Filomat 39:7 (2025), 2475–2482
https://doi.org/10.2298/FIL2507475K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On perfectness of Zaitov metrization of the hyperspace functor

Ljubiša D. R. Kočinac[a,*], **Dilorom R. Beshimova**[b], **Azad Ya. Ishmetov**[c]

[a] *Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia*
[b] *Bukhara State University, 170100 Bukhara, Uzbekistan*
[c] *Tashkent University of Architecture and Civil Engineering, Tashkent, Uzbekistan*

Abstract. Recently, A. Zaitov suggested a new metric on the space of all nonempty compact subsets of a given metric space. In the present paper we show that this metric turns the hyperspace functor $\exp$ into a perfect metrizable functor. Further, we establish that the hyperspace functor has many remarkable properties with respect to this metrization. In particular, this functor preserves isometric embeddings.

1. Introduction

For a topological space X we denote by $\exp X$ the set of all nonempty closed subsets of X, and by $\exp_c X$ the set of all nonempty compact closed subsets of X. The set $\exp X$ is equipped with several important topologies. The most famous, popular and investigated topology on $\exp X$ is the *Vietoris topology* V whose base is the collection of sets of the form

$$O\langle U_1, \ldots, U_n\rangle = \left\{F \in \exp X\colon\ F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \emptyset,\ i = 1, 2, \ldots, n\right\},$$

where $U_1, \ldots, U_n$ are open subsets of X ([2, 7, 11]).

For the space $(\exp X, \mathsf{V})$ we use the notation $\exp X$ and accept the name "space of closed subsets" or "hyperspace" of X. The space $\exp_c X$ is considered as a subspace of $\exp X$. For the space $\exp_c X$ we also keep the name "hyperspace of X"

Remark 1.1. It is easy to see that if X is a T_1-space, then the mapping $i : X \to \exp X$, which associates the point $x \in X$ with the singleton $i(x) = \{x\}$, is an embedding. So, one can assume X as a subspace of $\exp X$ or $\exp_c X$.

The preservation of "good" topological properties while passing to the space of closed subsets is one of the most important and interesting tasks concerning hyperspaces.

It is well known that if the space X is a compact Hausdorff space, then its hyperspace $\exp X$ is also a compact Hausdorff space.

2020 *Mathematics Subject Classification*. Primary 54B20; Secondary 18A22, 54E35.
Keywords. Hyperspace, metric, uniformly metrizable functor, perfect metrizable functor.
Received: 30 November 2024; Revised: 06 January 2025; Accepted: 10 January 2025
Communicated by Pratulananda Das
* Corresponding author: Ljubiša D. R. Kočinac
Email addresses: lkocinac@gmail.com (Ljubiša D. R. Kočinac), Drbeshimova@gmail.com (Dilorom R. Beshimova), ishmetov_azadbek@mail.ru (Azad Ya. Ishmetov)
ORCID iD: https://orcid.org/0000-0002-4870-7908 (Ljubiša D. R. Kočinac)

For compact Hausdorff spaces X and Y and a continuous mapping $f : X \to Y$ the equality

$$(\exp f)(A) = f(A), \qquad A \in \exp X, \tag{1}$$

defines a mapping $\exp f : \exp X \to \exp Y$ which is continuous. In this way we have that exp is a functor in the category of compact Hausdorff spaces and their continuous mappings [7].

Also, it is well known that for a metrizable space X the hyperspace $\exp_c X$ is also metrizable. In particular, for a compact metrizable space X its hyperspace exp X is also a compact metrizable space. For a metric space (X, d), the space $\exp_c X$ is equipped with the Hausdorff metric d_H (known also as Pompeiu–Hausdorff metric).

For a metric space (X, d), the Hausdorff metric in the space $\exp_c X$ is defined by the rule

$$d_H(A, B) = \inf\{\varepsilon \geq 0 : A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}, \quad (A, B \in \exp X),$$

where for $A \subset X$

$$A_\varepsilon = \bigcup_{a \in A} \{x \in X : d(a, x) \leq \varepsilon\}.$$

1.1. A new metric on the hyperspace

In [15], A. Zaitov suggested and briefly announced a new metric d_Z in the space $\exp_c X$ of all nonempty compact subsets of a given metric space (X, d). Some properties of this metric have been reported in the paper [14].

For a nonempty set X we put

$$\begin{aligned}
&X_1 = X_2 = X_3 = X,\\
&X_{123} = X_1 \times X_2 \times X_3 (= X^3),\\
&X_{ij} = X_i \times X_j (= X^2), \qquad 1 \leq i < j \leq 3,\\
&\pi_{ij}\colon\ X^3 \to X_i \times X_j, \qquad 1 \leq i < j \leq 3,\\
&\pi_k^{ij}\colon\ X_i \times X_j \to X_k, \qquad 1 \leq i < j \leq 3, \qquad k \in \{i,\ j\}.
\end{aligned}$$

Clearly, here π_{ij}, π_k^{ij} are corresponding projections.

Consider now a metric space (X, d). Define the function

$$d_Z : \exp_c X \times \exp_c X \to \mathbb{R}$$

by the formula

$$d_Z(F_1, F_2) = \inf\Big\{\sup\Big\{d(x, y) : (x, y) \in M\Big\} : M \subset X^2,\ \pi_i(M) = F_i,\ i = 1,\ 2\Big\}. \tag{2}$$

In [14] it was shown that d_Z is well defined and it is a metric on $\exp_c X$ whose restriction on $X \times X$ coincides with the metric d, and that it generates the Vietoris topology on $\exp_c X$. It was also shown that for a compact metric (resp., complete metric) space (X, d), the space $(\exp_c X, d_Z)$ is also a compact metric (resp., complete metric) space.

In the present paper we continue investigation of the hyperspace with the Zaitov metric, especially functorial properties of exp. We construct direct and inverse iterations of hyperspace functor and show that this metric turns the hyperspace functor into a uniformly metrizable functor. Then we get more strict result: the discussed metric turns the hyperspace functor into a perfect metrizable functor. Also, we prove that the hyperspace functor preserves isometric embeddings with respect to this metrization. At the end of the paper we pose a question with respect to the Zaitov metric arising in a natural way.

For a subset A of a space X, the symbol $[A]_X$ denotes the closure of A in X.

2. On properties of metrization d_Z

In this section we investigate the metrization d_Z.

Lemma 2.1. *Let (X, d) be a compact metric space. Then for every couple $F_1, F_2 \in \exp X$ and each $M \subset X^2$ with $\pi_i(M) = F_i$, $i = 1, 2$, we have $\pi_i([M]_{X^2}) = F_i$, $i = 1, 2$. Moreover, the set*

$$\Pi(F_1, F_2) = \left\{[M]_{X^2} \colon\ M \subset X^2,\ \pi_i(M) = F_i,\ i = 1, 2\right\}$$

is closed in $\exp X^2$.

Proof. Let $M \subset X^2$ be any set with $\pi_i(M) = F_i$, $i = 1, 2$. It is easy to see that $M \subset [M]_{X^2} \subset \pi_1^{-1}(F_1) \cap \pi_2^{-1}(F_2)$. This proves the first part of the lemma.

Since X is a compact metric space, the closure $[M]_{X^2}$ is a compact set. That is why $[M]_{X^2} \in \exp X^2$ and $\Pi(F_1, F_2) \subset \exp X^2$. Consider any convergent sequence

$$\Phi = \left\{M_n \in \exp X^2 \colon\ \pi_i(M_n) = F_i,\ i = 1, 2, n \in \mathbb{N}\right\},$$

$\lim\limits_{n\to\infty} \Phi \in \exp X^2$. Put $M_0 = \lim\limits_{n\to\infty} \Phi$. We have to show that $M_0 \in \Pi(F_1, F_2)$, i. e. $\pi_i(M_0) = F_i$, $i = 1, 2$. Note that $M_0 \subset \pi_1^{-1}(F_1) \cap \pi_2^{-1}(F_2)$. Consequently, for each $(x_1, x_2) \in M_0$ we have $\pi_i(x_i) = x_i$, $i = 1, 2$. It remains to check that π_i are onto mappings, $i = 1, 2$.

Fix an arbitrary point $x_1^0 \in F_1$. For each n there exists $x_2^n \in F_2$ with $(x_1^0, x_2^n) \in M_n$, i. e. $M_n \cap \pi_1^{-1}(x_1^0) \neq \emptyset$. As $\lim_{n\to\infty} M_n = M_0$, we have $M_0 \cap \pi_1^{-1}(x_1^0) \neq \emptyset$. Hence, $\pi_1(M_0) = F_1$. In the same way, one can show the equality $\pi_2(M_0) = F_2$ holds. So, $M_0 = \lim\limits_{n\to\infty} \Phi \in \Pi(F_1, F_2)$, and $\Pi(F_1, F_2)$ is closed.

Lemma 2.1 is proved. □

According to Lemma 2.1 for a compact metric space X with a metric d formula (2) acquires the shape

$$d_Z(F_1, F_2) = \min\{\max\{d(x, y) \colon\ (x, y) \in M\} \colon\ M \in \Pi(F_1, F_2)\}, \quad F_1, F_2 \in \exp X. \tag{3}$$

Equality (3) provides the following statement.

Corollary 2.2. *For each couple $F_1, F_2 \in \exp X$ there exists a set $M_{12} \in \Pi(F_1, F_2)$ such that*

$$d_Z(F_1, F_2) = \max\{d(x_1, x_2) \colon\ (x_1, x_2) \in M_{12}\}.$$

The next statement is quite obvious.

Proposition 2.3. *For a compact metric space (X, d) the equality*

$$\operatorname{diam}(\exp X,\ d_Z) = \operatorname{diam}(X,\ d)$$

holds.

Proposition 2.4. *For an isometric mapping $i_Y^X \colon\ (X, d^X) \to (Y, d^Y)$ of compact metric spaces the mapping $\left(\exp i_Y^X\right) \colon\ \left(\exp X, d_Z^X\right) \to \left(\exp Y, d_Z^Y\right)$ is an isometry.*

Proof. To prove this statement it is enough to use the definition of the metric d_Z and Proposition 2.3. □

In [6] it was shown the iterations of the continual hyperspace with respect to the Hausdorff metric d_H. We recall some notions and bring in certain facts and we will establish that the functor exp can be iterated by the metric d_Z.

Definition 2.5. ([12]) Recall that a functor F acting in the category $\mathscr{C}omp$ is called *seminormal* if it satisfies the following conditions: F is continuous (i. e. $F(\lim S) = \lim F(S)$); F is monomorphic (that is, preserves injectivity of mappings); F preserves intersections $\left(\text{of closed subsets } X_\alpha \subset X, \text{ i. e. } F\left(\bigcap\limits_\alpha X_\alpha\right) = \bigcap\limits_\alpha F(X_\alpha)\right)$; F preserves a point and the empty set (i. e. $F(1) = 1$, $F(\emptyset) = \emptyset$); here the symbol 1 means a singleton.

If the functor F is seminormal, then there is a unique natural transformation $\eta^F = \eta\colon \operatorname{Id} \to F$ of the identity functor into this functor. Moreover, this transformation is a monomorphism, i. e. for every compact Hausdorff space X the mapping $\eta_X\colon X \to F(X)$ is an embedding [3].

Definition 2.6. ([6]) A seminormal functor F acting in the category $\mathscr{MC}omp$ of metrizable compact spaces is called *metrizable* if each metric $d = d_X$ on an arbitrary compact space X can be associated with a metric $d_{F(X)}$ on the compact space $F(X)$ so that the conditions

(a) if a mapping $i\colon (X, d^X) \to (Y, d^Y)$ is an isometric embedding, then the mapping $F(i)\colon (F(X), d^X_{F(X)}) \to (F(Y, d^Y_{F(Y)})$ is also an isometric embedding;

(b) the embedding $\eta_X\colon (X, d) \to (F(X), d_{F(X)})$ is an isometry;

(c) $\operatorname{diam} F(X) = \operatorname{diam} X$

are fulfilled.

The correspondence $d = d_X \to d_{F(X)}$ is called the *metrization* of the functor F [5].

For a compact Hausdorff space X, a mapping $\eta_X\colon X \to \exp X$ defined as $\eta_X(x) = \{x\}$, with respect to Definition 2.6, Propositions 2.3 and 2.4 we have the following important result.

Corollary 2.7. *The functor* exp *acting in the category $\mathscr{MC}omp$ of compact metrizable spaces and their continuous mapping is metrizable according to the metric d_Z.*

In other words, the correspondence $d_X = d \to d_Z = d_{\exp(X)}$ is a metrization of the functor exp.

For a compact Hausdorff space X, we define the iterations

$$\exp^2 X = \exp(\exp X), \ldots, \exp^n X = \exp(\exp^{n-1} X), \ldots,$$

and for a continuous mapping $f\colon X \to Y$ of compact Hausdorff spaces we set

$$\exp^2 f = \exp(\exp f), \ldots, \exp^n f = \exp(\exp^{n-1} f), \ldots.$$

For a fixed metric d on a compact metric space X the metric d_Z on its hyperspace $\exp X$ we denote by d_1, i. e. $d_1 = d_Z$. Let $d_n = d_{\exp^n X} = \left(d_{\exp^{n-1} X}\right)_Z$ be the metric on the compact Hausdorff space $\exp^n X$ generated by the metrization $d \to d_Z$.

Next, we fix a compact Hausdorff space X and, setting $\eta_{\exp^{n-1} X}(\mathscr{X}) = \{\mathscr{X}\}$, $\mathscr{X} \in \exp^{n-1} X$, we define an embedding $\eta_{n-1,n} = \eta_{\exp^{n-1} X}\colon \exp^{n-1} X \to \exp^n X$. A direct sequence

$$X \xrightarrow{\eta_{0,1}} \exp X \xrightarrow{\eta_{1,2}} \exp^2 X \xrightarrow{\eta_{2,3}} \ldots \xrightarrow{\eta_{n-1,n}} \exp^n X \xrightarrow{\eta_{n,n+1}} \ldots. \tag{4}$$

arises. For natural numbers n, m with $n < m$, the embedding $\eta_{n,m}\colon \exp^n X \to \exp^m X$ is defined as the composition

$$\eta_{n,m} = \eta_{m-1,m} \circ \ldots \circ \eta_{n+1,n+2} \circ \eta_{n,n+1}.$$

The mappings

$$\eta_{n,m}\colon (\exp^n X, d_n) \to (\exp^m X, d_m)$$

are isometric embeddings. For each natural number n we set $\eta_n(\mathscr{X}) = \lim\limits_{m\to\infty} \eta_{n,m}(\mathscr{X})$, $\mathscr{X} \in \exp^n X$. It is easy to see that $\eta_n(\mathscr{X}) = \ldots\{\ldots\{\{\mathscr{X}\}\}\ldots\}\ldots$ for every $\mathscr{X} \in \exp^n X$. Let us introduce the notation

$$\exp^+ X = \{\eta_n(\mathscr{X})\colon \mathscr{X} \in \exp^n X,\ n = 1, 2, \ldots\}.$$

Then

$$\exp^+ X = \bigcup_{n=1}^{\infty} \eta_n(\exp^n X).$$

Since for any n the embedding $\eta_{n,n+1}$ is isometric, then for an arbitrary pair of elements $\mathscr{X}, \mathscr{Y} \in \exp^n X$ we have

$$\begin{aligned} d_n(\mathscr{X}, \mathscr{Y}) &= d_{n+1}(\{\mathscr{X}\}, \{\mathscr{Y}\}) = \\ &= d_{n+2}(\{\{\mathscr{X}\}\}, \{\{\mathscr{Y}\}\}) = \ldots = \\ &= d_{n+k}\Big(\underbrace{\{\ldots\{\{}_{k} \mathscr{X} \underbrace{\}\}\ldots\}}_{k}, \underbrace{\{\ldots\{\{}_{k} \mathscr{Y} \underbrace{\}\}\ldots\}}_{k}\Big) = \ldots. \end{aligned} \tag{5}$$

If $\mathscr{X}, \mathscr{Y} \in \exp^+ X$, then there exists n such that $\mathscr{X}, \mathscr{Y} \in \eta_n(\exp^n X)$. In turn, there are $\mathscr{Z}, \mathscr{T} \in \exp^n X$ with $\eta_n(\mathscr{Z}) = \mathscr{X}$ and $\eta_n(\mathscr{T}) = \mathscr{Y}$.

Now for an arbitrary pair of elements $\mathscr{X}, \mathscr{Y} \in \exp^+ X$ we put

$$d^+(\mathscr{X}, \mathscr{Y}) = d_n(\mathscr{Z}, \mathscr{T}), \tag{5$'$}$$

where $\mathscr{Z}, \mathscr{T} \in \exp^n X$ are elements such that $\eta_n(\mathscr{Z}) = \mathscr{X}$, $\eta_n(\mathscr{T}) = \mathscr{Y}$. By virtue of the equalities (5), the function $d_+\colon \exp^+ X \times \exp^+ X \to \mathbb{R}$ is defined correctly, and it is a metric on $\exp^+ X$. Thus, the metric space $(\exp^+ X, d_+)$ is obtained.

For a continuous mapping $f\colon X \to Y$ of metric compact spaces X and Y by the equality

$$(\exp^+ f)(\mathscr{X}) = \eta_n((\exp^n f)(\mathscr{Z})), \qquad \mathscr{X} \in \exp^+ X$$

we determine the mapping $\exp^+ f\colon \exp^+ X \to \exp^+ Y$, where $\mathscr{Z} \in \exp^n X$, $\eta_n(\mathscr{Z}) = \mathscr{X}$ and $\eta_n\colon \exp^n \to \exp^+$ is the limit of isometric embeddings $\eta_{n,m}$ at $n < m$ and $m \to \infty$.

Definition 2.8. ([5]) A metrizable functor F is said to be *uniformly metrizable* if some of its metrizations have the property

(d) for any continuous mapping $f\colon (X d^X) \to (Y, d^Y)$ the mapping $F^+(f)\colon (F^+(X), d_+^X) \to (F^+(Y), d_+^Y)$ is uniformly continuous.

A metrization of a functor with the property (d) is called *uniformly continuous*.

Theorem 2.9. *The functor* exp *is uniformly metrizable with respect to the metrization $d \to d_Z$.*

Proof. The proof of the theorem will be divided into three lemmas. The first lemma is the following statement.

Lemma 2.10. *For compact metric spaces (X, d^X) and (Y, d^Y), a continuous mapping $f\colon X \to Y$ and $M_{12} \in \Pi(F_1, F_2)$ such that*

$$d_Z^X(F_1, F_2) = \max\{d^X(x_1, x_2)\colon (x_1, x_2) \in M_{12}\}$$

one has

$$d_Z^Y((\exp f)(F_1), (\exp f)(F_2)) \leq \max\{d^Y(f(x_1), f(x_2))\colon (x_1, x_2) \in M_{12}\}.$$

Proof. We have $(\exp f)(F_i) = f(F_i)$, $i = 1, 2$, and $(f_1 \times f_2)(M_{12}) \in \Pi(f(F_1), f(F_2))$. Therefore,

$$\begin{aligned} d_Z^Y((\exp f)(F_1), (\exp f)(F_2)) &= d_Z^Y(f(F_1), f(F_2)) = \\ &= \min\{\max\{d^Y(y_1, y_2)\colon (y_1, y_2) \in N\}\colon N \in \Pi(f(F_1), f(F_2))\} \leq \\ &\leq \max\{d^Y(y_1, y_2)\colon (y_1, y_2) \in (f_1 \times f_2)(M_{12})\} = \\ &= \max\{d^Y(f(x_1), f(x_2))\colon (x_1, x_2) \in M_{12}\}. \end{aligned}$$

Lemma 2.10 is proved. □

The second lemma is the following statement.

Lemma 2.11. *If a mapping* $f\colon (X, d^X) \to (Y, d^Y)$ *is* (ε, δ)*-uniformly continuous then the mapping* $\exp f\colon (\exp X, d^X_Z) \to (\exp Y, d^X_Z)$ *is also* (ε, δ)*-uniformly continuous.*

Proof. It is enough to catch that for each $\sigma > 0$, an open σ-net $\{A_1, \dots, A_n\}$ of X and any couple of sets F_1, $F_2 \subset \bigcup_{j=1}^{k} A_{i_j}$ where $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, and $F_1 \cap A_{i_j} \neq \emptyset$, $F_2 \cap A_{i_j} \neq \emptyset$, $j = 1, \dots, k$, one has $d_Z(F_1, F_2) < \sigma$. Then the application of Lemma 2.10 finishes the proof of Lemma 2.11. □

According to Definition 2.8, Lemma 2.11 gives the following result that is the third lemma.

Lemma 2.12. *For any continuous mapping* $f\colon (X, d^X) \to (Y, d^Y)$ *the mapping*

$$\exp^+(f)\colon (\exp^+(X), d^X_+) \to (\exp^+(Y), d^Y_+)$$

is uniformly continuous.

Theorem 2.9 is completely proved. □

So, $\exp$ is a uniformly metrizable functor (with respect to the metrization $d \to d_Z$). Hence, by [6, Proposition 2] for a homeomorphism $f\colon (X, d^X) \to (Y, d^Y)$ of compact metric spaces, the mapping $\exp^+(f)\colon (\exp^+ X, d^X_+) \to (\exp^+ Y, d^Y_+)$ is a uniform homeomorphism. Therefore, topologically the metric space $(\exp^+ X, d_+)$ does not depend on the choice of the metric d on the (compact metric) space X. Consequently, the operation $\exp^+$ is a functor from the category of compact metric spaces into the category of metrizable spaces and uniformly continuous mappings.

Let us now associate with each compact metric space (X, d) the completion $(\exp^{++} X, d_{++})$ of the space $(\exp^+ X, d_+)$, and to every continuous mapping $f\colon (X, d^X) \to (Y, d^Y)$ a mapping $\exp^{++} f\colon \exp^{++} X \to \exp^{++} Y$, which is an expanding of the mapping $\exp^+ f$ to the completions of the spaces $(\exp^+ X, d^X_+)$ and $(\exp^+ Y, d^Y_+)$. Thus, we have defined the functor $\exp^{++}$ acting from the category of metric compact spaces into the category of complete metric spaces and uniformly continuous mappings. The functor $\exp^{++}$, like the functor $\exp^+$, is topologically invariant under the metric, and it can be considered as a functor acting from the category of compact metric spaces to the category of Polish spaces.

Definition 2.13. ([5]) A functor F is called *monadic* if there exist such natural transformations $\eta\colon \mathrm{Id} \to F$ (a monad unit) and $\psi\colon F^2 \to F$ (a monad multiplication), such that for each compact Hausdorff space X the following equalities hold:

$(M1)$ $\psi_X \circ F(\eta_X) = \mathrm{id}_{F(X)}$;
$(M2)$ $\psi_X \circ \eta_{F(X)} = \mathrm{id}_{F(X)}$;
$(M3)$ $\psi_X \circ F(\psi_X) = \psi_X \circ \psi_{F(X)}$.

In this case, the triple $\mathbf{F} = \langle F, \eta, \psi\rangle$ is called a *monad.*

If the previous definition only $(M1)$ and $(M2)$ hold, then the functor F is called *semimonadic*, and the triple $\mathbf{F} = \langle F, \eta, \psi\rangle$ is a *semi-monad.*

The exponential functor $\exp$ is monadic. Here, the mapping $\eta\colon \mathrm{Id} \to \exp$, which for a compact Hausdorff space X is defined by the equality $\eta_X(x) = \{x\}$, $x \in X$, is the unit of the monad, and the mapping $\psi\colon \exp^2 \to \exp$, which for a compact Hausdorff space X is given by $\psi_X(\mathscr{A}) = \cup\mathscr{A}$, $\mathscr{A} \in \exp^2 X$, is the multiplication of the monad, and the triple $\mathbf{H} = \langle \exp, \{\cdot\}, \cup\rangle$ is the exponential monad. However, M. M. Zarichny showed that the iterated exponential functor $\exp^2$ is semi-monadic, but not monadic ([4, Remark 2.1].

Note that $\mathbf{H} = \langle \exp, \{\cdot\}, \cup\rangle$ is the only (semi)monad including the functor $\exp$ (see [4, Theorem 2.1]).

For a natural number n let

$$\psi_{n+1,n} = \psi_{\exp^{n-1} X}\colon \exp^{n+1} X \to \exp^n X,$$

and for natural numbers n, m with $n < m$ we put

$$\psi m, n = \psi_{n+1,n} \circ \psi_{n+2,n+1} \circ \ldots \circ \psi_{m,m-1}.$$

The following inverse sequence appears

$$\mathsf{exp}\, X \xleftarrow{\psi_{2,1}} \exp^2 X \xleftarrow{\psi_{3,2}} \ldots \xleftarrow{\psi_{n,n-1}} \exp^n X \xleftarrow{\psi_{n+1,n}} \exp^{n+1} X \xleftarrow{\psi_{n+2,n+1}} \ldots \tag{6}$$

Let $\exp^{\omega} X$ be the limit of this sequence. Since $\psi_{n+1,n}$ are natural transformations, the operation $\exp^{\omega}$ is functorial. The functor $\exp^{\omega}$ acts in the category $\mathscr{C}omp$ of compact Hausdorff spaces and in its subcategory $\mathscr{MC}omp$ of metrizable compact spaces. The functor $\exp^{\omega}\colon \mathscr{C}omp \to \mathscr{C}omp$ is called the *infinite iteration* of the functor exp.

For $1 \le n < m$ we set $q_{n,m} = \mathsf{exp}(\eta_{n-1,m-1})$. From (M1) it follows

$$\psi_{n,m} \circ q_{n,m} = \mathrm{id}_{\exp^n(X)}\,. \tag{7}$$

Definition 2.14. ([6]) A uniformly metrizable semimonadic functor F is called *perfect metrizable* if some of its metrization, along with the properties (a), (b), (c), (d), have the properties

- (e) the mapping $\psi_{2,1}\colon (F^2(X), d_2) \to (F(X), d_1)$ is non-expending;
- (f) for any $a \in F^2(X)$ and $x \in X$ we have

$$d_1(\psi_{2,1}(a), \eta_{0,1}(x)) = d_2(a, \eta_{0,2}(x)). \tag{8}$$

Theorem 2.15. *The functor* exp *is perfect metrizable with respect to the metrization* $d \to d_Z$.

Proof. Let us check (e). Let $\mathscr{A}, \mathscr{B} \in \exp^2 X = \mathsf{exp}(\mathsf{exp}\, X)$ and $d_2(\mathscr{A}, \mathscr{B}) = \delta$. By Corollary 2.2 there exists a set $\mathscr{M}_{12} \in \Pi(\mathscr{A}, \mathscr{B}) \subset (\mathsf{exp}\, X)^2 = (\mathsf{exp}\, X) \times (\mathsf{exp}\, X)$ such that

$$\max\{d_1(F_1, F_2)\colon (F_1, F_2) \in \mathscr{M}_{12}\} = \delta.$$

Then $d_1(F_1, F_2) \le \delta$ for all $(F_1, F_2) \in \mathscr{M}_{12}$, and there is a pair $(F_{10}, F_{20}) \in \mathscr{M}_{12}$ such that $d_1(F_{10}, F_{20}) = \delta$. Let us estimate the distance $d_1(\psi_{2,1}(\mathscr{A}), \psi_{2,1}(\mathscr{B})) = d_1(\cup\mathscr{A}, \cup\mathscr{B})$. It is clear that $\left(\bigcup_{(F_1, F_2)\in\mathscr{M}_{12}} F_1 \times F_2\right) \in \Pi(\cup\mathscr{A}, \cup\mathscr{B}) \subset X^2$. That is why

$$d_1(\psi_{2,1}(\mathscr{A}), \psi_{2,1}(\mathscr{B})) \le \max\left\{d(x, y)\colon (x, y) \in \bigcup_{(F_1, F_2)\in\mathscr{M}_{12}} F_1 \times F_2\right\} = \delta,$$

i. e.

$$d_1(\psi_{2,1}(\mathscr{A}), \psi_{2,1}(\mathscr{B})) \le d_2(\mathscr{A}, \mathscr{B}). \tag{9}$$

Property (e) is probed.

Let us now check the condition (f). Note that for $x \in X$ we have $\eta_{0,1}(x) = \{x\} \in \mathsf{exp}\, X$ and $\eta_{0.2}(x) = \{\{x\}\} \in \exp^2 X$, and also for $\mathscr{A} \in \exp^2 X$ we have $\psi_{2,1}(\mathscr{A}) = \cup\mathscr{A} \in \mathsf{exp}\, X$. It is easy to see that the sets $\Pi(\mathscr{A}, \{\{x\}\})$ and $\Pi(\cup\mathscr{A}, \{x\})$ consist of a single element - the products $\mathscr{A} \times \{\{x\}\}$ and $\cup\mathscr{A} \times \{x\}$, respectively. That is why

$$d_1(\cup\mathscr{A}, \{x\}) = \max\left\{d(y, x)\colon (y, x) \in \cup\, \mathscr{A} \times \{x\}\right\},$$

$$d_2(\mathscr{A}, \{\{x\}\}) = \max\left\{d_1(F, \{x\})\colon\ , (F, \{x\}) \in \mathscr{A} \times \{\{x\}\}\right\}.$$

Let $(F_0, \{x\}) \in \mathscr{A} \times \{\{x\}\}$ be a pair such that $d_2(\mathscr{A}, \{\{x\}\}) = d_1(F_0, \{x\})$. Since $F_0 \subset \cup\mathscr{A}$, then $d_2(\mathscr{A}, \{\{x\}\}) \le d_1(\cup\mathscr{A}, \{x\})$. The reverse equality follows from the inequality (9). Condition (f), and thus Theorem 2.15, is proved. $\square$

We end the paper with one question.

Recall that every closed subset of a Tychonoff space can be represent as a unique idempotent probability measure [8, 10, 13, 16]. Moreover, any closed subset of a Tychonoff space is a unique order-preserving operator [9], [1]. Consequently, it arises in a natural way the following question.

Question 2.16. *Is the metric d_Z expendable over the space of idempotent probability measures (more widely, over the space of order-preserving functionals)?*

References

[1] F. S. Aktamov, *A variant of the Banach-Steinhaus theorem for weakly additive, order-oreserving operators*, AIP Conf. Proc. 2024, **3004** (2024), ID 030002, 6 pages.

[2] R. Engelking, *General Topology*, Mir, Moscow, 1989.

[3] V. V. Fedorchuk, *Certain geometric properties of covariant functors*, Russian Math. Surveys **39:5** (1984), 199–249.

[4] V. V. Fedorchuk, *Soft maps, multi-valued retractions, and functors*, Russian Math. Surveys **41:6** (1986), 149–197.

[5] V. V. Fedorchuk, *Fibrations of spaces of probability measures and the geometry of infinite iterations of some monadic functors*, Dokl. Math. **38** (1989), 35–38.

[6] V. V. Fedorchuk. *Triples of infinite iterates of metrizable functors*, Math. USSR Izv. **36** (1991), 411–433.

[7] V. V. Fedorchuk, V. V. Filippov, *General Topology. Basic Structures*, Fizmatlit, Moscow, 2006.

[8] A. Ya. Ishmetov, *Functor of idempotent probability measures with compact support and open mappings*, J. Math. Sci. **278** (2024), 463–475.

[9] Lj. D. R. Kočinac, F. S. Aktamov, A. A. Zaitov, *An open mapping theorem for order-preserving operators*, Kragujevac J. Math. **47** (2023), 1057–1063.

[10] Lj. D. R. Kočinac, A. A. Zaitov, M. R. Eshimbetov, *On the Čech-completeness of the space of τ-smooth idempotent probability measures*, Axioms **13** (2024), Art. ID 569.

[11] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[12] E. V. Shchepin, *Functors and uncountable powers of compacta*, Russian Math. Surveys **36** (1981), 1–71.

[13] A. O. Tagaymurotov, A. A. Zaitov, *On geometric form of a Hahn-Banach theorem's version for idempotent probability measures*, Lobachevskii J. Math. **44** (2023), 4893–4901.

[14] A. A. Zaitov, D. R. Beshimova, A. Ya. Ishmetov, *On the completeness of hyperspaces of complete metric spaces*, Uzbek Math. J. **68** (2024), 192–201.

[15] A. A. Zaitov, D. R. Beshimova, D. I. Jumaev, *On a metric on a hyperspace*, Borubaev readings IV, Bishkek, 2022.

[16] A. A. Zaitov, D. T. Eshkobilova, *Dugundji compacta and the space of idempotent probability measures*, Math. Notes **114** (2023), 433–442.