}{\beta-\alpha}\sum_{n=0}^{\infty} q^{n+1} F\left(\left(1-q^{n+1}\right)\alpha + q^{n+1}\left(\frac{5\alpha+\beta}{6}\right)\right) \\
&= \frac{6}{\beta-\alpha}\left(q - \frac{6}{20}\right) F\left(\frac{5\alpha+\beta}{6}\right) + \frac{36}{20(\beta-\alpha)}F(\alpha) - \frac{6(1-q)}{\beta-\alpha}\sum_{n=1}^{\infty} q^{n} F\left((1-q^{n})\alpha + q^{n}\left(\frac{5\alpha+\beta}{6}\right)\right) \\
&= \frac{6}{\beta-\alpha}\left(q - \frac{6}{20}\right) F\left(\frac{5\alpha+\beta}{6}\right) + \frac{36}{20(\beta-\alpha)}F(\alpha) \\
&\quad - \frac{6(1-q)}{\beta-\alpha}\left[\sum_{n=0}^{\infty} q^{n} F\left((1-q^{n})\alpha + q^{n}\left(\frac{5\alpha+\beta}{6}\right)\right) - F\left(\frac{5\alpha+\beta}{6}\right)\right] \\
&= \frac{6}{\beta-\alpha}\left[\frac{14}{20}F\left(\frac{5\alpha+\beta}{6}\right) + \frac{6}{20}F(\alpha)\right] - \frac{6(1-q)}{\beta-\alpha}\sum_{n=0}^{\infty} q^{n} F\left((1-q^{n})\alpha + q^{n}\left(\frac{5\alpha+\beta}{6}\right)\right) \\
&= \frac{6}{\beta-\alpha}\left[\frac{14}{20}F\left(\frac{5\alpha+\beta}{6}\right) + \frac{6}{20}F(\alpha)\right] - \frac{36}{(\beta-\alpha)^2}\int_\alpha^{\frac{5\alpha+\beta}{6}} F(\xi)_\alpha\, d_q\xi. \qquad (14)
\end{aligned}$$

Similarly,

$$L_2 = \int_0^1 \left(q\varpi - \frac{16}{20}\right) {}_\alpha D_q F\left(\frac{5-\varpi}{6}\alpha + \frac{1+\varpi}{6}\beta\right) d_q\varpi$$

$$= \frac{6}{\beta-\alpha}\left[\frac{4}{20}F\left(\frac{2\alpha+\beta}{3}\right)+\frac{16}{20}F\left(\frac{5\alpha+\beta}{6}\right)\right]-\frac{36}{(\beta-\alpha)^2}\int_{\frac{5\alpha+\beta}{6}}^{\frac{2\alpha+\beta}{3}}F(\xi)\,{}_{\frac{5\alpha+\beta}{6}}d_q\xi, \tag{15}$$

$$\begin{aligned} L_3 &= \int_0^1\left(q\varpi-\frac{2}{20}\right){}_{\alpha}D_qF\left(\frac{4-\varpi}{6}\alpha+\frac{2+\varpi}{6}\beta\right)d_q\varpi \\ &= \frac{6}{\beta-\alpha}\left[\frac{18}{20}F\left(\frac{\alpha+\beta}{2}\right)+\frac{2}{20}F\left(\frac{2\alpha+\beta}{3}\right)\right]-\frac{36}{(\beta-\alpha)^2}\int_{\frac{2\alpha+\beta}{3}}^{\frac{\alpha+\beta}{2}}F(\xi)\,{}_{\frac{2\alpha+\beta}{3}}d_q\xi, \end{aligned} \tag{16}$$

$$\begin{aligned} L_4 &= \int_0^1\left(q\varpi-\frac{18}{20}\right){}_{\alpha}D_qF\left(\frac{3-\varpi}{6}\alpha+\frac{3+\varpi}{6}\beta\right)d_q\varpi \\ &= \frac{6}{\beta-\alpha}\left[\frac{2}{20}F\left(\frac{\alpha+2\beta}{3}\right)+\frac{18}{20}F\left(\frac{\alpha+\beta}{2}\right)\right]-\frac{36}{(\beta-\alpha)^2}\int_{\frac{\alpha+\beta}{2}}^{\frac{\alpha+2\beta}{3}}F(\xi)\,{}_{\frac{\alpha+\beta}{2}}d_q\xi, \end{aligned} \tag{17}$$

$$\begin{aligned} L_5 &= \int_0^1\left(q\varpi-\frac{4}{20}\right){}_{\alpha}D_qF\left(\frac{2-\varpi}{6}\alpha+\frac{4+\varpi}{6}\beta\right)d_q\varpi \\ &= \frac{6}{\beta-\alpha}\left[\frac{16}{20}F\left(\frac{\alpha+5\beta}{6}\right)+\frac{4}{20}F\left(\frac{\alpha+2\beta}{3}\right)\right]-\frac{36}{(\beta-\alpha)^2}\int_{\frac{\alpha+2\beta}{3}}^{\frac{\alpha+5\beta}{6}}F(\xi)\,{}_{\frac{\alpha+2\beta}{3}}d_q\xi, \end{aligned} \tag{18}$$

and

$$\begin{aligned} L_6 &= \int_0^1\left(q\varpi-\frac{14}{20}\right){}_{\alpha}D_qF\left(\frac{1-\varpi}{6}\alpha+\frac{5+\varpi}{6}\beta\right)d_q\varpi \\ &= \frac{6}{\beta-\alpha}\left[\frac{6}{20}F(\beta)+\frac{14}{20}F\left(\frac{\alpha+5\beta}{6}\right)\right]-\frac{36}{(\beta-\alpha)^2}\int_{\frac{\alpha+5\beta}{6}}^{\beta}F(\xi)\,{}_{\frac{\alpha+5\beta}{6}}d_q\xi. \end{aligned} \tag{19}$$

Adding Eqs. (14) to (19), we obtain

$$\begin{aligned} &\frac{1}{20}\left[F(\alpha)+5F\left(\frac{5\alpha+\beta}{6}\right)+F\left(\frac{2\alpha+\beta}{3}\right)+6F\left(\frac{\alpha+\beta}{2}\right)+F\left(\frac{\alpha+2\beta}{3}\right)+5F\left(\frac{\alpha+5\beta}{6}\right)+F(\beta)\right] \\ &-\frac{1}{\beta-\alpha}\left[\int_{\alpha}^{\frac{5\alpha+\beta}{6}}F(\xi)_{\alpha}\,d_q\xi+\int_{\frac{5\alpha+\beta}{6}}^{\frac{2\alpha+\beta}{3}}F(\xi)_{\frac{5\alpha+\beta}{6}}\,d_q\xi+\int_{\frac{2\alpha+\beta}{3}}^{\frac{\alpha+\beta}{2}}F(\xi)_{\frac{2\alpha+\beta}{3}}\,d_q\xi\right. \\ &\left.+\int_{\frac{\alpha+\beta}{2}}^{\frac{\alpha+2\beta}{3}}F(\xi)_{\frac{\alpha+\beta}{2}}\,d_q\xi+\int_{\frac{\alpha+2\beta}{3}}^{\frac{\alpha+5\beta}{6}}F(\xi)_{\frac{\alpha+2\beta}{3}}\,d_q\xi+\int_{\frac{\alpha+5\beta}{6}}^{\beta}F(\xi)_{\frac{\alpha+5\beta}{6}}\,d_q\xi\right] \\ =\ &\frac{\beta-\alpha}{36}\left[L_1+L_2+L_3+L_4+L_5+L_6\right], \end{aligned} \tag{20}$$

which is the required result. Hence the proof of Lemma 3.1 is completed. □

Theorem 3.2. *Let $F : [\alpha,\beta] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a q-differentiable function on (α,β) with ${}_{\alpha}D^{q}F$ be continuous and integrable on $[\alpha,\beta]$. If $|{}_{\alpha}D^{q}F|$ is convex on $[\alpha,\beta]$, then we have the following inequality:*

$$\begin{aligned} &\left|Q(\alpha,\beta)-\mathbf{I}_q\right| \\ \leq\ &\frac{\beta-\alpha}{216}\Big[\left|{}_{\alpha}D_qF(\alpha)\right|\left(A_1(q)+A_2(q)+A_3(q)+A_4(q)+A_5(q)+A_6(q)\right) \\ &+\left|{}_{\alpha}D_qF(\beta)\right|\left(B_1(q)+B_2(q)+B_3(q)+B_4(q)+B_5(q)+B_6(q)\right)\Big], \end{aligned}$$

where

$$A_1(q) := \int_0^1 \left|q\varpi - \frac{6}{20}\right| (6-\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{6}{20}\left(\frac{(1+q)6-1}{1+q}\right) - \frac{6q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{6}{20}, \\ \frac{12\left(\frac{6}{20}\right)^2+6q}{1+q} - \frac{2\left(\frac{6}{20}\right)^3}{(1+q)(1+q+q^2)} - \frac{q}{1+q+q^2} - \frac{\frac{6}{20}((1+q)6-1)}{1+q}; & \frac{6}{20} \le q < 1 \end{cases}$$

$$A_2(q) := \int_0^1 \left|q\varpi - \frac{16}{20}\right| (5-\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{16}{20}\left(\frac{(1+q)5-1}{1+q}\right) - \frac{5q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{16}{20}, \\ \frac{10\left(\frac{16}{20}\right)^2+5q}{1+q} - \frac{2\left(\frac{16}{20}\right)^3}{(1+q)(1+q+q^2)} - \frac{q}{1+q+q^2} - \frac{\frac{16}{20}((1+q)5-1)}{1+q}; & \frac{16}{20} \le q < 1 \end{cases}$$

$$A_3(q) := \int_0^1 \left|q\varpi - \frac{2}{20}\right| (4-\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{2}{20}\left(\frac{(1+q)4-1}{1+q}\right) - \frac{4q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{2}{20}, \\ \frac{8\left(\frac{2}{20}\right)^2+4q}{1+q} - \frac{2\left(\frac{2}{20}\right)^3}{(1+q)(1+q+q^2)} - \frac{q}{1+q+q^2} - \frac{\frac{2}{20}((1+q)4-1)}{1+q}; & \frac{2}{20} \le q < 1 \end{cases}$$

$$A_4(q) := \int_0^1 \left|q\varpi - \frac{18}{20}\right| (3-\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{18}{20}\left(\frac{(1+q)3-1}{1+q}\right) - \frac{3q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{18}{20}, \\ \frac{6\left(\frac{18}{20}\right)^2+3q}{1+q} - \frac{2\left(\frac{18}{20}\right)^3}{(1+q)(1+q+q^2)} - \frac{q}{1+q+q^2} - \frac{\frac{18}{20}((1+q)3-1)}{1+q}; & \frac{18}{20} \le q < 1 \end{cases}$$

$$A_5(q) := \int_0^1 \left|q\varpi - \frac{4}{20}\right| (2-\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{4}{20}\left(\frac{(1+q)2-1}{1+q}\right) - \frac{2q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{4}{20}, \\ \frac{4\left(\frac{4}{20}\right)^2+2q}{1+q} - \frac{2\left(\frac{4}{20}\right)^3}{(1+q)(1+q+q^2)} - \frac{q}{1+q+q^2} - \frac{\frac{4}{20}((1+q)2-1)}{1+q}; & \frac{4}{20} \le q < 1 \end{cases}$$

$$A_6(q) := \int_0^1 \left|q\varpi - \frac{14}{20}\right| (1-\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{14}{20}\left(\frac{(1+q)-1}{1+q}\right) - \frac{q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{14}{20}, \\ \frac{2\left(\frac{14}{20}\right)^2+q}{1+q} - \frac{2\left(\frac{14}{20}\right)^3}{(1+q)(1+q+q^2)} - \frac{q}{1+q+q^2} - \frac{\frac{14}{20}((1+q)-1)}{1+q}; & \frac{14}{20} \le q < 1, \end{cases}$$

and

$$B_1(q) := \int_0^1 \left|q\varpi - \frac{6}{20}\right| (\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{6}{20}\left(\frac{1}{1+q}\right) - \frac{q}{1+q+q^2}; & 0 < q < \frac{6}{20}, \\ \frac{2\left(\frac{6}{20}\right)^3}{(1+q)(1+q+q^2)} + \frac{q}{1+q+q^2} - \frac{\frac{6}{20}+1}{1+q}; & \frac{6}{20} \le q < 1 \end{cases}$$

$$B_2(q) := \int_0^1 \left|q\varpi - \frac{16}{20}\right| (1+\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{16}{20}\left(\frac{(1+q)+1}{1+q}\right) - \frac{q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{16}{20}, \\ \frac{2\left(\frac{16}{20}\right)^2+q}{1+q} + \frac{2\left(\frac{16}{20}\right)^3}{(1+q)(1+q+q^2)} + \frac{q}{1+q+q^2} - \frac{\frac{16}{20}((1+q)+1)}{1+q}; & \frac{16}{20} \le q < 1 \end{cases}$$

$$B_3(q) := \int_0^1 \left|q\varpi - \frac{2}{20}\right| (2+\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{2}{20}\left(\frac{(1+q)2+1}{1+q}\right) - \frac{2q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{2}{20}, \\ \frac{4\left(\frac{2}{20}\right)^2+2q}{1+q} + \frac{2\left(\frac{2}{20}\right)^3}{(1+q)(1+q+q^2)} + \frac{q}{1+q+q^2} - \frac{\frac{2}{20}((1+q)2+1)}{1+q}; & \frac{2}{20} \le q < 1 \end{cases}$$

$$B_4(q) := \int_0^1 \left|q\varpi - \frac{18}{20}\right| (3+\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{18}{20}\left(\frac{(1+q)3+1}{1+q}\right) - \frac{3q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{18}{20}, \\ \frac{6\left(\frac{18}{20}\right)^2+3q}{1+q} + \frac{2\left(\frac{18}{20}\right)^3}{(1+q)(1+q+q^2)} + \frac{q}{1+q+q^2} - \frac{\frac{18}{20}((1+q)3+1)}{1+q}; & \frac{18}{20} \le q < 1 \end{cases}$$

$$B_5(q) := \int_0^1 \left|q\varpi - \frac{4}{20}\right| (4+\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{4}{20}\left(\frac{(1+q)4+1}{1+q}\right) - \frac{2q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{14}{20}, \\ \frac{8\left(\frac{4}{20}\right)^2+4q}{1+q} + \frac{2\left(\frac{4}{20}\right)^3}{(1+q)(1+q+q^2)} + \frac{q}{1+q+q^2} - \frac{\frac{4}{20}((1+q)4+1)}{1+q}; & \frac{14}{20} \le q < 1 \end{cases}$$

$$B_6(q) := \int_0^1 \left|q\varpi - \frac{14}{20}\right| (5+\varpi)\ {}_\alpha d_q\varpi = \begin{cases} \frac{14}{20}\left(\frac{(1+q)5+1}{1+q}\right) - \frac{5q}{1+q} + \frac{q}{1+q+q^2}; & 0 < q < \frac{4}{20}, \\ \frac{10\left(\frac{14}{20}\right)^2+5q}{1+q} + \frac{2\left(\frac{14}{20}\right)^3}{(1+q)(1+q+q^2)} + \frac{q}{1+q+q^2} - \frac{\frac{14}{20}((1+q)5+1)}{1+q}; & \frac{4}{20} \le q < 1, \end{cases}$$

and $Q(\alpha,\beta)$ is defined as in Lemma 3.1.

Proof. By Lemma 3.1, absolute property and convexity of $|{}_{\alpha}D^{q}F|$, we have

$$
\begin{aligned}
&\left|Q(\alpha,\beta)-\mathbf{I}_q\right| \\
\leq\ & \frac{\beta-\alpha}{36}\left[\int_0^1\left|q\varpi-\frac{6}{20}\right|\left|{}_{\alpha}D_qF\left(\frac{6-\varpi}{6}\alpha+\frac{\varpi}{6}\beta\right)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{16}{20}\right|\left|{}_{\alpha}D_qF\left(\frac{5-\varpi}{6}\alpha+\frac{1+\varpi}{6}\beta\right)\right|d_q\varpi\right.\\
&+\int_0^1\left|q\varpi-\frac{2}{20}\right|\left|{}_{\alpha}D_qF\left(\frac{4-\varpi}{6}\alpha+\frac{2+\varpi}{6}\beta\right)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{18}{20}\right|\left|{}_{\alpha}D_qF\left(\frac{3-\varpi}{6}\alpha+\frac{3+\varpi}{6}\beta\right)\right|d_q\varpi\\
&\left.+\int_0^1\left|q\varpi-\frac{4}{20}\right|\left|{}_{\alpha}D_qF\left(\frac{2-\varpi}{6}\alpha+\frac{4+\varpi}{6}\beta\right)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{14}{20}\right|\left|{}_{\alpha}D_qF\left(\frac{1-\varpi}{6}\alpha+\frac{5+\varpi}{6}\beta\right)\right|d_q\varpi\right]\\
\leq\ & \frac{\beta-\alpha}{36}\left[\int_0^1\left|q\varpi-\frac{6}{20}\right|\left(\frac{6-\varpi}{6}\right)\left|{}_{\alpha}D_qF(\alpha)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{6}{20}\right|\left(\frac{\varpi}{6}\right)\left|{}_{\alpha}D_qF(\beta)\right|d_q\varpi\right.\\
&+\int_0^1\left|q\varpi-\frac{16}{20}\right|\left(\frac{5-\varpi}{6}\right)\left|{}_{\alpha}D_qF(\alpha)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{16}{20}\right|\left(\frac{1+\varpi}{6}\right)\left|{}_{\alpha}D_qF(\beta)\right|d_q\varpi\\
&+\int_0^1\left|q\varpi-\frac{2}{20}\right|\left(\frac{4-\varpi}{6}\right)\left|{}_{\alpha}D_qF(\alpha)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{2}{20}\right|\left(\frac{2+\varpi}{6}\right)\left|{}_{\alpha}D_qF(\beta)\right|d_q\varpi\\
&+\int_0^1\left|q\varpi-\frac{18}{20}\right|\left(\frac{3-\varpi}{6}\right)\left|{}_{\alpha}D_qF(\alpha)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{18}{20}\right|\left(\frac{3+\varpi}{6}\right)\left|{}_{\alpha}D_qF(\beta)\right|d_q\varpi\\
&+\int_0^1\left|q\varpi-\frac{4}{20}\right|\left(\frac{2-\varpi}{6}\right)\left|{}_{\alpha}D_qF(\alpha)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{4}{20}\right|\left(\frac{4+\varpi}{6}\right)\left|{}_{\alpha}D_qF(\beta)\right|d_q\varpi\\
&\left.+\int_0^1\left|q\varpi-\frac{14}{20}\right|\left(\frac{1-\varpi}{6}\right)\left|{}_{\alpha}D_qF(\alpha)\right|d_q\varpi+\int_0^1\left|q\varpi-\frac{14}{20}\right|\left(\frac{5+\varpi}{6}\right)\left|{}_{\alpha}D_qF(\beta)\right|d_q\varpi\right]\\
=\ & \frac{\beta-\alpha}{36}\left[\left|{}_{\alpha}D_qF(\alpha)\right|\int_0^1\left(\left|q\varpi-\frac{6}{20}\right|\left(\frac{6-\varpi}{6}\right)+\left|q\varpi-\frac{16}{20}\right|\left(\frac{5-\varpi}{6}\right)+\left|q\varpi-\frac{2}{20}\right|\left(\frac{4-\varpi}{6}\right)\right.\right.\\
&\left.+\left|q\varpi-\frac{18}{20}\right|\left(\frac{3-\varpi}{6}\right)+\left|q\varpi-\frac{4}{20}\right|\left(\frac{2-\varpi}{6}\right)+\left|q\varpi-\frac{14}{20}\right|\left(\frac{1-\varpi}{6}\right)\right)d_q\varpi\\
&+\left|{}_{\alpha}D_qF(\beta)\right|\int_0^1\left(\left|q\varpi-\frac{6}{20}\right|\left(\frac{\varpi}{6}\right)+\left|q\varpi-\frac{16}{20}\right|\left(\frac{1+\varpi}{6}\right)+\left|q\varpi-\frac{2}{20}\right|\left(\frac{2+\varpi}{6}\right)\right.\\
&\left.\left.+\left|q\varpi-\frac{18}{20}\right|\left(\frac{3+\varpi}{6}\right)+\left|q\varpi-\frac{4}{20}\right|\left(\frac{4+\varpi}{6}\right)+\left|q\varpi-\frac{14}{20}\right|\left(\frac{5+\varpi}{6}\right)\right)d_q\varpi\right]\\
=\ & \frac{\beta-\alpha}{216}\left[\left|{}_{\alpha}D_qF(\alpha)\right|\left(A_1(q)+A_2(q)+A_3(q)+A_4(q)+A_5(q)+A_6(q)\right)\right.\\
&\left.+\left|{}_{\alpha}D_qF(\beta)\right|\left(B_1(q)+B_2(q)+B_3(q)+B_4(q)+B_5(q)+B_6(q)\right)\right].
\end{aligned}
$$

Hence, the proof of Theorem 3.2 is completed. □

Corollary 3.3. *Assume that $|{}_{\alpha}D^{q}F|$ is bounded, such that $|{}_{\alpha}D^{q}F|\leq M$, for $M\geq 0$. Then by Theorem 3.2, we have*

$$
\begin{aligned}
&\left|Q(\alpha,\beta)-\mathbf{I}_q\right| \\
\leq\ & \frac{M(\beta-\alpha)}{216}\left[\left(A_1(q)+A_2(q)+A_3(q)+A_4(q)+A_5(q)+A_6(q)\right)\right.\\
&\left.+\left(B_1(q)+B_2(q)+B_3(q)+B_4(q)+B_5(q)+B_6(q)\right)\right].
\end{aligned}
$$

Corollary 3.4. *If we take limit $q \to 1^-$ in Theorem 3.2, then we get the error bounds of the classical version of Weddle's type inequality for convex functions as follows:*

$$\begin{aligned}&\left|\frac{1}{20}\left[F(\alpha)+5F\left(\frac{\beta+5\alpha}{6}\right)+F\left(\frac{\beta+2\alpha}{3}\right)+6F\left(\frac{\beta+\alpha}{2}\right)\right.\right.\\&\left.\left.+F\left(\frac{2\beta+\alpha}{3}\right)+5F\left(\frac{5\beta+\alpha}{6}\right)+F(\beta)\right]-\mathbf{I}_q\right|\\\leq\ &\frac{13(\beta-\alpha)}{450}\left[|F'(\alpha)|+|F'(\beta)|\right].\end{aligned}$$

Theorem 3.5. *Let $F:[\alpha,\beta]\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a q-differentiable function on (α,β) with ${}_\alpha D^q F$ be continuous and integrable on $[\alpha,\beta]$. If $|{}_\alpha D^q F|^r$ is convex on $[\alpha,\beta]$ with $r\geq 1$, then we have the following inequality:*

$$\begin{aligned}&\left|Q(\alpha,\beta)-\mathbf{I}_q\right|\\\leq\ &\frac{\beta-\alpha}{36}\left(\frac{1}{6}\right)^{\frac{1}{r}}\left[\mathbb{k}_1^{1-\frac{1}{r}}\left(A_1(q)\left|{}_\alpha D_qF(\alpha)\right|^r+B_1(q)\left|{}_\alpha D_qF(\beta)\right|^r\right)^{\frac{1}{r}}+\mathbb{k}_2^{1-\frac{1}{r}}\left(A_2(q)\left|{}_\alpha D_qF(\alpha)\right|^r+B_2(q)\left|{}_\alpha D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\right.\\&+\mathbb{k}_3^{1-\frac{1}{r}}\left(A_3(q)\left|{}_\alpha D_qF(\alpha)\right|^r+B_3(q)\left|{}_\alpha D_qF(\beta)\right|^r\right)^{\frac{1}{r}}+\mathbb{k}_4^{1-\frac{1}{r}}\left(A_4(q)\left|{}_\alpha D_qF(\alpha)\right|^r+B_4(q)\left|{}_\alpha D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\\&\left.+\mathbb{k}_5^{1-\frac{1}{r}}\left(A_5(q)\left|{}_\alpha D_qF(\alpha)\right|^r+B_5(q)\left|{}_\alpha D_qF(\beta)\right|^r\right)^{\frac{1}{r}}+\mathbb{k}_6^{1-\frac{1}{r}}\left(A_6(q)\left|{}_\alpha D_qF(\alpha)\right|^r+B_6(q)\left|{}_\alpha D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\right],\end{aligned}$$

where $A_i's$ and $B_i's$ for all $i=1,2,3,4,5,6$, are provided as in Theorem 3.2, and

$$\mathbb{k}_1:=\int_0^1\left|q\varpi-\frac{6}{20}\right|\,{}_\alpha d_q\varpi=\begin{cases}\frac{6}{20}-\frac{q}{1+q};0<q<\frac{6}{20},\\\left(\frac{6}{20}\right)^2\frac{2}{1+q}+\frac{q}{1+q}-\frac{6}{20};\frac{6}{20}\leq q<1\end{cases}$$

$$\mathbb{k}_2:=\int_0^1\left|q\varpi-\frac{16}{20}\right|\,{}_\alpha d_q\varpi=\begin{cases}\frac{16}{20}-\frac{q}{1+q};0<q<\frac{16}{20},\\\left(\frac{16}{20}\right)^2\frac{2}{1+q}+\frac{q}{1+q}-\frac{16}{20};\frac{16}{20}\leq q<1\end{cases}$$

$$\mathbb{k}_3:=\int_0^1\left|q\varpi-\frac{2}{20}\right|\,{}_\alpha d_q\varpi=\begin{cases}\frac{2}{20}-\frac{q}{1+q};0<q<\frac{2}{20},\\\left(\frac{2}{20}\right)^2\frac{2}{1+q}+\frac{q}{1+q}-\frac{2}{20};\frac{2}{20}\leq q<1\end{cases}$$

$$\mathbb{k}_4:=\int_0^1\left|q\varpi-\frac{18}{20}\right|\,{}_\alpha d_q\varpi=\begin{cases}\frac{18}{20}-\frac{q}{1+q};0<q<\frac{18}{20},\\\left(\frac{18}{20}\right)^2\frac{2}{1+q}+\frac{q}{1+q}-\frac{18}{20};\frac{18}{20}\leq q<1\end{cases}$$

$$\mathbb{k}_5:=\int_0^1\left|q\varpi-\frac{4}{20}\right|\,{}_\alpha d_q\varpi=\begin{cases}\frac{4}{20}-\frac{q}{1+q};0<q<\frac{4}{20},\\\left(\frac{4}{20}\right)^2\frac{2}{1+q}+\frac{q}{1+q}-\frac{4}{20};\frac{4}{20}\leq q<1\end{cases}$$

$$\mathbb{k}_6:=\int_0^1\left|q\varpi-\frac{14}{20}\right|\,{}_\alpha d_q\varpi=\begin{cases}\frac{14}{20}-\frac{q}{1+q};0<q<\frac{14}{20},\\\left(\frac{14}{20}\right)^2\frac{2}{1+q}+\frac{q}{1+q}-\frac{14}{20};\frac{14}{20}\leq q<1.\end{cases}$$

Proof. By Lemma 3.1, power mean integral inequality and using convexity of $|{}_\alpha D^q F|^r$, we have

$$\begin{aligned}&\left|Q(\alpha,\beta)-\mathbf{I}_q\right|\\\leq\ &\frac{\beta-\alpha}{36}\left[\left(\int_0^1\left|q\varpi-\frac{6}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{6}{20}\right|\left|{}_\alpha D_qF\left(\frac{6-\varpi}{6}\alpha+\frac{\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}}\right.\\&+\left(\int_0^1\left|q\varpi-\frac{16}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{16}{20}\right|\left|{}_\alpha D_qF\left(\frac{5-\varpi}{6}\alpha+\frac{1+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}}\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1\left|q\varpi-\frac{2}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{2}{20}\right|\left|{}_\alpha D_qF\left(\frac{4-\varpi}{6}\alpha+\frac{2+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|q\varpi-\frac{18}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{18}{20}\right|\left|{}_\alpha D_qF\left(\frac{3-\varpi}{6}\alpha+\frac{3+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|q\varpi-\frac{4}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{4}{20}\right|\left|{}_\alpha D_qF\left(\frac{2-\varpi}{6}\alpha+\frac{4+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|q\varpi-\frac{14}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{14}{20}\right|\left|{}_\alpha D_qF\left(\frac{1-\varpi}{6}\alpha+\frac{5+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}}\Bigg]\\
\leq\;& \frac{\beta-\alpha}{36}\Bigg[\left(\int_0^1\left|q\varpi-\frac{6}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{6}{20}\right|\left(\frac{6-\varpi}{6}\left|{}_\alpha D_qF(\alpha)\right|^r+\frac{\varpi}{6}\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|q\varpi-\frac{16}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{16}{20}\right|\left(\frac{5-\varpi}{6}\left|{}_\alpha D_qF(\alpha)\right|^r+\frac{1+\varpi}{6}\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|q\varpi-\frac{2}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{2}{20}\right|\left(\frac{4-\varpi}{6}\left|{}_\alpha D_qF(\alpha)\right|^r+\frac{2+\varpi}{6}\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|q\varpi-\frac{18}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{18}{20}\right|\left(\frac{3-\varpi}{6}\left|{}_\alpha D_qF(\alpha)\right|^r+\frac{3+\varpi}{6}\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|q\varpi-\frac{4}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{4}{20}\right|\left(\frac{2-\varpi}{6}\left|{}_\alpha D_qF(\alpha)\right|^r+\frac{4+\varpi}{6}\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|q\varpi-\frac{14}{20}\right|d_q\varpi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{14}{20}\right|\left(\frac{1-\varpi}{6}\left|{}_\alpha D_qF(\alpha)\right|^r+\frac{5+\varpi}{6}\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\Bigg]\\
=\;& \frac{\beta-\alpha}{36}\left(\frac{1}{6}\right)^{\frac{1}{r}}\Bigg[\Bbbk_1^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{6}{20}\right|\left((6-\varpi)\left|{}_\alpha D_qF(\alpha)\right|^r+\varpi\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\\
&+\Bbbk_2^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{16}{20}\right|\left((5-\varpi)\left|{}_\alpha D_qF(\alpha)\right|^r+(1+\varpi)\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)\\
&+\Bbbk_3^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{2}{20}\right|\left((4-\varpi)\left|{}_\alpha D_qF(\alpha)\right|^r+(2+\varpi)\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)\\
&+\Bbbk_4^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{18}{20}\right|\left((3-\varpi)\left|{}_\alpha D_qF(\alpha)\right|^r+(3+\varpi)\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)\\
&+\Bbbk_5^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{4}{20}\right|\left((2-\varpi)\left|{}_\alpha D_qF(\alpha)\right|^r+(4+\varpi)\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)\\
&+\Bbbk_6^{1-\frac{1}{r}}\left(\int_0^1\left|q\varpi-\frac{14}{20}\right|\left((1-\varpi)\left|{}_\alpha D_qF(\alpha)\right|^r+(5+\varpi)\left|{}_\alpha D_qF(\beta)\right|^r\right)d_q\varpi\right)\Bigg].
\end{aligned}
$$

After some simple calculations, we get the required result. Hence, the proof of Theorem 3.5 is completed. □

Corollary 3.6. *Assume that $|{}_\alpha D^qF|$ is bounded, such that $|{}_\alpha D^qF| \leq M$, for $M \geq 0$. Then by Theorem 3.5, we have*

$$
\begin{aligned}
&\left|Q(\alpha,\beta)-\mathbf{I}_q\right|\\
\leq\;& \frac{M(\beta-\alpha)}{36}\left(\frac{1}{6}\right)^{\frac{1}{r}}\Big[\Bbbk_1^{1-\frac{1}{r}}\left(A_1(q)+B_1(q)\right)^{\frac{1}{r}}+\Bbbk_2^{1-\frac{1}{r}}\left(A_2(q)+B_2(q)\right)^{\frac{1}{r}}+\Bbbk_3^{1-\frac{1}{r}}\left(A_3(q)+B_3(q)\right)^{\frac{1}{r}}
\end{aligned}
$$

$$+\Bbbk_4^{1-\frac{1}{r}}\left(A_4(q)+B_4(q)\right)^{\frac{1}{r}}+\Bbbk_5^{1-\frac{1}{r}}\left(A_5(q)+B_5(q)\right)^{\frac{1}{r}}+\Bbbk_6^{1-\frac{1}{r}}\left(A_6(q)+B_6(q)\right)^{\frac{1}{r}}\Bigg].$$

Corollary 3.7. *If we take limit $q\to 1^-$ in Theorem 3.5, then we get the error bounds of the classical version of Weddle's type inequality for convex functions as follows:*

$$\begin{aligned}&\left|\frac{1}{20}\left[F(\alpha)+5F\left(\frac{\beta+5\alpha}{6}\right)+F\left(\frac{\beta+2\alpha}{3}\right)+6F\left(\frac{\beta+\alpha}{2}\right)\right.\right.\\&\left.\left.+F\left(\frac{2\beta+\alpha}{3}\right)+5F\left(\frac{5\beta+\alpha}{6}\right)+F(\beta)\right]-\mathbf{I}_q\right|\\\le\;&\frac{13(\beta-\alpha)}{225}\left[\frac{|F'(\alpha)|^r+|F'(\beta)|^r}{2}\right]^{\frac{1}{r}}.\end{aligned}$$

Theorem 3.8. *Let $F:[\alpha,\beta]\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a q-differentiable function on (α,β) with ${}_{\alpha}D^{q}F$ be continuous and integrable on $[\alpha,\beta]$. If $|{}_{\alpha}D^{q}F|^r$ is convex on $[\alpha,\beta]$ with $r,p>1$ and $\frac{1}{r}+\frac{1}{p}=1$, then we have the following inequality:*

$$\begin{aligned}&\left|Q(\alpha,\beta)-\mathbf{I}_q\right|\\\le\;&\frac{\beta-\alpha}{36}\left(\frac{1}{6}\right)^{\frac{1}{r}}\left[\Bbbk_7^{\frac{1}{p}}\left(\frac{6(1+q)-1}{1+q}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{1}{1+q}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\right.\\&+\Bbbk_8^{\frac{1}{p}}\left(\frac{5(1+q)-1}{1+q}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{(1+q)+1}{1+q}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\\&+\Bbbk_9^{\frac{1}{p}}\left(\frac{4(1+q)-1}{1+q}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{2(1+q)+1}{1+q}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\\&+\Bbbk_{10}^{\frac{1}{pr}}\left(\frac{3(1+q)-1}{1+q}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{3(1+q)+1}{1+q}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\\&+\Bbbk_{11}^{\frac{1}{p}}\left(\frac{2(1+q)-1}{1+q}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{4(1+q)+1}{1+q}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\\&\left.+\Bbbk_{12}^{\frac{1}{p}}\left(\frac{q}{1+q}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{5(1+q)+1}{1+q}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)^{\frac{1}{r}}\right],\end{aligned}$$

where

$$\Bbbk_7:=\int_0^1\left|q\varpi-\frac{6}{20}\right|^p\,{}_{\alpha}d_q\varpi,$$

$$\Bbbk_8:=\int_0^1\left|q\varpi-\frac{16}{20}\right|^p\,{}_{\alpha}d_q\varpi,$$

$$\Bbbk_9:=\int_0^1\left|q\varpi-\frac{2}{20}\right|^p\,{}_{\alpha}d_q\varpi,$$

$$\Bbbk_{10}:=\int_0^1\left|q\varpi-\frac{18}{20}\right|^p\,{}_{\alpha}d_q\varpi,$$

$$\Bbbk_{11}:=\int_0^1\left|q\varpi-\frac{4}{20}\right|^p\,{}_{\alpha}d_q\varpi,$$

$$\Bbbk_{12}:=\int_0^1\left|q\varpi-\frac{14}{20}\right|^p\,{}_{\alpha}d_q\varpi.$$

Proof. By Lemma 3.1, Hölder's inequality and using convexity of $|_{\alpha}D^{q}F|^{r}$, we have

$$
\begin{aligned}
&\left|Q(\alpha,\beta)-\mathbf{I}_q\right| \\
\leq\ & \frac{\beta-\alpha}{36}\left[\left(\int_0^1\left|q\varpi-\frac{6}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{\alpha}D_qF\left(\frac{6-\varpi}{6}\alpha+\frac{\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}}\right. \\
&+\left(\int_0^1\left|q\varpi-\frac{16}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{\alpha}D_qF\left(\frac{5-\varpi}{6}\alpha+\frac{1+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}} \\
&+\left(\int_0^1\left|q\varpi-\frac{2}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{\alpha}D_qF\left(\frac{4-\varpi}{6}\alpha+\frac{2+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}} \\
&+\left(\int_0^1\left|q\varpi-\frac{18}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{\alpha}D_qF\left(\frac{3-\varpi}{6}\alpha+\frac{3+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}} \\
&+\left(\int_0^1\left|q\varpi-\frac{4}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{\alpha}D_qF\left(\frac{2-\varpi}{6}\alpha+\frac{4+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}} \\
&\left.+\left(\int_0^1\left|q\varpi-\frac{14}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{\alpha}D_qF\left(\frac{1-\varpi}{6}\alpha+\frac{5+\varpi}{6}\beta\right)\right|^r d_q\varpi\right)^{\frac{1}{r}}\right] \\
\leq\ & \frac{\beta-\alpha}{36}\left[\left(\int_0^1\left|q\varpi-\frac{6}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{6-\varpi}{6}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{\varpi}{6}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\right. \\
&+\left(\int_0^1\left|q\varpi-\frac{16}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{5-\varpi}{6}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{1+\varpi}{6}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}} \\
&+\left(\int_0^1\left|q\varpi-\frac{2}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{4-\varpi}{6}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{2+\varpi}{6}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}} \\
&+\left(\int_0^1\left|q\varpi-\frac{18}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{3-\varpi}{6}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{3+\varpi}{6}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}} \\
&+\left(\int_0^1\left|q\varpi-\frac{4}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{2-\varpi}{6}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{4+\varpi}{6}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}} \\
&\left.+\left(\int_0^1\left|q\varpi-\frac{14}{20}\right|^p d_q\varpi\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1-\varpi}{6}\left|{}_{\alpha}D_qF(\alpha)\right|^r+\frac{5+\varpi}{6}\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\right] \\
=\ & \frac{\beta-\alpha}{36}\left(\frac{1}{6}\right)^{\frac{1}{r}}\left[\Bbbk_7^{\frac{1}{p}}\left(\int_0^1\left|q\varpi-\frac{6}{20}\right|\left((6-\varpi)\left|{}_{\alpha}D_qF(\alpha)\right|^r+\varpi\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)^{\frac{1}{r}}\right. \\
&+\Bbbk_8^{\frac{1}{p}}\left(\int_0^1\left((5-\varpi)\left|{}_{\alpha}D_qF(\alpha)\right|^r+(1+\varpi)\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right) \\
&+\Bbbk_9^{\frac{1}{p}}\left(\int_0^1\left((4-\varpi)\left|{}_{\alpha}D_qF(\alpha)\right|^r+(2+\varpi)\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right) \\
&+\Bbbk_{10}^{\frac{1}{p}}\left(\int_0^1\left((3-\varpi)\left|{}_{\alpha}D_qF(\alpha)\right|^r+(3+\varpi)\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right) \\
&+\Bbbk_{11}^{\frac{1}{p}}\left(\int_0^1\left((2-\varpi)\left|{}_{\alpha}D_qF(\alpha)\right|^r+(4+\varpi)\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)
\end{aligned}
$$

$$+\mathbb{k}_{12}^{\frac{1}{p}}\left(\int_0^1\left((1-\varpi)\left|{}_{\alpha}D_qF(\alpha)\right|^r+(5+\varpi)\left|{}_{\alpha}D_qF(\beta)\right|^r\right)d_q\varpi\right)\Bigg].$$

After some simple calculations, we get the required result. Hence, the proof of Theorem 3.8 is completed. □

Corollary 3.9. *Assume that $|{}_{\alpha}D^qF|$ is bounded, such that $|{}_{\alpha}D^qF|\leq M$, for $M\geq 0$. Then by Theorem 3.8, we have*

$$\begin{aligned}&\left|Q(\alpha,\beta)-\mathbf{I}_q\right|\\ \leq\;& \frac{M(\beta-\alpha)}{36}\left[\mathbb{k}_7^{\frac{1}{p}}+\mathbb{k}_8^{\frac{1}{p}}+\mathbb{k}_9^{\frac{1}{p}}+\mathbb{k}_{10}^{\frac{1}{pr}}+\mathbb{k}_{11}^{\frac{1}{p}}+\mathbb{k}_{12}^{\frac{1}{p}}\right].\end{aligned}$$

Corollary 3.10. *If we take limit $q\to 1^-$ in Theorem 3.8, then we get the error bounds of the classical version of Weddle's type inequality for convex functions as follows:*

$$\begin{aligned}&\left|\frac{1}{20}\left[F(\alpha)+5F\left(\frac{\beta+5\alpha}{6}\right)+F\left(\frac{\beta+2\alpha}{3}\right)+6F\left(\frac{\beta+\alpha}{2}\right)\right.\right.\\ &\left.\left.+F\left(\frac{2\beta+\alpha}{3}\right)+5F\left(\frac{5\beta+\alpha}{6}\right)+F(\beta)\right]-\mathbf{I}_q\right|\\ \leq\;&(\beta-\alpha)\left(\frac{2^{-1-2p}\times 15^{-1-p}\left(1+2^{1+p}+3^{1+p}+7^{1+p}+8^{1+p}+9^{1+p}\right)}{1+p}\right)^{\frac{1}{p}}\left(\frac{|F'(\alpha)|^r+\left|F'(\beta)\right|^r}{2}\right)^{\frac{1}{r}}.\end{aligned}$$

4. Numerical Example and Graphical Analysis

In this section, numerical example with 2D and 3D graphical behavior of newly established inequalities is presented.

Example 4.1. *Assume that a function $F:[\alpha,\beta]=[0,1]\to\mathbb{R}$ is defined as $F(\varkappa)=\varkappa^6$. Then F is q-differentiable*

$${}_{\alpha}D_qF(x)={}_{0}D_qF(x)=[6]_q\,x^5,$$

$$\begin{aligned}{}_{\alpha}D_qF(\alpha)&={}_0D_qF(0)=0,\\ {}_{\alpha}D_qF(\beta)&={}_0D_qF(1)=[6]_q\,.\end{aligned}$$

Take LHS of Theorem 3.2, we have

$$\left|Q(\alpha,\beta)-\mathbf{I}_q\right|,$$

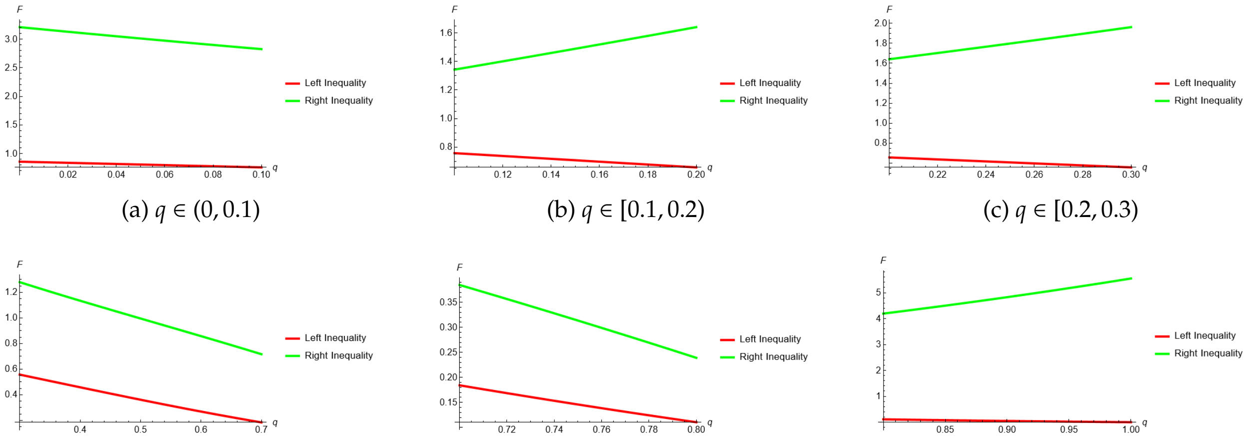
where

$$Q(\alpha,\beta)=\frac{1}{20}\left[F(\alpha)+5F\left(\frac{5\alpha+\beta}{6}\right)+F\left(\frac{2\alpha+\beta}{3}\right)+6F\left(\frac{\alpha+\beta}{2}\right)+F\left(\frac{\alpha+2\beta}{3}\right)+5F\left(\frac{\alpha+5\beta}{6}\right)+F(\beta)\right].$$

Now, take RHS of Theorem 3.2, we have

$$\begin{aligned}&\frac{\beta-\alpha}{216}\left[\left|{}_{\alpha}D_qF(\alpha)\right|(A_1(q)+A_2(q)+A_3(q)+A_4(q)+A_5(q)+A_6(q))\right.\\ &\left.+\left|{}_{\alpha}D_qF(\beta)\right|(B_1(q)+B_2(q)+B_3(q)+B_4(q)+B_5(q)+B_6(q))\right].\end{aligned}$$

Remark 4.2. *Clearly from Figure 2 and Table 1, when q approaches to zero we get better results as compared to classical Weddle's rule for $q=1$.*



(a) $q \in (0, 0.1)$ (b) $q \in [0.1, 0.2)$ (c) $q \in [0.2, 0.3)$

(d) $q \in (0.3, 0.7]$ (e) $q \in [0.7, 0.8)$ (f) $q \in [0.8, 1)$

Figure 1: Comparative analysis of the inequalities of Theorem 3.2, of Example 4.1: 2D plots for different ranges of q between 0 and 1.

q	Left inequality	Right inequality
0.01	0.84712400	1.089180
0.14	0.71712500	1.459240
0.28	0.57722200	1.894720
0.42	0.43846500	2.387400
0.56	0.30485700	2.958900
0.70	0.18404800	3.363297
0.84	0.08410390	4.435550
0.98	0.00878387	5.395010

Table 1: Comparative analysis of inequalities for Theorem 3.2, when $q \in (0, 1)$.

5. Applications

Weddle's type inequalities are extremely useful and efficient in numerical integration and examining the particular means of real numbers. In the case of quadrature formulas, they provide important error bounds of the approximations of integrals and the approximations' accuracies. Besides, in the context of special means, they provide us with complex and binding relations, which are basic in many mathematical studies. Thus, the type inequalities presented by Weddle help expand various contexts of theoretical and applied mathematics.

5.1. *Quadrature Formula*

In this subsection, we discuss the possible use of the inequalities of Weddle's type is derived above in quadrature formulas. Therefore, using Weddle's inequalities increase the efficiency and accuracy of the quadrature formula in numerical integration. Thus, adding these inequalities will help to optimize the computations in the quadrature formula and the properties of functions such as convexity. Weddle's type inequalities systematically arrange the selection of sampling points and weights used in the quadrature formula to enhance precision while estimating the definite integrals. It is also the application of the theory contained in Weddle's inequalities and a pertinent example of its application in numerical analysis and computational mathematics.

We prove the following propositions for the error bounds of Weddle's rule.

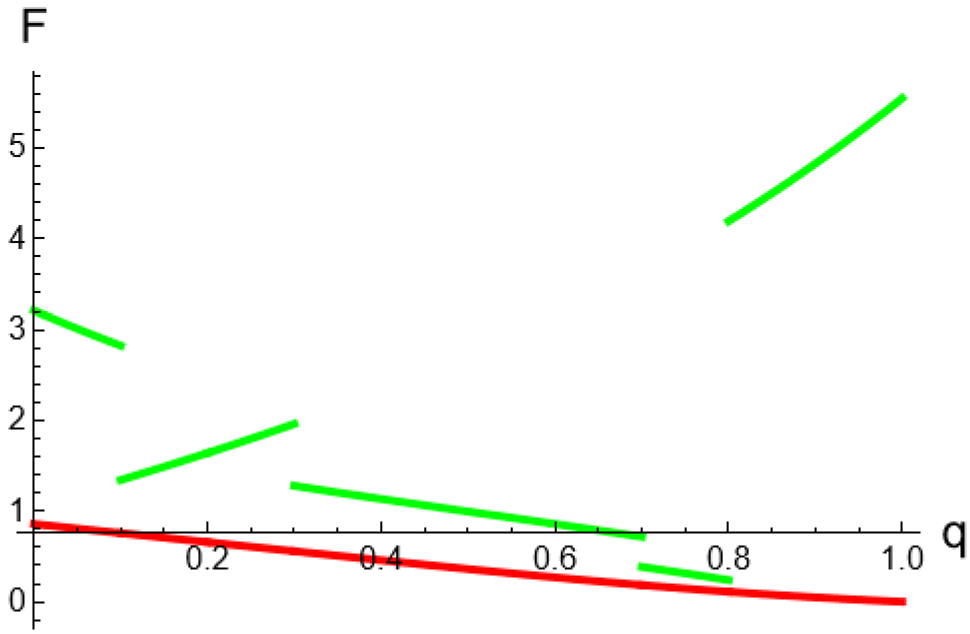


Figure 2: Combine piecewise 2D plot when q lies between 0 and 1.

Proposition 5.1. *Suppose that $F : [\alpha, \beta] \to \mathbb{R}$ be a q-differentiable function on (α, β), then we have*

$$(\beta - \alpha)\mathbf{I}_q = S_n(I_n, F) + \mathbb{R}_n(I_n, F),$$

where

$$S_n(I_n, F) := \frac{1}{20}\sum_{i=0}^{n-1}(\xi_{i+1} - \xi_i)\left[F(\xi_i) + 5F(\xi_i + h) + F(\xi_i + 2h) + 6F(\xi_i + 3h) + F(\xi_i + 4h) + 5F(\xi_i + 5h) + F(\xi_{i+1})\right],$$

and I_n is the partition given by

$$I_n : \alpha = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_{n-1} < \xi_n = \beta,$$

$$h_i := \frac{\xi_{i+1} - \xi_i}{6},\ i = 0, 1, 2, \ldots, n-1.$$

The remainder term satisfies the following condition:

$$|\mathbb{R}_n(I_n, F)| \leq \sum_{i=0}^{n-1}\frac{(\xi_{i+1} - \xi_i)^2}{216}\Big[\left|{}_{\alpha}D_qF(\xi_i)\right|(A_1(q) + A_2(q) + A_3(q) + A_4(q) + A_5(q) + A_6(q)) + \left|{}_{\alpha}D_qF(\xi_{i+1})\right|(B_1(q) + B_2(q) + B_3(q) + B_4(q) + B_5(q) + B_6(q))\Big],$$

for all i =0,1,2, . . . , n-1, where $A_i's$ and $B_i's$ $(i = 1, 2, 3, 4, 5, 6)$ are defined as in Theorem 3.2.

Proof. Let us set things according to the situation, by Theorem 3.2, we have

$$\alpha = \xi_i, \beta = \xi_{i+1}, h_i = \frac{\xi_{i+1} - \xi_i}{6},$$

where $i = 0, 1, 2, \ldots, n-1$. Then, we get the following estimation:

$$\begin{aligned}
&\Big|\frac{1}{20}\left(\xi_{i+1}-\xi_i\right)\left[F\left(\xi_i\right)+5F\left(\xi_i+h\right)+F\left(\xi_i+2h\right)\right.\\
&\left.+6F\left(\xi_i+3h\right)+F\left(\xi_i+4h\right)+5F\left(\xi_i+5h\right)+F\left(\xi_{i+1}\right)\right]-(\beta-\alpha)\mathbf{I}_q\Big|\\
\leq\;&\frac{\left(\xi_{i+1}-\xi_i\right)^2}{216}\Big[\left|{}_\alpha D_q F\left(\xi_i\right)\right|\left(A_1(q)+A_2(q)+A_3(q)+A_4(q)+A_5(q)+A_6(q)\right)\\
&+\left|{}_\alpha D_q F\left(\xi_{i+1}\right)\right|\left(B_1(q)+B_2(q)+B_3(q)+B_4(q)+B_5(q)+B_6(q)\right)\Big],
\end{aligned}$$

for all $i = 0, 1, 2, \ldots, n-1$. After summing and applying triangular inequality, we obtain

$$\begin{aligned}
&\Big|\frac{1}{20}\sum_{i=0}^{n-1}\left(\xi_{i+1}-\xi_i\right)\left[F\left(\xi_i\right)+5F\left(\xi_i+h\right)+F\left(\xi_i+2h\right)\right.\\
&\left.+6F\left(\xi_i+3h\right)+F\left(\xi_i+4h\right)+5F\left(\xi_i+5h\right)+F\left(\xi_{i+1}\right)\right]-(\beta-\alpha)\mathbf{I}_q\Big|\\
\leq\;&\sum_{i=0}^{n-1}\frac{\left(\xi_{i+1}-\xi_i\right)^2}{216}\Big[\left|{}_\alpha D_q F\left(\xi_i\right)\right|\left(A_1(q)+A_2(q)+A_3(q)+A_4(q)+A_5(q)+A_6(q)\right)\\
&+\left|{}_\alpha D_q F\left(\xi_{i+1}\right)\right|\left(B_1(q)+B_2(q)+B_3(q)+B_4(q)+B_5(q)+B_6(q)\right)\Big],
\end{aligned}$$

which is the required proof of the Proposition 5.1. □

Remark 5.2. *Other propositions can be similarly verified for different cases and are left to the interested readers.*

5.2. Special Means

Special means, including the arithmetic mean, geometric mean, harmonic mean, and others, are fundamental in various mathematical applications. Weddle's type inequalities offer insights into the relationships and bounds between these means:

- **Arithmetic Mean:**

$$A\left(\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_n\right) := \frac{\varpi_1+\varpi_2+\varpi_3+\ldots+\varpi_n}{n}.$$

- **Generalized Logarithmic Mean:**

$$L_r\left(\varpi_1, \varpi_2\right) := \left[\frac{\varpi_2^{r+1}-\varpi_1^{r+1}}{(r+1)(\varpi_2-\varpi_1)}\right]^{\frac{1}{r}},\ \varpi_1 \neq \varpi_2, \forall\, \varpi_1, \varpi_2 \in \mathbb{R}^+;\ r \in \mathbb{Z}\setminus\{-1, 0\}.$$

Proposition 5.3. *Assume that $\alpha, \beta \in \mathbb{R}, 0 < \alpha < \beta$ and $n \in \mathbb{N}, n \geq 2$. Then, we have*

$$\begin{aligned}
&\Big|\frac{1}{10}A\left(\alpha^n, \beta^n\right)+\frac{1}{4}A^n\left(\alpha, \alpha, \alpha, \alpha, \alpha, \beta\right)+\frac{1}{20}A^n\left(\alpha, \alpha, \alpha, \beta\right)+\frac{3}{10}A^n\left(\alpha, \beta\right)\\
&+\frac{1}{20}A^n\left(\alpha, \beta, \beta, \beta\right)+\frac{1}{4}A^n\left(\alpha, \beta, \beta, \beta, \beta, \beta\right)-L_n^n\left(\alpha, \beta\right)\Big|\\
\leq\;& n\frac{(\beta-\alpha)}{216}A\left(\alpha^{n-1}, \beta^{n-1}\right).
\end{aligned}\tag{21}$$

Proof. By applying Theorem 3.2 for $f(x) = x^n$ with $q \to 1^-$, we can obtain the result (21). □

Remark 5.4. *Other propositions can be similarly verified for different cases and left to interested readers.*

6. Conclusion

In conclusion, this study provides a significant contribution to the theory of convex functions by extending classical Weddle's type inequalities into the quantum calculus framework. The derivation of new conditions and the algebraic classification of quantum transformations enhance the understanding of the interplay between convexity and quantum operations, laying the groundwork for further exploration in both theoretical and applied contexts. The numerical and graphical analysis presented offer practical insights into real-world applications, while the connections drawn to existing inequalities underscore the broader relevance of the results within convex analysis and optimization. This work not only deepens the foundational theory of convex functions but also paves the way for future advancements in mathematical modeling and problem-solving in diverse scientific fields.

References

[1] M. A. Ali, H. Budak, M. Feckan and S. Khan, *A new version of q-Hermite-Hadamard's midpoint and trapezoid type inequalities for convex functions*, Math. Slovaca, **73**(2) (2023), 369–386.

[2] M. A. Ali, H. Budak and Z. Zhang, *A new extension of quantum Simpson's and quantum Newton's type inequalities for quantum differentiable convex functions*, Math. Methods Appl. Sci., **45**(4) (2022), 1845–1863.

[3] N. Alp, M. Z. Sarikaya, M. Kunt and I. Iscan, *q-Hermite-Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, J. King Saud Univ. Sci., **30**(2) (2018), 193–203.

[4] S. Bermudo, P. Kórus and J. E. Nápoles Valdés, *On q-Hermite-Hadamard inequalities for general convex functions*, Acta Math. Hungar., **162**(2020), 364–374.

[5] H. Budak, *Some trapezoid and midpoint type inequalities for newly defined quantum integrals*, Proyecciones, **40**(1) (2021), 199–215.

[6] S. Charsreechai, M. A. Ali, M. A. Ashraf, T. Sitthiwirattaham, S. Etemad, M. D. L. Sen and S. Rezapour, *On new estimates of q-Hermite-Hadamard inequalities with applications in Quantum Calculus*, Axioms, **12**(1) (2023), Art. 49.

[7] K. S. Chung, W. S. Nam and S. T. Kang, *New q-derivative and q-logarithm*, Int. J. Theor. Phys., **33** (1994), 2019–2029.

[8] R. J. Dwilewicz, *A short history of convexity*, Differ. Geom. Dyn. Syst., (2009).

[9] T. S. Du and T. C. Zhou, *On the fractional double integral inclusion relations having exponential kernel's via interval-valued co-ordinated convex mappings*, Chaos Solitons Fractals, **156**(2022), Art. 19.

[10] T. S. Du, C. Y. Luo and Z. J. Cao, *On the Bullen-type inequalities via generalized fractional integral and their applications*, Fractals, **29**(7) (2021), Art. 20.

[11] T. Ernst, *The history of q-calculus and new method*, Dep. Math., Uppsala Univ., (2000).

[12] T. Ernst, *A new method of q-calculus*, Acta Univ. Upsaliensis, (2002).

[13] T. Ernst, *A comprehensive treatment of q-calculus*, Springer, (2012).

[14] H. Gauchman, *Integral inequalities in q-calculus*, Comput. Math. Appl., **47**(2004), 281–300.

[15] H. Exton, *q-Hypergeometric functions and applications*, Halstead Press, New York, (1983).

[16] J. W. Green, *Recent applications of convex functions*, Amer. Math. Monthly, **61**(7P1) (1954), 449–454.

[17] F. H. Jackson, *On q-functions and certain difference operators*, Earth Environ. Sci. Trans. R. Soc. Edinb., **46**(2) (1909), 253–281.

[18] V. Kac and P. Cheung, *Quantum Calculus*, Springer, (2002).

[19] S. Iftikhar, M. U. Awan and H. Budak, *Exploring quantum Simpson-type inequalities for convex functions: A novel investigation*, Symmetry, **15**(7) (2023), Art. 1312.

[20] S. Mandelbrojt and L. Schwartz, *Jacques Hadamard (1865–1963)*, Bull. Amer. Math. Soc., **71**(1) (1971), 107–129.

[21] A. Matković and J. Pečarić, *A variant of Jensen inequality for convex functions of several variables*, J. Math. Inequal., **1**(1) (2007), 45–51.

[22] B. Meftah, A. Souahi and M. Merad, *Quantum Simpson-like type inequalities for q-differentiable convex functions*, J. Anal., **32** (2024), 1331–1365.

[23] E. F. Robertson, *Jacques Hadamard (1865–1963)*, MacTutor Hist. Math. Arch., Univ. St Andrews.

[24] P. K. Sarpong and J. A. Prah, *Applications of convex functions and concave functions*, Dama Int. J. Res., (2018).

[25] J. Soontharanon, M. A. Ali, H. Budak, K. Nonlaopon and Z. Abdullah, *Simpson's and Newton's type inequalities for* (α, m)*-convex functions via quantum calculus*, Symmetry, **14**(4) (2022), Art. 736.

[26] J. Tariboon and S. K. Ntouyas, *Quantum calculus on finite interval and applications to impulsive difference equations*, Adv. Differ. Equ., (2013), Art. 19.

[27] M. Toseef, Z. Zhang and M. A. Ali, *On q-Hermite-Hadamard-Mercer and Midpoint-Mercer inequalities for general convex functions with their computational analysis*, Int. J. Geom. Methods Mod. Phys., DOI: 10.1142/S0219887824503195.

[28] Y. Zhang, T. S. Du, H. Wang and Y. J. Shen, *Different types of quantum integral inequalities via* (α, m)*-convexity*, J. Inequal. Appl., **264** (2018), Art. 1–24.