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Ulam type stabilities for (k,ψ)-fractional order quadratic integral equations

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Abstract. The primary objective of this paper is to comprehensively establish the Hyers-Ulam, generalized Hyers-Ulam, Hyers-Ulam-Rassias, and generalized Hyers–Ulam stability properties for (k, ψ)-fractional order quadratic integral equations. These stability concepts play a crucial role in understanding the persistence, resilience, and response of solutions to small perturbations, providing insight into the behavior and reliability of solutions within complex systems. Our analysis is grounded in the application of Gronwall's lemma, an essential tool that we adapt specifically for the unique structure of (k, ψ)-fractional order systems. This approach not only enriches the theoretical understanding of stability within these fractional order integral equations but also broadens the applicability of Gronwall's lemma to new contexts.To substantiate our findings, we provide two illustrative examples, carefully chosen to demonstrate the stability characteristics across a range of conditions and parameter settings. These examples are further supplemented by detailed 2D and 3D graphical representations generated in MATLAB, allowing for a visual examination of stability and solution dynamics. These visualizations not only complement the analytical proofs but also offer an intuitive validation of the stability results. Through this integrated approach the paper aims to present a well-rounded and thorough assessment of stability in (k, ψ)-fractional order quadratic integral equations.

1. Introduction And Preliminaries

Fractional calculus is an extension of classical calculus that investigates derivatives and integrals of any (non-integer) order. Fractional integro-differential equations extend traditional calculus by allowing non-integer derivatives and integrals, making them suitable for capturing memory and heredity effects in dynamic systems. These equations are effective in capturing non-Newtonian properties in blood flow dynamics and accurately predicting viscoelastic behavior in polymer rheology ([1], [3]). Fractional equations are useful in electrodynamics for complex media, control theory for system stability, and signal processing for filtering complicated frequency patterns. They provide a flexible framework that outperforms standard models [20].

Fractional calculus allows for the definition of non-integer (fractional) order derivatives and integrals, such as half-derivatives, quarter-integrals, and so on, whereas classical calculus focuses on integer-order

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derivatives and integrals, such as first, second, and third derivatives. It allows for more flexible explanations of complex processes by expanding the standard idea of differentiation and integration to non-integer orders ([11], [12]. It has a wide range of applications, including signal processing, control theory, engineering, and physics. The theory's ability to model memory-rich, non-local systems in the real world makes it both sophisticated and versatile.

S. M. Ulam [13] introduced the concept of stability in 1940, and it has since become a key topic in mathematical analysis. The goal was to find a functional equation with an approximation as close to the exact solution as possible. D. H. Hyers (see [14]) addressed this issue for Banach spaces in 1945, focusing on the additive Cauchy equation $f(x + y) = f(x) + f(y)$, resulting in the Hyers–Ulam stability. Th. M. Rassias [18] expanded on Hyers' work in the 1970s, introducing additional concepts and building the Hyers-Ulam-Rassias stability. Rassias' contributions [19] greatly enlarged the area of stability research, resulting in various generalizations and applications see([15], [21], [22]).

Nonlinear integral equations are a foundation of mathematical analysis, with substantial research in a variety of contexts due to their theoretical importance and practical applications. Nonlinear quadratic integral equations are commonly employed to describe complicated events in domains like as physics, biology, and engineering (see [8], [9]). These equations, however complex due to their nonlinearity, give a solid framework for expressing intricate real-world interactions. The Gronwall lemma, highlighted in works like as([2], [7], and [10]), is an invaluable tool in this domain, especially for determining the stability of integral equations. This lemma enables a systematic approach to bounded solutions, as well as a clear method to achieving stability in nonlinear systems. Its efficiency in ensuring Hyers-Ulam stability highlights its critical role in the study of nonlinear integral equations.

The investigation of Hyers–Ulam stability, generalized Hyers–Ulam stability, Hyers–Ulam–Rassias and generalized Hyers–Ulam–Rassias stability using Gronwall lemma for(k,ψ)-fractional order qudratic integral equation remains relatively unexplored. The previously mentioned works encouraged us to continue on the same path. As a result, we have considered the following (k,ψ)-fractional order quadratic integral equation:

$$
\begin{aligned}
\chi(\xi) \quad = \quad & \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð,\chi(ð))dð \\
\times \quad & \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð,\chi(ð))dð \\
+ \quad & \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i^*}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð,\chi(ð))dð, \xi \in \Delta,
\end{aligned}
\tag{1}
$$

where $\Delta = [\xi_0, \xi_f], 0 \le \xi_0 < \xi_f < \infty$, $\psi, \phi, \psi_i : \Delta \to \mathbb{R}$ are increasing functions with $\psi'(\xi), \phi'(\xi), \psi_i'(\xi) \neq 0 \; \forall \xi \in \Delta$, $i = 1, \ldots, m \in \mathbb{N}$ and $\zeta_1 \in (0, k_1), \zeta_2 \in (0, k_2), \beta_i \in (0, k_i^*)$, $i = 1, \ldots m$. Also $\aleph_1$, $\aleph_2$ *and* $\Re_i$; $i = 1, \ldots m$ are continuous functions.

These equations have substantial applications in a variety of domains [5]. In population dynamics, for example, they describe biological population development by altering the rate of increase in response to both current population levels and previous events, representing genetic or memory influences. They are used in physics and control theory to represent systems with nonlinear feedback and time-varying processes, such as viscoelastic materials or diffusion processes. Quadratic fractional integral equations are useful for modeling systems like nonlinear filtering and memory-influenced signal recovery. For further information, we recommend ([6], [16]).

We now offer the formal definition of the aforementioned stabilities for the nonlinear (k,ψ)-fractional quadratic integral equation.

To verify the Hyers-Ulam, generalized Hyes–Ulam, Hyers-Ulam-Rassias and generalized Hyers-Ulam-Rassias stability of the (k,ψ)-fractional order quadratic integral equation, the definitions given are quite important.

Definition 1.1. *In term of* ε, *equation* (1) *is considered Hyers–Ulam stable if a constant* ¥ > 0 *exists such that, for every solution* $\chi(\xi) \in C(\Delta, \mathbb{R})$ *of the inequality*

$$\Bigg| \chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi(ð))dð \times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi(ð))dð - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð, \chi(ð))dð \Bigg| \leq \varepsilon, \tag{2}$$

For equation (1), *there is a solution* $\chi^\diamond(\xi) \in C(\Delta, \mathbb{R})$ *such that:*

$$|\chi(\xi) - \chi^\diamond(\xi)| \leq ¥.\varepsilon, \forall \xi \in \Delta .$$

Definition 1.2. *The equation* (1) *is generalized Hyers-Ulam stable if there exists* $\mathfrak{M} \in C(\mathbb{R}+, \mathbb{R}+)$, $\mathfrak{M}(0) = 0$, *such that for each solution* $\chi(\xi) \in C(\Delta, \mathbb{R})$ *of the inequality*

$$\Bigg| \chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi(ð))dð \times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi(ð))dð - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð, \chi(ð))dð \Bigg| \leq \varepsilon, \tag{3}$$

There exists a solution $\chi^\diamond(\xi) \in C(\Delta, \mathbb{R})$ *of the equation* (1) *with*

$$|\chi(\xi) - \chi^\diamond(\xi)| \leq \mathfrak{M}(\varepsilon)¥, \forall \xi \in \Delta = [\xi_0, \xi] .$$

Definition 1.3. *In term of* $\Omega(\xi)$, *equation* (1) *is considered Hyers–Ulam–Rassias stable if a constant* ¥ > 0 *exists such that, for every solution* $\chi(\xi) \in C(\Delta, \mathbb{R})$ *of the inequality*

$$\Bigg| \chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi(ð))dð \times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi(ð))dð - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð, \chi(ð))dð \Bigg| \leq \Omega(\xi), \tag{4}$$

For equation (1), *there is a solution* $\chi^\diamond(\xi) \in C(\Delta, \mathbb{R})$ *such that:*

$$|\chi(\xi) - \chi^\diamond(\xi)| \leq ¥.\Omega(\xi), \forall \xi \in \Delta .$$

Definition 1.4. *The equation* (1) *is generalized Hyers-Ulam-Rassias stable with* $\Omega(\xi)$, *if there exists* ¥ > 0 *such that* *for each solution* $\chi(\xi) \in C(\Delta, \mathbb{R})$ *of the inequality*

$$\Bigg|\chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(\eth)(\psi(\xi) - \psi(\eth))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(\eth, \chi(\eth))d\eth \times \int_{\xi_0}^{\xi} \frac{\phi'(\eth)(\phi(\xi) - \phi(\eth))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(\eth, \chi(\eth))d\eth - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(\eth)(\psi_i(\xi) - \psi_i(\eth))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(\eth, \chi(\eth))d\eth\Bigg| \leq \Omega(\xi).\varepsilon, \tag{5}$$

there exists a solution $\chi^\diamond(\xi) \in C(\Delta, \mathbb{R})$ *with:*

$$|\chi(\xi) - \chi^\diamond(\xi)| \leq ¥.\Omega(\xi).\varepsilon, \forall \xi \in \Delta = [\xi_0, \xi] .$$

In this article, the (k,ψ)-fractional order quadratic integral equation (NFQIE) is systematically examined. A focus on significant stability results such as the Hyers-Ulam and Hyers-Ulam-Rassias stabilities, which are supported by Gronwall's lemma, the main findings are described in depth in sections 2, 3, 4, and 5. Within the fractional framework, each component offers a foundation for comprehending the stability and robustness of NFQIE. Illustrative examples and graphs are included in Section 6 to graphically illustrate the precision, efficacy, and relevance of these findings. These visual aids validate the suggested approaches within the specified domain by elucidating the theory and connecting the mathematical findings with practical interpretations.

2. Hyers–Ulam Stability Of (k,ψ)-fractional order quadratic integral equation

Hyers–Ulam stability result for (k,ψ)-fractional order quadratic integral equation (1) in Theorem 2.1 in the finite interval case is presented in this section.

Theorem 2.1. *We suppose that:*
(i) $\aleph_1, \aleph_2$ *and* $\Re_i;\ i = 1, \ldots m$ *are continuous functions.*
(ii) $\exists$ *positive constant* Ł$_1$, Ł$_2$ *and* Ł$_i$ *for* $i = 1, \cdots m$ *such that:*

$$|\aleph_1(\xi, \eth_1) - \aleph_1(\xi, \eth_2)| \leq Ł_1|\eth_1 - \eth_2|, \forall \eth_1, \eth_2 \in \mathbb{R},$$

$$|\aleph_2(\xi, \eth_1) - \aleph_2(\xi, \eth_2)| \leq Ł_2|\eth_1 - \eth_2|, \forall \eth_1, \eth_2 \in \mathbb{R},$$

and

$$|\Re_i(\xi, \eth_1) - \Re_i(\xi, \eth_2)| \leq Ł_i|\eth_1 - \eth_2|, \forall \eth_1, \eth_2 \in \mathbb{R}.$$

then,
(a) The equation (1) *has in* $C(\Delta, \mathbb{R})$ *a unique solution* $\chi^\diamond$*;*
(b) For each $\varepsilon > 0$ *, if* $\chi \in C(\Delta, \mathbb{R})$ *is a solution of inequility*

$$\Bigg|\chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(\eth)(\psi(\xi) - \psi(\eth))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(\eth, \chi(\eth))d\eth \times \int_{\xi_0}^{\xi} \frac{\phi'(\eth)(\phi(\xi) - \phi(\eth))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(\eth, \chi(\eth))d\eth - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(\eth)(\psi_i(\xi) - \psi_i(\eth))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(\eth, \chi(\eth))d\eth\Bigg| \leq \varepsilon, \forall\ \xi \in \Delta,$$

Then
$|\chi(\xi) - \chi^{\diamond}(\xi)| \leq ¥.\varepsilon, \ \forall \ \xi \in \Delta,$
where; $¥ = \exp(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i)$ *for* $i = 1, \cdots m$
i.e. equation (1) *is Hyers–Ulam stable.*

Proof. (a) It is simple to prove the theorem (see [2]). As a result, we will not provide the proof of this theorem,
(b) Consider
$|\chi(\xi) - \chi^{\diamond}(\xi)|$

$$\begin{aligned}
|\chi(\xi) - \chi^{\diamond}(\xi)| &= \Bigg|\chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi^{\diamond}(ð))dð \\
&\quad \times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi^{\diamond}(ð))dð \\
&\quad - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \mathfrak{R}_i(ð, \chi^{\diamond}(ð))dð\Bigg| \\
&\leq \Bigg|\chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi(ð))dð \\
&\quad \times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi(ð))dð \\
&\quad - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \mathfrak{R}_i(ð, \chi(ð))dð\Bigg| \\
&\quad + \Bigg|\int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi(ð))dð \\
&\quad \times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi(ð))dð \\
&\quad + \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \mathfrak{R}_i(ð, \chi(ð))dð \\
&\quad - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi^{\diamond}(ð))dð \\
&\quad \times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi^{\diamond}(ð))dð \\
&\quad - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \mathfrak{R}_i(ð, \chi^{\diamond}(ð))dð\Bigg| \\
&\leq \varepsilon + \int_{\xi_0}^{\xi} \left|\frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)}\right|.|\aleph_1(ð, \chi(ð)) - \aleph_1(ð, \chi^{\diamond}(ð))|dð \\
&\quad \times \int_{\xi_0}^{\xi} \left|\frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)}\right|.|\aleph_2(ð, \chi(ð)) - \aleph_2(ð, \chi^{\diamond}(ð))|dð
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\int_{\xi_0}^{\xi}\left|\frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)}\right|.|\Re_i(ð,\chi(ð))-\Re_i(ð,\chi^\diamond(ð))|dð\\
\leq\ & \varepsilon+\int_{\xi_0}^{\xi}\left|\frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)}\right|.Ł_1|\chi(ð)-\chi^\diamond(ð)|dð\\
&\times\int_{\xi_0}^{\xi}\left|\frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)}\right|.Ł_2|\chi(ð)-\chi^\diamond(ð)|dð\\
&+\sum_{i=1}^{m}\int_{\xi_0}^{\xi}\left|\frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)}\right|.Ł_i|\chi(ð)-\chi^\diamond(ð)|dð
\end{aligned}
$$

Using Gronwall lemma (see [10])

$$
\begin{aligned}
\leq\ & \varepsilon.\exp\Bigg(Ł_1\int_{\xi_0}^{\xi}\left|\frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)}\right|.|\chi(ð)-\chi^\diamond(ð)|dð\\
&\times Ł_2\int_{\xi_0}^{\xi}\left|\frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)}\right|.|\chi(ð)-\chi^\diamond(ð)|dð\\
&+\sum_{i=1}^{m}Ł_i\int_{\xi_0}^{\xi}\left|\frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)}\right|.|\chi(ð)-\chi^\diamond(ð)|dð\Bigg)\\
\leq\ & \varepsilon.\exp(Ł_1.\hbar_1\times Ł_2.\hbar_2+Ł_i.\hbar_i)
\end{aligned}
$$

$$|\chi(\xi)-\chi^\diamond(\xi)|\leq\varepsilon.¥\quad\forall\xi\in\Delta,$$

where;

$$¥=\exp(Ł_1.\hbar_1\times Ł_2.\hbar_2+Ł_i.\hbar_i).$$

which means that equation (1) is Hyers–Ulam stable. □

3. Generalized Hyers–Ulam stability of (k,ψ)-fractional order quadratic integral equation

Here, we provide the Hyers–Ulam stability conclusion for the (k,ψ)-fractional order quadratic integral equation (1) in the infinite interval case of Theorem 3.1.

Theorem 3.1. *We suppose that:*
(i) $\aleph_1,\aleph_2$ *and* $\Re_i;\ i=1,\dots m$ *are continuous functions.*
(ii) $\exists$ *positive constant* $Ł_1,Ł_2$ *and* $Ł_i$ *such that:*

$$|\aleph_1(\xi,ð_1)-\aleph_1(\xi,ð_2)|\leq Ł_1|ð_1-ð_2|,\forall ð_1,ð_2\in\mathbb{R},$$

$$|\aleph_2(\xi,ð_1)-\aleph_2(\xi,ð_2)|\leq Ł_2|ð_1-ð_2|,\forall ð_1,ð_2\in\mathbb{R},$$

and

$$|\Re_i(\xi,ð_1)-\Re_i(\xi,ð_2)|\leq Ł_i|ð_1-ð_2|,\forall ð_1,ð_2\in\mathbb{R}.$$

then,
(a) The equation (1) *has in* $C([\xi_0,\infty),\mathbb{R})$ *a unique solution* $\chi^\diamond$*;*

(b) For each $\varepsilon > 0$, if $\chi \in C([\xi_0, \infty), \mathbb{R})$ is a solution of inequality

$$\left|\chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(\eth)(\psi(\xi)-\psi(\eth))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(\eth, \chi(\eth))d\eth \right.$$
$$\times \int_{\xi_0}^{\xi} \frac{\phi'(\eth)(\phi(\xi)-\phi(\eth))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(\eth, \chi(\eth))d\eth$$
$$\left. - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(\eth)(\psi_i(\xi)-\psi_i(\eth))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \mathfrak{R}_i(\eth, \chi(\eth))d\eth \right| \leq \varepsilon, \forall\ \xi \in [\xi_0, \infty),$$

Then
$|\chi(\xi) - \chi^{\diamond}(\xi)| \leq ¥.\mathfrak{M}(\varepsilon),\ \forall\ \xi \in [\xi_0, \infty)$,
where;
$¥ = \exp(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i)$ *i.e. equation* (1) *is generalized Hyers–Ulam stable.*

Proof. (a) The theorem is easy to prove. Therefore, the proof of this theorem will not be presented here (see [4],[2]).
(b) Consider

$$|\chi(\xi) - \chi^{\diamond}(\xi)|$$
$$\leq \varepsilon + \int_{\xi_0}^{\xi} \left| \frac{\psi'(\eth)(\psi(\xi)-\psi(\eth))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \right| .|\aleph_1(\eth, \chi(\eth)) - \aleph_1(\eth, \chi^{\diamond}(\eth))|d\eth$$
$$\times \int_{\xi_0}^{\xi} \left| \frac{\phi'(\eth)(\phi(\xi)-\phi(\eth))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \right| .|\aleph_2(\eth, \chi(\eth)) - \aleph_2(\eth, \chi^{\diamond}(\eth))|d\eth$$
$$+ \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \left| \frac{\psi_i'(\eth)(\psi_i(\xi)-\psi_i(\eth))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \right| .|\mathfrak{R}_i(\eth, \chi(\eth)) - \mathfrak{R}_i(\eth, \chi^{\diamond}(\eth))|d\eth$$

Using Gronwall lemma (see [10])

$$\leq \varepsilon. \exp\left(Ł_1 \int_{\xi_0}^{\infty} \left| \frac{\psi'(\eth)(\psi(\xi)-\psi(\eth))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \right| .|\chi(\eth) - \chi^{\diamond}(\eth)|d\eth \right.$$
$$\times Ł_2 \int_{\xi_0}^{\infty} \left| \frac{\phi'(\eth)(\phi(\xi)-\phi(\eth))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \right| .|\chi(\eth) - \chi^{\diamond}(\eth)|d\eth$$
$$\left. + \sum_{i=1}^{m} Ł_i \int_{\xi_0}^{\infty} \left| \frac{\psi_i'(\eth)(\psi_i(\xi)-\psi_i(\eth))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \right| .|\chi(\eth) - \chi^{\diamond}(\eth)|d\eth \right)$$
$$\leq \varepsilon. \exp(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i),$$

$$|\chi(\xi) - \chi^{\diamond}(\xi)| \leq \varepsilon.¥ \quad \forall \xi \in [\xi_0, \infty)$$

where;

$$¥ = \exp(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i).$$

which means that equation (1) is generalized Hyers–Ulam stable.
□

4. Hyers–Ulam–Rassias stability Of (k,ψ)-fractional order quadratic integral equation

This section gives the Hyers–Ulam stability result for the (k,ψ)-fractional order quadratic integral equation (1) in Theorem 4.1 in the finite interval case.

Theorem 4.1. *We suppose that:*
(i) $\aleph_1, \aleph_2$ *and* $\Re_i;\ i = 1, \ldots m$ *are continuous functions.*
(ii) $\exists$ *positive constant* Ł$_1$, Ł$_2$ *and* Ł$_i$ *such that:*

$$|\aleph_1(\xi, ð_1) - \aleph_1(\xi, ð_2)| \leq Ł_1|ð_1 - ð_2|, \forall ð_1, ð_2 \in \mathbb{R},$$

$$|\aleph_2(\xi, ð_1) - \aleph_2(\xi, ð_2)| \leq Ł_2|ð_1 - ð_2|, \forall ð_1, ð_2 \in \mathbb{R},$$

and

$$|\Re_i(\xi, ð_1) - \Re_i(\xi, ð_2)| \leq Ł_i|ð_1 - ð_2|, \forall ð1, ð_2 \in \mathbb{R}.$$

(iii) $\Omega(\xi)$ *is a non–decreasing function.*
then,
(a) The equation (1) *has in* $C(\Delta, \mathbb{R})$ *a unique solution* $\chi^\diamond$;
(b) If $\chi \in C(\Delta, \mathbb{R})$ *is such that*

$$\Bigg|\chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi(ð))dð$$
$$\times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi(ð))dð$$
$$- \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð, \chi(ð))dð\Bigg| \leq \Omega(\xi),\ \forall\ \xi \in \Delta,$$

Then

$$|\chi(\xi) - \chi^\diamond(\xi)| \leq ¥.\Omega(\xi)\ \forall\ \xi \in \Delta,$$

where;
$¥ = \exp\,(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i)$.
i.e. equation (1) *is Hyers–Ulam–Rassias stable.*

Proof. (a) It is well known result (see [2]).
(b)Consider
$|\chi(\xi) - \chi^\diamond(\xi)|$

$$|\chi(\xi) - \chi^\diamond(\xi)| = \Bigg|\chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi^\diamond(ð))dð$$
$$\times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi^\diamond(ð))dð$$
$$- \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð, \chi^\diamond(ð))dð\Bigg|$$

$$\leq \left| \chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð,\chi(ð))dð \right.$$
$$\times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð,\chi(ð))dð$$
$$\left. -\sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð,\chi(ð))dð \right|$$
$$+\left| \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð,\chi(ð))dð \right.$$
$$\times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð,\chi(ð))dð$$
$$+\sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð,\chi(ð))dð$$
$$-\int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð,\chi^{\diamond}(ð))dð$$
$$\times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð,\chi^{\diamond}(ð))dð$$
$$\left. -\sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð,\chi^{\diamond}(ð))dð \right|$$
$$\leq \Omega(\xi) + \int_{\xi_0}^{\xi} \left| \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \right| .|\aleph_1(ð,\chi(ð)) - \aleph_1(ð,\chi^{\diamond}(ð))|dð$$
$$\times \int_{\xi_0}^{\xi} \left| \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \right| .|\aleph_2(ð,\chi(ð)) - \aleph_2(ð,\chi^{\diamond}(ð))|dð$$
$$+\sum_{i=1}^{m} \int_{\xi_0}^{\xi} \left| \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \right| .|\Re_i(ð,\chi(ð)) - \Re_i(ð,\chi^{\diamond}(ð))|dð$$
$$\leq \Omega(\xi) + \int_{\xi_0}^{\xi} \left| \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \right| .Ł_1|\chi(ð)) - \chi^{\diamond}(ð))|dð$$
$$\times \int_{\xi_0}^{\xi} \left| \frac{\phi'(ð)(\phi(\xi)-\phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \right| .Ł_2|\chi(ð) - \chi^{\diamond}(ð)|dð$$
$$+\sum_{i=1}^{m} \int_{\xi_0}^{\xi} \left| \frac{\psi_i'(ð)(\psi_i(\xi)-\psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \right| .Ł_i|\chi(ð)) - \chi^{\diamond}(ð))|dð$$

Using Gronwall lemma (see [10])

$$\leq \Omega(\xi).\exp\left(Ł_1 \int_{\xi_0}^{\xi} \left| \frac{\psi'(ð)(\psi(\xi)-\psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \right| .|\chi(ð) - \chi^{\diamond}(ð)|dð \right.$$

$$\times Ł_2 \int_{\xi_0}^{\xi} \left| \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2 \Gamma_{k_2}(\zeta_1)} \right| . |\chi(ð) - \chi^{\diamond}(ð)| dð$$

$$+ \sum_{i=1}^{m} Ł_i \int_{\xi_0}^{\xi} \left| \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^* \Gamma_{k_i^*}(\beta_i)} \right| . |\chi(ð) - \chi^{\diamond}(ð)| dð \Bigg)$$

$$\leq \Omega(\xi). \exp\left(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i\right)$$

$$|\chi(\xi) - \chi^{\diamond}(\xi)| \leq \Omega(\xi).¥ \quad \forall \xi \in \Delta$$

where;

$$¥ = \exp\left(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i\right), \forall i = 1, \cdots m.$$

which means that equation (1) is Hyers-Ulam-Rassias stable. □

5. Generalized Hyers-Ulam-Rassias Stability Of (k,ψ)-fractional order quadratic integral equation

This section gives the Hyers–Ulam stability result for the (k,ψ)-fractional order quadratic integral equation (1) in Theorem 5.1 in the infinite interval case.

Theorem 5.1. *We suppose that:*
(i) $\aleph_1, \aleph_2$ *and* $\Re_i$; $i = 1, \ldots m$ *are continuous functions.*
(ii) $\exists$ *positive constant* $Ł_1, Ł_2$ *and* $Ł_i$ *such that:*

$$|\aleph_1(\xi, ð_1) - \aleph_1(\xi, ð_2)| \leq Ł_1 |ð_1 - ð_2|, \forall ð1, ð_2 \in \mathbb{R},$$

$$|\aleph_2(\xi, ð_1) - \aleph_2(\xi, ð_2)| \leq Ł_2 |ð_1 - ð_2|, \forall ð1, ð_2 \in \mathbb{R},$$

and

$$|\Re_i(\xi, ð_1) - \Re_i(\xi, ð_2)| \leq Ł_i |ð_1 - ð_2|, \forall ð1, ð_2 \in \mathbb{R}.$$

(iii) $\Omega(\xi)$ *is a non–decreasing function.*
then,
(a) The equation (1) *has in* $C([\xi, \infty), \mathbb{R})$ *a unique solution* $\chi^{\diamond}$*;*
(b) If $\chi \in C([\xi, \infty), \mathbb{R})$ *is such that*

$$\Bigg| \chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1 \Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi(ð)) dð$$

$$\times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2 \Gamma_{k_2}(\zeta_1)} \quad \aleph_2(ð, \chi(ð)) dð$$

$$- \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^* \Gamma_{k_i^*}(\beta_i)} \Re_i(ð, \chi(ð)) dð \Bigg| \leq \Omega(\xi).\varepsilon, \ \forall \ \xi \in \Delta,$$

Then

$$|\chi(\xi) - \chi^{\diamond}(\xi)| \leq ¥.\varepsilon \Omega(\xi) \ \forall \ \xi \in \Delta = [\xi_0, \infty),$$

where;

$$¥ = \exp\left(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i\right).$$

i.e. equation (1) *is generalized Hyers–Ulam–Rassias stable.*

Proof. (a) It is a well known result (see [10]).
(b)Consider

$$\begin{aligned}
&|\chi(\xi) - \chi^{\diamond}(\xi)| \\
&= \Bigg|\chi(\xi) - \int_{\xi_0}^{\xi} \frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)} \aleph_1(ð, \chi^{\diamond}(ð))dð \\
&\quad \times \int_{\xi_0}^{\xi} \frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)} \aleph_2(ð, \chi^{\diamond}(ð))dð \\
&\quad - \sum_{i=1}^{m} \int_{\xi_0}^{\xi} \frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)} \Re_i(ð, \chi^{\diamond}(ð))dð\Bigg|
\end{aligned}$$

Using Gronwall lemma (see [10])

$$\begin{aligned}
&\leq \Omega(\xi)\varepsilon.\exp\Bigg(Ł_1 \int_{\xi_0}^{\infty} \left|\frac{\psi'(ð)(\psi(\xi) - \psi(ð))^{\frac{\zeta_1}{k_1}-1}}{k_1\Gamma_{k_1}(\zeta_1)}\right|.|\chi(ð) - \chi^{\diamond}(ð)|dð \\
&\times Ł_2 \int_{\xi_0}^{\infty} \left|\frac{\phi'(ð)(\phi(\xi) - \phi(ð))^{\frac{\zeta_2}{k_2}-1}}{k_2\Gamma_{k_2}(\zeta_1)}\right|.|\chi(ð) - \chi^{\diamond}(ð)|dð \\
&+ \sum_{i=1}^{m} Ł_i \int_{\xi_0}^{\infty} \left|\frac{\psi_i'(ð)(\psi_i(\xi) - \psi_i(ð))^{\frac{\beta_i}{k_i}-1}}{k_i^*\Gamma_{k_i^*}(\beta_i)}\right|.|\chi(ð) - \chi^{\diamond}(ð)|dð\Bigg) \\
&\leq \Omega(\xi)\varepsilon.\exp\left(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i\right) \\
&\leq \Omega(\xi)\varepsilon.¥ \quad \forall \xi \in [\xi_0, \infty),
\end{aligned}$$

$$|\chi(\xi) - \chi^{\diamond}(\xi)| \leq \Omega(\xi)\varepsilon.¥. \quad \forall \xi \in [\xi_0, \infty),$$

where;

$$¥ = \exp\left(Ł_1.\hbar_1 \times Ł_2.\hbar_2 + Ł_i.\hbar_i\right), \forall i = 1, \cdots m.$$

which means that equation (1) is generalized Hyers-Ulam-Rassias stable. □

6. Illustrative Examples

In this section, we present two examples to illustrate the results obtained earlier.

Example 6.1. *Let us consider the following NQFIE:*

$$\begin{aligned}
\chi(\xi) = &\int_0^{\xi} \frac{(\xi - ð)^{\frac{1}{5}-1}}{5\Gamma_5(1)} \frac{1}{20}.\chi(ð)dð \times \int_0^{\xi} \frac{(\xi - ð)^{\frac{2}{5}-1}}{5\Gamma_5(2)} \frac{1}{30}.\chi(ð)dð \\
&+ \sum_{i=1}^{2} \int_0^{\xi} \frac{(\xi - ð)^{\frac{\beta_i}{k_{i^*}}-1}}{k_{i^*}\Gamma_{k_i^*}(\beta_i)} \frac{1}{40i}.\chi(ð)dð, \ \xi \in \Delta = [0, 1],
\end{aligned} \tag{6}$$

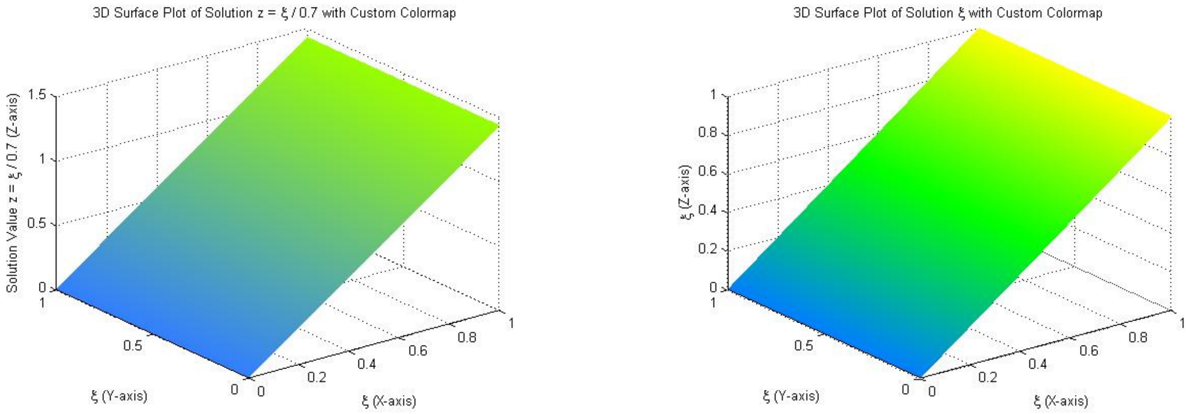


Figure 1: In the interval $[0, 1]$, the 3D charts offer a thorough and complete perspective of the approximate solution ξ and the exact solution ξ_0 for equation (6). The plots successfully show how each solution changes in connection to two independent variables by displaying ξ_0 and ξ in three dimensions. This enables a close investigation of their relationship and interaction. This method provides a clear grasp of how well the approximate solution ξ matches the exact solution ξ_0 over the interval by highlighting both the similarities and differences in their trajectories. Finally, by offering important insights on the precision and efficacy of the approximation over the designated range, this visualization improves our understanding of the dynamic relationship between the exact and approximate solution.

The form of equation (6) *is* (1)*, as we have observed. Thus, we have obtained the following information by comparing equation* (1) *with equation* (6)*:*

$$\psi(\xi) = \phi(\xi) = \psi_i^*(\xi) = \xi,$$

$$\psi(ð) = \phi(ð) = \psi_i^*(ð) = ð \textit{ where } i = 1, 2.$$

and their respective derivatives are:

$$\psi'(\xi), \phi'(\xi), \psi_i^{*'}(\xi) \neq 0,$$

$$\psi'(ð) = \phi'(ð) = \psi_i^{*'}(ð) = 1,$$

also $k_1 = k_2 = k_{i^*} = 5, \beta_1 = 3, \beta_2 = 4, \zeta_1 = 1, \zeta_2 = 2.$

we now check all the conditions of theorem 2.1, for any $\xi \in [0, 1]$ *and for* $ð_1, ð_2 \in \mathbb{R}$ *we have,*

$$|\aleph_1(\xi, ð_1) - \aleph_1(\xi, ð_2)| \leq \frac{1}{20}|ð_1 - ð_2|, \ \forall ð_1, ð_2 \in \mathbb{R},$$

$$|\aleph_2(\xi, ð_1) - \aleph_2(\xi, ð_2)| \leq \frac{1}{30}|ð_1 - ð_2|, \ \forall ð_1, ð_2 \in \mathbb{R},$$

and

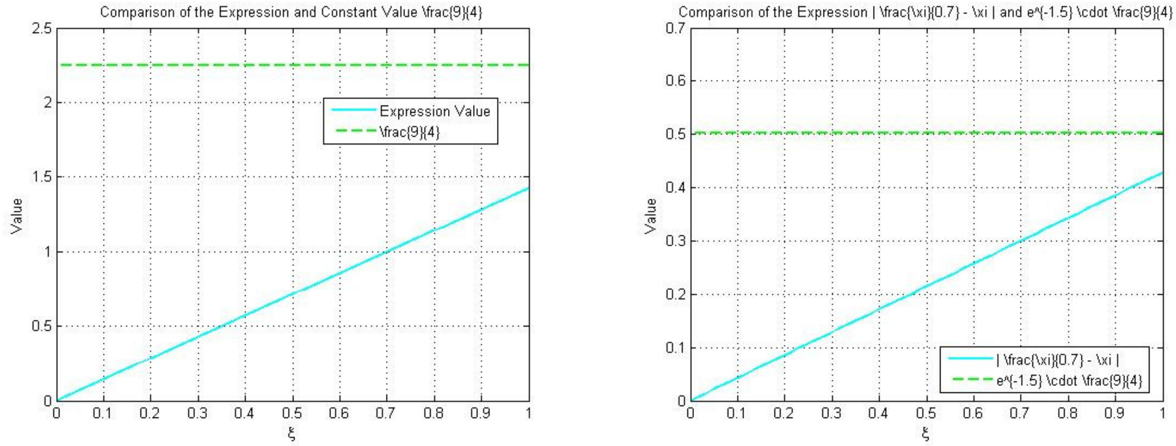
$$|\mathfrak{R}_i(\xi, ð_1) - \mathfrak{R}_i(\xi, ð_2)| \leq \frac{1}{40i}|ð_1 - ð_2|, \ \forall ð_1, ð_2 \in \mathbb{R}.$$

Moreover if we choose approximate solution $\chi(\xi) = \frac{\xi}{0.7}$*, it follows,*

$$= \left| \frac{\xi}{0.7} - \left(\int_0^{\xi} \frac{(\xi - ð)^{\frac{1}{5}-1}}{5\Gamma_5(1)} \frac{1}{20} . \chi(ð) dð \times \int_0^{\xi} \frac{(\xi - ð)^{\frac{2}{5}-1}}{5\Gamma_5(2)} \frac{1}{30} . \chi(ð) dð \right.\right.$$

$$\left.\left. + \sum_{i=1}^{2} \int_0^{\xi} \frac{(\xi - ð)^{\frac{\beta_i}{k_{i^*}}-1}}{k_{i^*}\Gamma_{k_i^*}(\beta_i)} \frac{1}{40i} . \chi(ð) dð \right) \right| \leq \varepsilon, \forall \xi \in \Delta = [0, 1],$$

$$= \left| \frac{\xi}{0.7} - \frac{1}{70}\left(\frac{-5\xi^{\frac{6}{5}}}{6} + 5\xi^{\frac{6}{5}} \right) \times \frac{1}{105}\left(\frac{-5\xi^{\frac{7}{5}}}{7} + \frac{5\xi^{\frac{7}{5}}}{2} \right) - \frac{1}{280}\left(\frac{-5\xi^{\frac{8}{5}}}{7} + \frac{5\xi^{\frac{8}{5}}}{3} \right) - \frac{1}{1680}\left(\frac{-5\xi^{\frac{9}{5}}}{9} + \frac{5\xi^{\frac{9}{5}}}{4} \right) \right| \leq \frac{9}{4} = \varepsilon, \forall \xi \in \Delta = [0,1]. \tag{7}$$



(a) (b)

Figure 2: On the left side, 2D plot of the inequality (7) illustrating a comparison between the two functions: $\left| \frac{\xi}{0.7} - \frac{1}{70}\left(\frac{-5\xi^{\frac{6}{5}}}{6} + 5\xi^{\frac{6}{5}} \right) \times \frac{1}{105}\left(\frac{-5\xi^{\frac{7}{5}}}{7} + \frac{5\xi^{\frac{7}{5}}}{2} \right) - \frac{1}{280}\left(\frac{-5\xi^{\frac{8}{5}}}{7} + \frac{5\xi^{\frac{8}{5}}}{3} \right) - \frac{1}{1680}\left(\frac{-5\xi^{\frac{9}{5}}}{9} + \frac{5\xi^{\frac{9}{5}}}{4} \right) \right|$ denoted by a blue line, and $\varepsilon = \frac{9}{4}$, denoted by a green line. This plot illustrates the comparison of the functions by showcasing their relative magnitudes and behavior within the specified domain.
On the right side, 2D plot of the inequality (8) illustrating a comparison between the two functions: $\left| \frac{\xi}{0.7} - \xi \right|$, denoted by a blue line, and $e^{-1.5}.\frac{9}{4}$ denoted by a green line, examining these comparisons offers a deeper understanding of how each function interacts with the others within the given interval.

This demonstrate the Hyers–Ulam Stability of equation (6). *Furthermore, considering the exact solution* $\chi^{\diamond}(\xi) = \xi$ *we have;*

$$|\chi(\xi) - \chi^{\diamond}(\xi)| = \left| \frac{\xi}{0.7} - \xi \right| \leq e^{-1.5}.\frac{9}{4}, \ \forall\ \xi \in [0,1]. \tag{8}$$

Example 6.2. *Let us consider the following NQFIE:*

$$\chi(\xi) = \int_0^{\xi} \frac{2\eth(\xi^2 - \eth^2)^{\frac{1}{7}-1}}{7\Gamma_7(1)} \frac{1}{11}.\chi(\eth)d\eth \times \int_0^{\xi} \frac{2\eth(\xi^2 - \eth^2)^{\frac{2}{7}-1}}{7\Gamma_7(2)} \frac{1}{15}.\chi(\eth)d\eth + \sum_{i=1}^{2} \int_0^{\xi} \frac{2\eth(\xi^2 - \eth^2)^{\frac{\beta_i}{k_{i^*}}-1}}{k_{i^*}\Gamma_{k_i^*}(\beta_i)} \frac{1}{23i}.\chi(\eth)d\eth, \ \xi \in \Delta = [0,1], \tag{9}$$

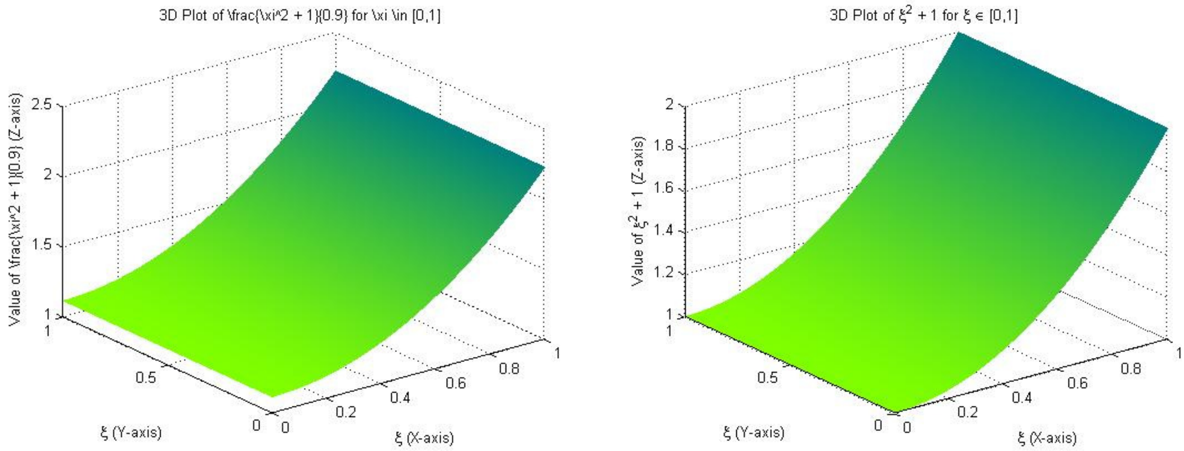


Figure 3: The 3D plots offer a detailed and insightful visualization of both the exact solution ξ_0 and the approximate solution ξ for equation (6), over the interval $[0, 1]$. These illustrations make it easier to comprehend how each solution behaves and changes inside the specified domain. The charts effectively illustrate the evolution of these solutions in connection to two independent variables by presenting ξ_0 and ξ in three-dimensional space. This method captures the parallels and discrepancies in the development of the exact and approximate solutions, enabling us to closely analyze their relationship and interaction. In the end, this comprehensive depiction improves our comprehension of the dynamic interplay between the exact and approximate answers, illuminating the approximation's efficacy within the designated range.

The form of equation (6) *is* (1)*, as we have observed. Thus, we have obtained the following information by comparing equation* (1) *with equation* (6)*:*

$$\psi(\xi) = \phi(\xi) = \psi_i^*(\xi) = \xi^2,$$

$$\psi(\eth) = \phi(\eth) = \psi_i^*(\eth) = \eth^2 \textit{ where } i = 1, 2.$$

and their respective derivatives are:

$$\psi'(\xi), \phi'(\xi), \psi_i^{*'}(\xi) \neq 0,$$

$$\psi'(\eth) = \phi'(\eth) = \psi_i^{*'}(\eth) = 2\eth,$$

also $k_1 = k_2 = k_{i^*} = 7, \beta_1 = 3, \beta_2 = 4, \zeta_1 = 1, \zeta_2 = 2.$

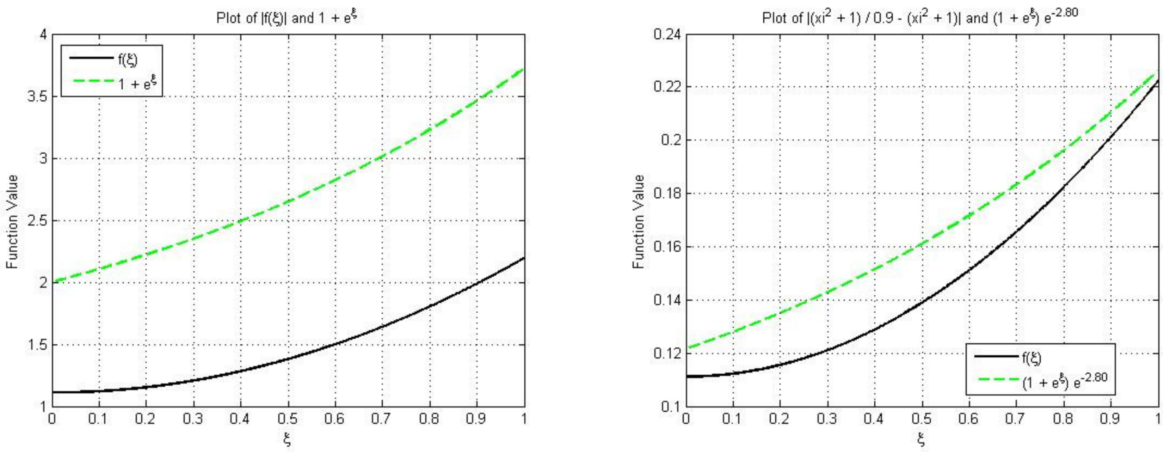
we now check all the conditions of theorem 4.1, for any $\xi \in [0, 1]$ *and for* $\eth_1, \eth_2 \in \mathbb{R}$ *we have,*

$$|\aleph_1(\xi, \eth_1) - \aleph_1(\xi, \eth_2)| \leq \frac{1}{11}|\eth_1 - \eth_2|, \ \forall \eth_1, \eth_2 \in \mathbb{R},$$

$$|\aleph_2(\xi, \eth_1) - \aleph_2(\xi, \eth_2)| \leq \frac{1}{15}|\eth_1 - \eth_2|, \ \forall \eth_1, \eth_2 \in \mathbb{R},$$

and

$$|\mathfrak{R}_i(\xi, \eth_1) - \mathfrak{R}_i(\xi, \eth_2)| \leq \frac{1}{23i}|\eth_1 - \eth_2|, \ \forall \eth_1, \eth_2 \in \mathbb{R}.$$



(a) **(b)**

Figure 4: On the left side, 2D plot of the inequality (10) illustrating a comparison between the two functions: $\left|\frac{\xi^2+1}{0.9} - \frac{1}{69.3}\left(7\xi^{\frac{2}{7}} - \frac{7\xi^{\frac{16}{7}}}{8} + 7\xi^{\frac{16}{7}}\right) \times \frac{1}{94.5}\left(\frac{7\xi^{\frac{4}{7}}}{2} - \frac{7\xi^{\frac{18}{7}}}{9} + \frac{7\xi^{\frac{18}{7}}}{2}\right) - \frac{1}{289.8}\left(\frac{7\xi^{\frac{6}{7}}}{3} - \frac{7\xi^{\frac{20}{7}}}{10} + \frac{7\xi^{\frac{20}{7}}}{3}\right) - \frac{1}{1738.8}\left(\frac{7\Im^{\frac{8}{7}}}{4} - \frac{7\xi^{\frac{22}{7}}}{11} + \frac{7\xi^{\frac{22}{7}}}{4}\right)\right|$ denoted by a black line, and $\Omega(\xi) = 1 + e^{\xi}$, denoted by a green line. This plot illustrates the comparison of the functions by showcasing their relative magnitudes and behavior within the specified domain.
On the right side, 2D plot of the inequality (11) illustrating a comparison between the two functions: $\left|\frac{\xi^2+1}{0.9} - \xi^2 + 1\right|$, denoted by a black line, and $= (1 + e^{\xi}).e^{-0.5}$ denoted by a green line, examining these comparisons offers a deeper understanding of how each function interacts with the others within the given interval.

Moreover if we choose approximate solution $\chi(\xi) = \frac{\xi^2+1}{0.9}$, *it follows,*

$$= \left|\frac{\xi^2+1}{0.9} - \left(\int_0^{\xi} \frac{2\eth(\xi^2-\eth^2)^{\frac{1}{7}-1}}{7\Gamma_7(1)} \frac{1}{11}.\chi(\eth)d\eth \times \int_0^{\xi} \frac{2\eth(\xi^2-\eth^2)^{\frac{2}{7}-1}}{7\Gamma_7(2)} \frac{1}{15}.\chi(\eth)d\eth \right.\right.$$
$$\left.\left. + \sum_{i=1}^{2} \int_{\xi_0}^{\xi} \frac{2\eth(\xi^2-\eth^2)^{\frac{\beta_i}{k_{i^*}}-1}}{k_{i^*}\Gamma_{k_i^*}(\beta_i)} \frac{1}{23i}.\chi(\eth)d\eth\right)\right| \leq 1 + e^{\xi} = \Omega(\xi), \forall \xi \in \Delta = [0,1],$$

$$= \left|\frac{\xi^2+1}{0.9} - \frac{1}{69.3}\left(7\xi^{\frac{2}{7}} - \frac{7\xi^{\frac{16}{7}}}{8} + 7\xi^{\frac{16}{7}}\right) \times \frac{1}{94.5}\left(\frac{7\xi^{\frac{4}{7}}}{2} - \frac{7\xi^{\frac{18}{7}}}{9} + \frac{7\xi^{\frac{18}{7}}}{2}\right)\right.$$
$$\left. - \frac{1}{289.8}\left(\frac{7\xi^{\frac{6}{7}}}{3} - \frac{7\xi^{\frac{20}{7}}}{10} + \frac{7\xi^{\frac{20}{7}}}{3}\right) - \frac{1}{1738.8}\left(\frac{7\Im^{\frac{8}{7}}}{4} - \frac{7\xi^{\frac{22}{7}}}{11} + \frac{7\xi^{\frac{22}{7}}}{4}\right)\right|, \tag{10}$$

$$\leq 1 + e^{\xi} = \Omega(\xi), \forall \xi \in \Delta = [0,1].$$

This demonstrate the Hyers–Ulam–Rassias Stability of equation (9). *Furthermore, considering the exact solution* $\chi^{\diamond}(\xi) = \xi^2 + 1$ *we have;*

$$|\chi(\xi) - \chi^{\diamond}(\xi)| = \left|\frac{\xi^2+1}{0.9} - \xi^2 + 1\right| \leq (1 + e^{\xi}).e^{-0.5}, \ \forall \ \xi \in [0,1]. \tag{11}$$

7. Conclusion

The examination of the stability of the (k,ψ)-Fractional Order Quadratic Integral Equation (NFQIE) has advanced significantly with this work. The Ulam-Hyers, Generalized Ulam-Hyers, Ulam-Hyers-Rassias, and Generalized Ulam-Hyers-Rassias stability criteria have been systematically applied to these equations for the first time. By applying these well-established stability ideas to fractional-order integral equations, a new method is presented that advances our knowledge of how these equations behave under perturbations.

Ethics declarations

Conflict of interest

The authors declare that they have no conflicts of interest.

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