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Some remarks on generalized Schwarz-Pick type inequality for harmonic quasiconformal mappings with simply connected ranges

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Abstract. The main result of this paper is a generalized Schwarz-Pick type inequality for ordinary harmonic quasiconformal mappings of the unit disk onto arbitrary simply connected domains in the complex plane. This result extends some of our earlier findings, as well as those presented in the excellent article [2]. Additionally, by analyzing the properties of the hyperbolic metric on simply connected hyperbolic domains in the complex plane, we establish the co-Lipschitz continuity of these mappings and determine the corresponding bi-Lipschitz constant with respect to the hyperbolic metric.

1. Introduction

Let R be an arbitrary Riemannian surface with a complex structure defined by the atlas (U_ν, h_ν). Denote by $z_\nu \in V_\nu = h_\nu(U_\nu) \subset \mathbb{C}$ the local parameter on that surface associated with the chart (U_ν, h_ν). Assume that the surface R is also equipped with a Riemannian metric, represented in terms of the local parameters as $ds^2 = \rho_\nu(z_\nu)|dz_\nu|^2$, where ρ_ν is a positive C^2 function on V_ν compatible with the complex structure on R. Specially, whenever $z_\mu \in V_\mu = h_\mu(U_\mu) \subset \mathbb{C}$ and $z_\nu \in V_\nu = h_\nu(U_\nu) \subset \mathbb{C}$ are local parameters on R with $U_\mu \cap U_\nu \neq \emptyset$, then $\rho_\mu(z_\mu) = \rho_\nu(A(z_\mu))|A'(z_\mu)|^2$, $z_\mu \in U_\mu \cap U_\nu$, where $z_\nu = A(z_\mu) = (h_\nu \circ h_\mu^{-1})(z_\mu)$ is the mapping describing a conformal (one-to-one and analytic) transition between local parameters z_μ and z_ν on R.

Note that in the neighborhood of each point on a Riemann surface R, the Riemannian metric can be viewed as a positive multiple of the Euclidean metric. To emphasize the conformal invariance of this metric under changes in local parameters, we will refer to it as the conformal metric on R and, whenever convenient, denote it by $ds^2 = \rho(z)|dz|^2$. Additionally, the associated representative, i.e., the function $\rho : z \mapsto \rho(z)$, will be referred to as the density of the conformal metric on R.

It is well known that if $\gamma : [0,1] \to R$ is an arbitrary rectifiable curve on the surface R, then the length of that curve with respect to the given conformal metric $ds^2 = \rho(z)|dz|^2$ on R is defined as the nonnegative quantity $|\gamma|_\rho = \int_\gamma \sqrt{\rho(z)}|dz|$. If we choose any two points P_1 and P_2 on R and define $d_R(P_1, P_2) = \inf |\gamma|_\rho$, where the infimum is taken over all rectifiable curves γ on R that join P_1 and P_2, then a distance function d_R, induced by the conformal metric $ds^2 = \rho(z)|dz|^2$, is defined on R. This implies that the surface R can also be regarded as a metric space.

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Let $ds^2 = \rho(z)|dz|^2$ be a conformal metric on the Riemann surface R. Consider a chart (U, h) and the local parameter $z \in V = h(U) \subset \mathbb{C}$ around an arbitrary point $P_0 \in R$, $h(P_0) = z_0 \in V$, on R and, without loss of generality, denote by ρ the representation of the given metric in terms of the local parameter z. The real number

$$K_\rho(z_0) = -\frac{1}{2}\frac{(\triangle \log \rho)(z_0)}{\rho(z_0)} \tag{1}$$

is referred to as the Gaussian curvature of the conformal metric $ds^2 = \rho(z)|dz|^2$ at the point P_0 on R. It can be shown that this definition is independent of the choice of local parameter on R near P_0. Indeed, if a different chart $(\tilde{U}, \tilde{h})$ also contains P_0, with corresponding local parameter $\tilde{z} \in \tilde{V} = \tilde{h}(\tilde{U}) \subset \mathbb{C}$ around P_0 and $\tilde{h}(P_0) = \tilde{z}_0$, i.e. if $ds^2 = \tilde{\rho}(\tilde{z})|d\tilde{z}|^2$ represents the conformal metric in terms of that parameter, then it holds that

$$\begin{aligned}(\triangle \log \rho)(z) &= (\triangle \log((\tilde{\rho} \circ A)|A'(z)|^2))(z)\\ &= (\triangle \log(\tilde{\rho} \circ A))(z) + 2(\triangle \log |A'|)(z)\\ &= (\triangle \log \tilde{\rho})(A(z))|A'(z)|^2,\end{aligned}$$

in a suitably chosen neighborhood $V' \subset V$ of the point z_0, as the function $z \mapsto \log |A'(z)|$, $z \in V'$, is harmonic, where $\tilde{z} = A(z)$ is the mapping establishing the conformal transition between parameters. Thus, since $\tilde{z}_0 = A(z_0)$ and $\rho(z_0) = \tilde{\rho}(A(z_0))|A'(z_0)|^2$, it follows that $K_\rho(z_0) = K_{\tilde{\rho}}(\tilde{z}_0)$. Therefore, regardless of the choice of local parameter used, we refer to the Gaussian curvature of the conformal metric $ds^2 = \rho(z)|dz|^2$ as the function defined on R by formula (1).

Example 1.1. *Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in $\mathbb{C}$. Consider the conformal metric $ds^2 = \lambda_{\mathbb{D}}(z)|dz|^2$ on $\mathbb{D}$, where the density function $\lambda_{\mathbb{D}}$ is defined as*

$$\lambda_{\mathbb{D}}(z) = \left(\frac{2}{1-|z|^2}\right)^2, \ z \in \mathbb{D}. \tag{2}$$

For any $z \in \mathbb{D}$, we have

$$\begin{aligned}(\triangle \log \lambda_{\mathbb{D}})(z) &= 4(\log \lambda_{\mathbb{D}})_{z\bar{z}}(z) = -8(\log(1-|z|^2))_{z\bar{z}}(z)\\ &= 8\left(\frac{\bar{z}}{1-|z|^2}\right)_{\bar{z}}(z) = \frac{8}{(1-|z|^2)^2},\end{aligned}$$

which implies that $(\triangle \log \lambda_{\mathbb{D}})(z) = 2\lambda_{\mathbb{D}}(z)$. Thus, $K_{\lambda_{\mathbb{D}}}(z) = -1$, for all $z \in \mathbb{D}$. Consequently, the conformal metric $ds^2 = \lambda_{\mathbb{D}}(z)|dz|^2$ has constant negative Gaussian curvature on $\mathbb{D}$. Moreover, the associated distance function for this metric on $\mathbb{D}$ is given by

$$d_{\mathbb{D}}(z_1, z_2) = \log \frac{1 + \left|\frac{z_1 - z_2}{1 - \bar{z}_2 z_1}\right|}{1 - \left|\frac{z_1 - z_2}{1 - \bar{z}_2 z_1}\right|}, \ z_1, \ z_2 \in \mathbb{D}. \tag{3}$$

Definition 1.2. *The hyperbolic metric on the unit disk is a conformal metric $ds^2 = \lambda_{\mathbb{D}}(z)|dz|^2$, where the density function $\lambda_{\mathbb{D}}$ is given by (2). The function $d_{\mathbb{D}}$ is called the hyperbolic distance on the unit disk $\mathbb{D}$.*

For more details on the hyperbolic metric see [1, 2, 6, 7, 9, 14].

Let Ω and Ω' be domains (open and connected) in the complex plane $\mathbb{C}$.

Definition 1.3. *A mapping $f : \Omega \to \Omega'$, of class C^2 in Ω, is called harmonic with respect to a conformal metric $ds^2 = \rho(w)|dw|^2$ on Ω' (where ρ is a positive function of the class C^2 on Ω') if*

$$f_{z\bar{z}}(z) + \frac{\rho_w(f(z))}{\rho(f(z))} f_z(z) f_{\bar{z}}(z) = 0, \tag{4}$$

for all $z \in \Omega$. Here, f_z and $f_{\bar{z}}$ denote the partial derivatives of f with respect to z and $\bar{z}$, respectively, and $f_{z\bar{z}}$ represents the mixed second-order partial derivative of f in Ω, with $f_{z\bar{z}} = (f_z)_{\bar{z}} = (f_{\bar{z}})_z$.

In the special case when the conformal metric on Ω' is the Euclidean metric, i.e. when $\rho \equiv 1$ on Ω', the condition (4) reduces to the usual harmonicity condition $(\triangle f)(z) = 4f_{z\bar{z}}(z) = 0$, for all $z \in \Omega$. Thus, f is a Euclidean harmonic (or just harmonic) mapping in this case.

Definition 1.4. *A orientation-preserving C^1 diffeomorphism $f : \Omega \to \Omega' = f(\Omega)$ is said to be a regular k-quasiconformal mapping (or simply k-quasiconformal) if there exists a constant $k \in [0, 1)$ such that*

$$|\mu_f(z)| = \left|\frac{f_{\bar{z}}(z)}{f_z(z)}\right| \leqslant k, \tag{5}$$

for all $z \in \Omega$, where $\mu_f(z) = \frac{f_{\bar{z}}(z)}{f_z(z)}$, $z \in \Omega$, is the complex dilatation of f.

Remark 1.5. *Note that the previous definition is correct since the mapping f is orientation-preserving, i.e. the Jacobian of that mapping is positive. Hence, $J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |f_z(z)|^2(1 - |\mu_f(z)|^2) > 0$, which implies that $f_z(z) \neq 0$. The smallest constant k satisfying property (5) is referred to as the quasiconformality constant of the mapping f. Additionally, many authors use the constant K instead of k to represent the quasiconformal constant of a k-quasiconformal mapping f, where $K = \dfrac{1+k}{1-k} \geqslant 1$. Throughout the text, if f is a $k-$quasiconformal mapping, the constant K will always be equal to $K = \frac{1+k}{1-k}$ and be a number not less then 1.*

Note that when $K = 1$, i.e. $k = 0$, the mapping f is conformal, because in this case $f_{\bar{z}} \equiv 0$ on Ω.

An initial partial characterization of harmonic quasiconformal diffeomorphisms of the unit disk onto itself was provided in [11], where it was shown that such mappings are co-Lipschitz. Furthermore, a complete characterization of these diffeomorphisms in terms of their boundary functions was established in [15] (see also [8]), using Mori's inequality for quasiconformal mappings. Specifically, every quasiconformal mapping of the unit disk onto itself extends to a homeomorphism of their corresponding closures. Notably, these mappings have been proven to be bi-Lipschitz with respect to the Euclidean metric and, more significantly for our purposes, they are also quasi-isometries with respect to the hyperbolic metric (see [9, 17]).

Theorem 1.6 ([9]). *Let f be a k-quasiconformal harmonic diffeomorphism of the unit disc $\mathbb{D}$ onto itself. Then f is a quasi-isometry of the unit disc $\mathbb{D}$ with respect to the hyperbolic metric. In addition, f is a (K^{-1}, K) bi-Lipschitz with respect to the hyperbolic metric.*

Theorem 1.7 ([9]). *Let f be a k-quasiconformal harmonic diffeomorphism of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto itself. Then f is a quasi-isometry of $\mathbb{H}$ with respect to the hyperbolic metric on $\mathbb{H}$. More specifically, f is a (K^{-1}, K) bi-Lipschitz with respect to the hyperbolic metric on $\mathbb{H}$, but also with respect to the Euclidean metric, i.e.*

$$|z_1 - z_2|/K \leqslant |f(z_1) - f(z_2)| \leqslant K|z_1 - z_2|,$$

for all z_1, $z_2 \in \mathbb{H}$. The estimates are sharp.

For further properties and characterizations of the harmonic quasiconformal mappings, that act between various subdomains of the complex plane $\mathbb{C}$, we refer to [2, 5, 7–9, 11–14].

Let R be an arbitrary Riemann surface of hyperbolic type, meaning that its universal covering surface is the unit disk. By using the Uniformization theorem, the surface R can be uniquely equipped with a conformal metric of constant Gaussian curvature equal to -1, which we shall refer to as the hyperbolic metric on the surface R. In that case we will use the notation $ds^2 = \lambda_R(z)dz^2$, for the hyperbolic metric on R, and d_R for the corresponding hyperbolic distance function on R, which is induced by the hyperbolic metric with the density λ_R. Moreover, if $\tau : \mathbb{D} \to R$ is the projection that realizes the covering of the surface R by the unit disk $\mathbb{D}$, then the mapping τ is a local isometry with respect to the distances induced by the corresponding hyperbolic metrics.

Example 1.8. *It is trivial to deduce that the metric density function of the hyperbolic metric on the upper half plane* $\mathbb{H}$ *is given by the formula* $\lambda_{\mathbb{H}}(z) = \frac{1}{y^2}$, *where* $z = x + iy$, $x \in \mathbb{R}$, $y > 0$, *is a point in* $\mathbb{H}$. *Thus, the conformal metric* $ds^2 = \frac{1}{y^2}|dz|^2$, $z = x + iy$, $y > 0$, *is the hyperbolic metric on the upper half plane* $\mathbb{H}$. *On the other hand, if we consider the mapping*

$$s(z) = \frac{e^{i\frac{\pi}{2}z} - 1}{e^{i\frac{\pi}{2}z} + 1} = i\tan\left(\frac{\pi}{4}z\right), \ z \in \mathbb{S} = (-1, 1) \times \mathbb{R},$$

which establishes a conformal isomorphism between the strip $\mathbb{S}$ *and the unit disk* $\mathbb{D}$ *(the mapping* s^{-1} *could be considered as a covering mapping), then it is straightforward to compute that*

$$\lambda_{\mathbb{S}}(z) = \frac{4|s'(z)|^2}{(1 - |s(z)|^2)^2} = \left(\frac{\pi}{2(|\cos(\frac{\pi}{4}z)|^2 - |\sin(\frac{\pi}{4}z)|^2)}\right)^2 = \left(\frac{\pi}{2}\right)^2 \frac{1}{\cos^2\left(\frac{\pi}{2}\operatorname{Re} z\right)},$$

for every $z \in \mathbb{S}$, *so the hyperbolic metric on* $\mathbb{S}$ *is given by* $ds^2 = \left(\frac{\pi}{2}\right)^2 \frac{1}{\cos^2\left(\frac{\pi}{2}\operatorname{Re} z\right)}|dz|^2$, $z \in \mathbb{S}$.

In [2] authors formulated and proved the following result that is important in our approach.

Theorem 1.9 ([2]). *Let* Ω *be a simply connected convex domain of the hyperbolic type in the complex plane* $\mathbb{C}$. *If* f *is a harmonic and* k*-quasiconformal mapping of the unit disk* $\mathbb{D}$ *onto* Ω, *then the inequalities*

$$\frac{K+1}{2K} \leqslant \sqrt{\frac{\lambda_{\Omega}(f(z))}{\lambda_{\mathbb{D}}(z)}}|f_z(z)| \leqslant \frac{K+1}{2}$$

hold, for all $z \in \mathbb{D}$, *where* λ_{Ω} *is the hyperbolic metric density function on* Ω. *The estimates are sharp.*

2. Hyperbolic partial derivatives of a C^1 mapping and some estimates

Let $G \subset \mathbb{C}$ be an arbitrary simply connected domain in $\mathbb{C}$, distinct from $\mathbb{C}$, i.e. its boundary in $\mathbb{C}$ contains at least two points. By the Riemann mapping theorem, the domain G is conformally equivalent to the unit disk $\mathbb{D}$. Denote by $g : G \to \mathbb{D}$ the mapping that establishes this conformal isomorphism. Then, it is easy to verify that the corresponding hyperbolic metric on G is given by

$$ds^2 = \lambda_G(w)|dw|^2, \text{ where } \lambda_G(w) = \frac{4|g'(w)|^2}{(1 - |g(w)|^2)^2}, \ w \in G.$$

Observe that if $\tilde{g} : G \to \mathbb{D}$ is another mapping that establishes a conformal isomorphism between G and $\mathbb{D}$, then

$$\frac{|\tilde{g}'(w)|}{1 - |\tilde{g}(w)|^2} = \frac{|g'(w)|}{1 - |g(w)|^2},$$

because $\tilde{g} \circ g^{-1}$ is a conformal automorphism of the disk $\mathbb{D}$. Hence, the hyperbolic metric defined in this way does not depend on the choice of the mapping g. For simplicity, we will always use such a domain G in the following text.

Definition 2.1. *Let* $G, G' \subset \mathbb{C}$ *be simply connected domains, distinct from* $\mathbb{C}$, *and let* $f : G \to G'$ *be a* C^1 *mapping. The hyperbolic partial derivatives of* f, *with respect to* z *and* $\bar{z}$, *in the domain* G *are defined as*

$$\|\partial f\|(z) = \sqrt{\frac{\lambda_{G'}(f(z))}{\lambda_G(z)}}|f_z(z)|, \quad \|\bar{\partial} f\|(z) = \sqrt{\frac{\lambda_{G'}(f(z))}{\lambda_G(z)}}|f_{\bar{z}}(z)|. \tag{6}$$

It follows directly from the classical Schwarz-Pick lemma that if f is an analytic mapping between G and G', then $\|\partial_z f\|(z) \leqslant 1$, for all $z \in G$, with equality holding if and only if f is a conformal isomorphism. Similarly, if $f : G \to G'$ is a C^1 mapping and $h : \mathbb{D} \to G$ is the Riemann mapping, i.e. a one-to-one analytic mapping of the unit disk $\mathbb{D}$ onto the domain G, then

$$\|\partial(f \circ h)\|(\zeta) = \sqrt{\frac{\lambda_{G'}((f \circ h)(\zeta))}{\lambda_{\mathbb{D}}(\zeta)}}|(f \circ h)_\zeta(\zeta)|$$

$$= \sqrt{\frac{\lambda_{G'}(f(z))}{\lambda_G(z)}}\sqrt{\frac{\lambda_G(z)}{\lambda_{\mathbb{D}}(\zeta)}}|f_z(z)||h'(\zeta)|$$

$$= \sqrt{\frac{\lambda_{G'}(f(z))}{\lambda_G(z)}}|f_z(z)|\frac{|(h^{-1})'(z)|(1 - |\zeta|^2)}{1 - |h^{-1}(z)|^2}|h'(\zeta)| = \|\partial f\|(z),$$

because h is a conformal isomorphism. Similarly, we have $\|\bar{\partial}_z(f \circ h)\|(\zeta) = \|\bar{\partial}_z f\|(z)$. Therefore, it suffices to restrict our attention to a mappings $f : \mathbb{D} \to G$, where $G \neq \mathbb{C}$ is a simply connected domain in $\mathbb{C}$.

Our goal now is to utilize estimates of the hyperbolic derivatives to characterize the behavior of harmonic quasiconformal mappings from the unit disk to certain hyperbolic target domains in the complex plane.

Lemma 2.2. *Let $f : \mathbb{D} \to G$, where G is a simply connected hyperbolic domain in $\mathbb{C}$, be a C^1 mapping that preserves orientation. If there exists some $M > 0$ such that $\|\partial f\|(z)(1 + |\mu_f(z)|) \leqslant M$ for every $z \in \mathbb{D}$, then*

$$d_G(f(z_1), f(z_2)) \leqslant M\, d_{\mathbb{D}}(z_1, z_2),$$

for every $z_1, z_2 \in \mathbb{D}$.

Proof. Let z_1 and z_2 be arbitrary points in the disk $\mathbb{D}$, and let $\gamma : [0, 1] \to \mathbb{D}$ be the geodesic line with respect to the hyperbolic metric in $\mathbb{D}$ connecting z_1 and z_2. Then,

$$d_G(f(z_1), f(z_2)) \leqslant \int_{f\circ\gamma} \sqrt{\lambda_G(w)}|dw|$$

$$\leqslant \int_\gamma \sqrt{\frac{\lambda_G(f(z))}{\lambda_{\mathbb{D}}(z)}}\sqrt{\lambda_{\mathbb{D}}(z)}(|f_z(z)| + |f_{\bar{z}}(z)|)|dz|$$

$$= \int_\gamma \|\partial f\|(z)\sqrt{\lambda_{\mathbb{D}}(z)}(1 + |\mu_f(z)|)|dz|$$

$$\leqslant M \int_\gamma \sqrt{\lambda_{\mathbb{D}}(z)}|dz| = M\, d_{\mathbb{D}}(z_1, z_2),$$

which proves the statement. □

Using a similar argument, namely that $|dw| \geqslant (|f_z(z)| - |f_{\bar{z}}(z)|)|dz|$ holds because f is orientation-preserving, it can be easily shown that the following lemma holds.

Lemma 2.3. *Let $f : \mathbb{D} \to G$, where G is a simply connected hyperbolic domain in $\mathbb{C}$, be an orientation-preserving C^1 homeomorphism. If there exists some $m > 0$ such that $\|\partial f\|(z)(1 - |\mu_f(z)|) \geqslant m$ for every $z \in \mathbb{D}$, then*

$$d_G(f(z_1), f(z_2)) \geqslant m\, d_{\mathbb{D}}(z_1, z_2),$$

for every $z_1, z_2 \in \mathbb{D}$.

3. Further results and comments

Among the numerous generalizations of the Schwarz-Pick lemma, we now highlight one of the most elegant and remarkable results, which applies specifically to mappings between Riemann surfaces. This result follows directly from the Schwarz lemma for Kähler manifolds, established by S. T. Yau (see [18]) and further discussed in the excellent article [16].

Theorem 3.1 ([16]). *Let R and S be two Riemann surfaces without boundary, with conformal metrics $ds^2 = \rho(z)|dz|^2$ and $ds^2 = \sigma(w)|dw|^2$, respectively. If the metric $ds^2 = \rho(z)|dz|^2$ on the surface R is complete metric, and if for the corresponding Gaussian curvatures the conditions $K_\rho(z) \geqslant -a_1$ and $K_\sigma(w) \leqslant -a_2$ hold, for some real constants $a_1 \geqslant 0$ and $a_2 > 0$, then for any analytic mapping $F : R \to S$ between these surfaces we have*

$$\sigma(F(z))|F'(z)|^2 \leqslant \frac{a_1}{a_2}\rho(z),$$

for each $z \in R$. In particular, if $a_1 = 0$, then F is a locally constant mapping.

Example 3.2. *Let $\Omega = \mathbb{C} \setminus [0, +\infty)$. Denote by s the regular branch of the multivalued function $S : z \mapsto \sqrt{z}$, in the region Ω, determined by the condition $s(-1) = i$. Then, by the mapping $g(z) = \frac{s(z)-i}{s(z)+i}$, $z \in \Omega$, which is a conformal isomorphism from the region Ω onto the unit disk $\mathbb{D}$, the hyperbolic metric is induced on Ω. Thus, the hyperbolic metric on Ω has the form $ds^2 = \lambda_\Omega(z)|dz|^2 = \frac{|dz|^2}{4|z|\,\mathrm{Im}(s(z))}$, $z \in \Omega$. Consider the mappings $f(z) = Kx + iy$ and $\tilde{f}(z) = \frac{1}{K}x + iy$, $z = x + iy \in \Omega$, $K > 1$. These mappings are obviously k-quasiconformal, with $k = \frac{K-1}{K+1}$, and harmonic in Ω. Obviously, $f(\Omega) = \tilde{f}(\Omega) = \Omega$. It is also not difficult to verify that $f_z(z) = \frac{K+1}{2}$ and $\tilde{f}_z(z) = \frac{K+1}{2K}$ hold, for each $z \in \Omega$. However, a simple inspection (see [2]) reveals that $\|\partial_z f\|(z) > \frac{K+1}{2}$, as well as $\|\partial_z \tilde{f}\|(z) < \frac{K+1}{2K}$, whenever $z = x + iy \in \Omega$ and $x > 0$, $y > 0$.*

The previous example illustrates that, by applying the comment following Definition 2.1, for simply connected domains in the complex plane of hyperbolic type that are not convex, we may anticipate results differing from those presented in [2] (see also Theorem 1.9). So, let $G \subset \mathbb{C}$ be a simply connected region distinct from $\mathbb{C}$. We further examine the properties of the hyperbolic metric on G, demonstrating that the previous statements, with modified constants, also hold in the case of arbitrary hyperbolic domains in $\mathbb{C}$. To this end, we highlight a result that addresses this (see [4]).

Lemma 3.3. *Let $G \subset \mathbb{C}$ be a simply connected region distinct from $\mathbb{C}$. Then, for the density function of the hyperbolic metric λ_G, that is defined on G, the following inequality holds:*

$$\left|(\log \lambda_G)_{zz}(z)\right| \leqslant \frac{16 + 9\sqrt{3}}{8}\lambda_G(z), \ z \in G. \tag{7}$$

Now we are ready to state and to prove the main result of this paper.

Theorem 3.4. *Let f be a k-quasiconformal harmonic mapping of the unit disk $\mathbb{D}$ onto G, where $G \subset \mathbb{C}$ is an arbitrary simply connected region distinct from $\mathbb{C}$. If the conformal metric $ds^2 = \lambda_G(f(z))|f_z(z)|^2|dz|^2$ is complete on $\mathbb{D}$, then $\|\partial f\|(z) \geqslant \left(1 + k^2 + \frac{16+9\sqrt{3}}{2}k\right)^{-1/2}$, for each $z \in \mathbb{D}$. Additionally, $d_G(f(z_1), f(z_2)) \geqslant m(k)\, d_{\mathbb{D}}(z_1, z_2)$, for each $z \in \mathbb{D}$, meaning the mapping f is $m(k)$ co-Lipschitz mapping with respect to the hyperbolic metric, where $m(k) = \dfrac{1-k}{\sqrt{1 + k^2 + \frac{16+9\sqrt{3}}{2}k}}$.*

Proof. For the density of the conformal metric $\sigma(z) = \lambda_G(f(z))|f_z(z)|^2$, $z \in \mathbb{D}$, we find that

$$K_\sigma(z) = -\left(1 + |\mu_f(z)|^2 + 4\mathrm{Re}\left(\frac{(\log \lambda_G)_{ww}(f(z))}{\lambda_G(f(z))}\frac{f_{\bar{z}}(z)}{\overline{f_z(z)}}\right)\right), \tag{8}$$

for each $z \in \mathbb{D}$. Therefore, by applying inequality (7), we conclude that

$$\left| Re\left(\frac{(\log \lambda_G)_{ww}(f(z))}{\lambda_G(f(z))} \frac{f_{\bar{z}}(z)}{\overline{f_z(z)}} \right) \right| \leqslant \frac{16 + 9\sqrt{3}}{8} |\mu_f(z)|, \tag{9}$$

for each $z \in \mathbb{D}$. Thus, using (8), we obtain

$$K_\sigma(z) \geqslant -\left(1 + |\mu_f(z)|^2 + \frac{16 + 9\sqrt{3}}{2} |\mu_f(z)| \right) \geqslant -\left(1 + k^2 + \frac{16 + 9\sqrt{3}}{2} k \right), \tag{10}$$

for every $z \in \mathbb{D}$. Finally, since the density $\sigma(z) = \lambda_G(f(z))|f_z(z)|^2$, $z \in \mathbb{D}$, induces a complete metric on $\mathbb{D}$, applying Theorem 3.1 to the identity mapping and the corresponding metric densities, we find that

$$\lambda_{\mathbb{D}}(z) \leqslant (1 + k^2 + \frac{16 + 9\sqrt{3}}{2} k) \lambda_G(f(z))|f_z(z)|^2, \; z \in \mathbb{D}. \tag{11}$$

Consequently, when we calculate the corresponding hyperbolic derivative,

$$\|\partial f\|(z) = \sqrt{\frac{\lambda_G(f(z))}{\lambda_{\mathbb{D}}(z)}} |f_z(z)| \geqslant \left(1 + k^2 + \frac{16 + 9\sqrt{3}}{2} k \right)^{-1/2}, \tag{12}$$

for each $z \in \mathbb{D}$. The rest is trivial consequence of the Lemma 2.3 and the inequality $|\mu_f(z)| \leqslant k$, $z \in \mathbb{D}$. □

Remark 3.5. *Determining when equality in inequality (10) holds, and whether the best lower bound for the curvature K_σ is always negative, is challenging. However, K_σ cannot be non-negative on $\mathbb{D}$. If it were, by Theorem 3.1, if we consider identity map $F : \mathbb{D} \ni z \mapsto w = F(z) = z \in \mathbb{D}$ and chose conformal metrics $ds^2 = \sigma(z)|dz|^2 = \lambda_G(f(z))|f_z(z)|^2|dz|^2$ and $ds^2 = \lambda_{\mathbb{D}}(w)|dw|^2$, where $z, w \in \mathbb{D}$, than, trivially we will obtain that F is constant, which is not true. Thus,* $\inf\{K_\sigma(z) : z \in \mathbb{D}\} < 0$.

It would be of particular interest, under the same conditions, to determine the corresponding Lipschitz constant for the hyperbolic metric using an alternative approach. The result in [6], obtained by the author and to be elaborated in a forthcoming paper, provides such a constant only for small values of $0 \leqslant k < k_0 \approx 0.06331$. Fundamentally, these results represent variations of the Schwarz-Pick type theorems adapted to surfaces equipped with a conformal metric of non-positive Gaussian curvature. A significantly more challenging problem, however, would be to derive appropriate bi-Lipschitz constants in the Euclidean case. Since the Euclidean metric has Gaussian curvature identically equal to zero, this case lies beyond the scope of the present work.

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