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Study of Ricci and Riemann soliton structures on concircularly ϕ-recurrent Sasakian manifolds

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Abstract. The object of this paper is to study some properties of Ricci soliton structures on concircularly ϕ -recurrent Sasakian manifolds. Initially, it is proved that a concircularly ϕ - recurrent Sasakian manifold is an Einstein manifold and as a consequence of this, the Weyl conformal curvature tensor satisfies $W(\xi, X)Y = 0$. Further, the characterization of the vector field admitting Ricci and Riemann soliton have been studied. Additionally, the three-dimensional locally concircularly ϕ-recurrent Sasakian manifolds have been considered with an example and also it has been shown that such a manifold admitting almost Ricci soliton reduces to Ricci soliton.

1. Introduction

A concircular transformation in an n-dimensional Riemannian manifold M which maps every geodesic circle in M to another geodesic circle. In this context, a geodesic circle [12] is defined as a curve whose first curvature is constant and the second curvature is identically zero. The geometry of concircular transformations, i.e. the concircular geometry provides a more general framework for studying transformation than the conformal geometry or inversive geometry of the Euclidean space in the sense that change of metric and preservation is focused on the geodesic circles rather than circle diffeomorphism. One important invariant in concircular transformation is the concircular curvature tensor.

Definition 1.1. *The concircular curvature tensor C in a Riemannian manifold (M^{2n+1}, g) is given by [27]*

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y] \tag{1}$$

where R is the Riemann curvature tensor and r is the scalar curvature.

Many authors ([5], [22], [21]) have studied the notion of local symmetry of a Riemannian manifold to a different extent. As a weaker version of local symmetry, T. Takahashi [23] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry, De et al. [7] introduced the notion of ϕ-recurrent Sasakian manifold. Since then several works have been done by many authors ([18], [1],[20],[8], [9]) on different types of Riemannian manifolds.

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The notion of Ricci flow was introduced by [13] in 1982 to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold. On the manifold M, a Ricci soliton is a triple (g, V, λ) defined by

$$£_V g + 2S = 2\lambda g \tag{2}$$

where $£_V$ is the Lie derivative along the vector field V, called the potential vector field V, g is the Riemannian metric, S is a Ricci tensor, and λ is a constant. A Ricci soliton is a generalization of an Einstein metric. It will be called *Shrinking*, *steady* or *expanding* according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. Otherwise, it will be called indefinite. When the vector field V is the gradient of a smooth function $f : M^n \to R$, then the manifold will be called gradient Ricci Soliton.

Ricci solitons have been studied by several authors such as ([17], [6], [16], [10], [11]) and many others. Recently, almost Ricci soliton was introduced by Pigola et.al [19], where they modified the definition of Ricci soliton by adding the condition on the parameter λ to be a variable in (1).
A triplet (g, X, λ) on Riemannian manifold (M^n, g) is called a Riemann soliton [15] if there exists a real constant λ such that

$$\frac{1}{2} £_X g \wedge g + R = \lambda G \tag{3}$$

where $G = \frac{1}{2} g \wedge g$, $£_X$ denotes the Lie derivative operator in the direction of the vector field X, and $\wedge$ is the Kulkarni Nomizu product.

A vector field φ on Riemannian manifold (M^n, g) is called a $\varphi(Ric)$ vector field if it satisfies [14]

$$\nabla_X \varphi = \mu QX \tag{4}$$

where μ is a constant and Q is the Ricci operator defined by $Ric(X, Y) = g(QX, Y)$.

Motivated by the above studies, in this paper, we study Ricci soliton structures in a concircularly ϕ-recurrent Sasakian manifold. The paper is organized as follows: In sections 2 and 3, we give a brief introduction to concircularly ϕ-recurrent Sasakian manifold and some features of it. Sections 4 and 5 deal with the study of Ricci Soliton and Riemann Soliton in concircularly ϕ-recurrent Sasakian manifolds. In section 6, we study 3-dimensional locally concircularly ϕ-recurrent Sasakian manifolds, and in section 7, we show that an almost Ricci soliton reduces to a Ricci soliton in a concircularly ϕ-recurrent Saskian manifold. Finally, in section 8, we constructed an example to show the existence of a three-dimensional concircularly ϕ-recurrent Sasakian manifold.

2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold with the structure (ϕ, ξ, η, g), Then the following relations hold [2]:

$$\phi^2 X = -X + \eta(X)\xi, \tag{5}$$

$$a)\ \eta(\xi) = 1, \quad b)\ g(X, \xi) = \eta(X), \quad c)\ \eta(\phi X) = 0, \tag{6}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$

$$R(\xi, X)Y = (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{8}$$

$$a)\, \nabla_X \xi = -\phi X \quad b)\, (\nabla_X \eta)(Y) = g(X, \phi Y), \tag{9}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{11}$$

$$\eta(R(X, Y)Z = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{12}$$

$$S(X, \xi) = 2n\, \eta(X) \tag{13}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n\, \eta(X)\eta(Y) \tag{14}$$

for all vector fields X, Y, Z, where ∇ denotes the operator of covariant differentiation with respect to g. ϕ is a skew-symmetric tensor field of type (1,1) and ξ is called the characteristic vector field. S is the Ricci tensor of type (0,2) and R is the Riemannian curvature tensor of the manifold.

Definition 2.1 *A Sasakian manifold is said to be a locally ϕ-symmetric manifold if*

$$\phi^2((\nabla_W R)(X, Y)Z) = 0 \tag{15}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.2 *A Sasakian manifold is said to be a locally concircularly ϕ-symmetric manifold if*

$$\phi^2((\nabla_W C)(X, Y)Z) = 0 \tag{16}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.3 *A Sasakian manifold is said to be a concircularly ϕ-recurrent Sasakian manifold if there exists a non-zero 1-form A such that*

$$\phi^2((\nabla_W C)(X, Y)Z) = A(W)C(X, Y)Z \tag{17}$$

for arbitrary vector fields X,Y,Z,W.

If the 1-form A vanishes, then the manifold reduces to a ϕ-symmetric manifold.

3. Concircularly ϕ-recurrent Sasakian manifold

Let us consider a concircularly ϕ-recurrent Sasakian manifold. Then by virtue of (5) and (17), we get

$$-(\nabla_W C)((X, Y)Z + \eta((\nabla_W C)(X, Y)Z)\xi = A(W)C(X, Y)Z \tag{18}$$

from which it follows that

$$-g(\nabla_W C)((X, Y)Z, U) + \eta((\nabla_W C)(X, Y)Z)\eta(U) = A(U)g(C(X, Y)Z, U) \tag{19}$$

Let $\{e_i\}$, i=1,2,...,2n+1 be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (19) and taking summation over i, $1 \leq i \leq 2n + 1$ we get

$$(\nabla_W S)(Y, Z) = \frac{dr(W)}{(2n+1)} g(Y, Z) - \frac{dr(W)}{2n(2n+1)}[g(Y, Z) - \eta(Y)\eta(Z)] + A(W)[S(Y, Z) - \frac{r}{(2n+1)} g(Y, Z)] \tag{20}$$

Replacing Z=ξ in (20) and using (6) and (13), we have

$$(\nabla_W S)(Y, \xi) = \frac{dr(W)}{(2n+1)} \eta(Y) + A(W)[2n - \frac{r}{(2n+1)}]\eta(Y) \tag{21}$$

Now we have

$$(\nabla_W S)(Y, \xi) = (\nabla_W S)(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi) \tag{22}$$

From equations (9) and (13), we have

$$(\nabla_W S)(Y, \xi) = -2ng(Y, \phi W) + S(Y, \phi W) \tag{23}$$

In view of (21) and (23), we obtain

$$S(Y, \phi W) = 2ng(Y, \phi W) + \frac{dr(W)}{(2n+1)}\eta(Y) + A(W)[2n - \frac{r}{(2n+1)}]\eta(Y) \tag{24}$$

Replacing Y by ϕY in (24) and using (6), we have

$$S(\phi Y, \phi W) = 2ng(\phi Y, \phi W) \tag{25}$$

Using (7) and (14) in (25), we have

$$S(Y, W) = 2ng(Y, W) \ \ \forall \, Y, W \tag{26}$$

Hence we can state the following theorem:

Theorem 3.1. *A concircularly ϕ - recurrent Sasakian manifold is an Einstein manifold.*

Weyl ([25], [26]) constructed a generalized curvature tensor on Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric. The Weyl conformal curvature tensor is defined by

$$\begin{aligned} W(X, Y)Z = R(X, Y)Z - \frac{1}{(2n-1)}[g(QY, Z)X - g(QX, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ + \frac{r}{(2n)(2n-1)}[g(Y, Z)X - g(X, Z)Y] \end{aligned} \tag{27}$$

for X, Y, Z ∈ TM, where TM is the Lie algebra of differentiable vector fields in (M^{2n+1}, g). R and r are the Riemann curvature tensor and scalar curvature of M respectively and Q is the Ricci operator satisfying the relation $S(X, Y) = g(QX, Y)$.

From the above definition, it can be seen that

$$(div\, W)(X, Y)Z = (\frac{2n-2}{2n-1})[\{(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\} - \frac{1}{4n}\{g(Y, Z)dr(X) - g(X, Z)dr(Y)\}] \tag{28}$$

Since the manifold becomes Einstein, therefore div W = 0, where W denotes the Weyl conformal curvature tensor.
In [4], authors proved that if a Sasakian manifold satisfies div W = 0, then $W(\xi, X)Y = 0$.
Thus we obtain the following:
Corollary 3.1. *In a concircularly ϕ-recurrent Sasakian manifold, the Weyl conformal curvature tensor satisfies $W(\xi, X)Y = 0$.*

4. Ricci Soliton

Suppose that a concircularly ϕ-recurrent Sasakian manifold admits a Ricci soliton (g, ξ, λ).
Then (2) holds and using Theorem 3.1, we have

$$(\pounds_\xi g)(X, Y) + 4ng(X, Y) = 2\lambda g(X, Y) \tag{29}$$

Substituting $Y = \xi$ in (29), we get

$$g(\nabla_{\xi}\xi, X) + 4n\eta(X) = 2\lambda\eta(X) \tag{30}$$

Putting X=ξ in (30), we obtain

$$\lambda = 2n \tag{31}$$

Thus from (30) we have,

$$g(\nabla_{\xi}\xi, X) = 0 \quad \forall X. \tag{32}$$

which implies $\nabla_{\xi}\xi$= 0, the integral curves of the vector field ξ are geodesic. Therefore we have the following proposition:

Proposition 4.1. *Let (M^{2n+1}, g) be a concircularly ϕ-recurrent Sasakian manifold with a Ricci Soliton (g, ξ, λ) such that the vector field ξ is the characteristic vector field of M. Then the integral curves of ξ are geodesic on M and the soliton is expanding.*

Next, we consider the vector field ξ to be a $\xi(Ric)$ vector field, then from the Lie derivative and (4) we have,

$$(\pounds_{\xi}g)(X, Y) = g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X) = 2\mu Ric(X, Y) \tag{33}$$

Combining (2) and (33), we obtain

$$(\mu + 1)Ric(X, Y) = \lambda g(X, Y) \tag{34}$$

Putting $X = \xi$ in (34), we obtain

$$(\mu + 1)Ric(\xi, Y) = \lambda\eta(Y) \tag{35}$$

Substituting $Y = \xi$ in (35), we have

$$2n(\mu + 1) = \lambda \tag{36}$$

From (36),
if μ = -1, then $\lambda = 0$ which implies the Ricci soliton is steady.
If μ > -1, then $\lambda > 0$ which implies the Ricci soliton is expanding.
If μ < -1, then $\lambda < 0$ which implies the Ricci soliton is shrinking.

So we can state the following theorem:

Theorem 4.2 *Let (M^{2n+1}, g) be a concircularly ϕ-recurrent Sasakian manifold with a Ricci Soliton (g,ξ,λ) and the vector field ξ is a $\xi(Ric)$ vector field. Then the Ricci soliton is steady, expanding or shrinking according to whether μ = -1, 1 or < 1.*

5. Riemann Soliton

The Riemann soliton equation in (3) can be expressed as

$$\begin{aligned} 2R(X, Y, Z, W) + \{g(X, W)(\pounds_{\xi}g)(Y, Z) + g(Y, Z)(\pounds_{\xi}g)(X, W) - g(X, Z)(\pounds_{\xi}g)(Y, W) - g(Y, W)(\pounds_{\xi}g)(X, Z)\} \\ = 2\lambda[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)] \end{aligned} \tag{37}$$

Contracting (37) over X and W, we obtain

$$(\pounds_\xi g)(Y,Z) + \frac{2}{(2n-1)}S(Y,Z) - \frac{2}{(2n-1)}[2n\lambda - div\,\xi]g(Y,Z) = 0 \tag{38}$$

Putting $Y = \xi$ in (38),

$$(\pounds_\xi g)(\xi,Z) + \frac{2}{(2n-1)}S(\xi,Z) - \frac{2}{(2n-1)}[2n\lambda - div\,\xi]g(\xi,Z) = 0 \tag{39}$$

Now using Theorem 3.1 and (39), we have

$$g(\nabla_\xi \xi, Z) = \frac{2}{(2n-1)}[2n(\lambda-1) - div\,\xi] \tag{40}$$

Replacing Z by ξ in (40), it becomes

$$div\,\xi = 2n(\lambda-1) \tag{41}$$

Substituting (41) in (40), we find

$$g(\nabla_\xi \xi, Z) = 0 \;\; for\, all\, Z \tag{42}$$

Thus we have the following theorem:

Theorem 5.1. *In a concircularly ϕ-recurrent Sasakian manifold with Riemann Soliton $(g,\xi.\lambda)$, $div\,\xi = 2n(\lambda-1)$ and the integral curves of ξ are geodesic.*

6. On a 3-dimensional locally concircular ϕ-recurrent Sasakian manifolds

The curvature tensor in a three-dimensional Sasakian manifold has the following form [3]

$$\begin{aligned} R(X,Y)Z = &\frac{(r-4)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &+ \frac{(6-r)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \end{aligned} \tag{43}$$

Taking covariant differentiation of (43), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z = &\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &- \frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &+ \frac{(6-r)}{2}[g(Y,Z)(\nabla_W\eta)(X)\xi + g(Y,Z)\eta(X)(\nabla_W\xi) - g(X,Z)(\nabla_W\eta)(Y)\xi - g(X,Z)\eta(Y)(\nabla_W\xi) \\ &+ (\nabla_W\eta)(Y)\eta(Z)X + (\nabla_W\eta)(Z)\eta(Y)X - (\nabla_W\eta)(X)\eta(Z)Y - (\nabla_W\eta)(Z)\eta(X)Y] \end{aligned} \tag{44}$$

Taking X,Y,Z,W orthogonal to ξ, and using (9), we have

$$(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{(6-r)}{2}[g(Y,Z)g(X,\phi W)\xi - g(X,Z)g(W,\phi Y)\xi] \tag{45}$$

From (45), it follows that

$$\phi^2(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y] \tag{46}$$

Now taking X, Y, Z, W orthogonal to ξ and using (5) and (6), we obtain

$$\phi^2(\nabla_W R)(X,Y)Z = -\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \tag{47}$$

Now differentiating covariantly (1) with respect to W (for n = 3), we get

$$(\nabla_W C)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y] \tag{48}$$

Applying ϕ^2 to both sides of (48), we have

$$\phi^2(\nabla_W C)(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z - \frac{dr(W)}{6}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y] \tag{49}$$

Using (17), (47) and (5) in (49), we obtain

$$\begin{aligned} A(W)C(X,Y)Z = &-\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &- \frac{dr(W)}{6}[-g(Y,Z)X + g(X,Z)\eta(X)\xi + g(X,Z)Y - g(X,Z)\eta(Y)\xi] \end{aligned} \tag{50}$$

Taking X, Y, Z, W orthogonal to ξ in (50), we get

$$C(X,Y)Z = -\frac{dr(W)}{3A(W)}[g(Y,Z)X - g(X,Z)Y] \tag{51}$$

Putting W = $\{e_i\}$ in (51), where $\{e_i\}$, i = 1, 2, 3 is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \le i \le 3$, we obtain

$$C(X,Y)Z = -\frac{dr(e_i)}{3A(e_i)}[g(Y,Z)X - g(X,Z)Y] \tag{52}$$

Using (1) in (52), we have

$$R(X,Y)Z = \lambda_1[g(Y,Z)X - g(X,Z)Y] \tag{53}$$

where $\lambda_1 = (\frac{r}{6} - \frac{dr(e_i)}{3A(e_i)})$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ_1 will be a constant on the manifold. Therefore, M^3 is of constant curvature λ_1.

Thus we can state the following theorem:

Theorem 6.1. *A 3-dimensional locally concircularly $\phi-$recurrent Sasakian manifold is of constant curvature.*

If the scalar curvature is constant then from (47) and (49), we can say that
$\phi^2(\nabla_W C)(X,Y)Z = 0$

i.e. A locally three-dimensional concircularly ϕ - recurrent Sasakian manifold becomes ϕ - symmetric.

It is known from Watanabe's result [24] that a three dimensional Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant. Hence we can state the following theorem:

Theorem 6.2. *A 3-dimensional Sasakian manifold is locally concircualrly ϕ-recurrent if and only if it is locally ϕ-symmetric.*

7. Almost Ricci Soliton

Suppose that a 3-dimensional locally concircularly ϕ-recurrent Sasakian manifold admits an almost Ricci Soliton defined by (2) where λ is a smooth function. Using Theorem 6.1 and equation (53) we have,

$$S(Y,Z) = \lambda_1 g(Y,Z) \tag{54}$$

where λ_1 is a constant.

From equation (2) and (54) we get,

$$(\pounds_V g)(Y,Z) = 2(\lambda - \lambda_1)g(Y,Z) \tag{55}$$

Differentiating the above equation with respect to X and using (6) we have,

$$(\nabla_X \pounds_V g) = 2d\lambda(X)g(Y,Z) \tag{56}$$

Now we recall the following formula [28],

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y)Z) - g((\pounds_V \nabla)(X,Z)Y) \tag{57}$$

for any vector fields X, Y, and Z on M. From this we can easily deduce

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y)Z) + g((\pounds_V \nabla)(X,Z)Y) \tag{58}$$

Since $\pounds_V \nabla$ is a symmetric tensor of type (1, 2) it follows from (58) that

$$g((\pounds_V \nabla)(X,Y)Z = \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) - \frac{1}{2}(\nabla_Z \pounds_V g)(X,Y) \tag{59}$$

Using (56) in (59), we have

$$g((\pounds_V \nabla)(X,Y)Z) = d\lambda(X)g(Y,Z) + d\lambda(Y)g(X,Z) - d\lambda(Z)g(X,Y) \tag{60}$$

Substituting X = Y = e_i in the above equation and removing Z from both sides, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking Σ_i, $1 \leq i \leq 3$, we get

$$(\pounds_V \nabla)(e_i, e_i) = -D\lambda \tag{61}$$

where $d\lambda(X) = g(D\lambda, X)$, D denotes the gradient operator with respect to g. Now differentiating (2) and using (58) we determine

$$g(\pounds_V \nabla)(X,Y)Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \tag{62}$$

Taking X = Y = e_i (where $\{e_i\}$ is an orthonormal frame) in (62) and summing over i, we obtain

$$(\pounds_V \nabla)(e_i, e_i) = 0 \tag{63}$$

Combining (61) and (63), we have

$$D\lambda = 0 \tag{64}$$

This implies that λ is constant which leads to the following theorem:

Theorem 7.1. *An almost Ricci soliton on a 3-dimensional locally concircularly ϕ - recurrent Sasakian manifold reduces to Ricci soliton.*

8. Example

In this section we give an example to show the existence of a three dimensional concircularly ϕ - recurrent Sasakian manifold.

We consider the three dimensional manifold $M = \{(x, y, z) \in R^3, (x, y, z) \neq 0\}$ are standard coordinates of R^3.

The vector fields

$$e_1 = \frac{\partial}{\partial y}, \ e_2 = -\frac{\partial}{\partial z}, \ e_3 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$
Further, let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = -e_2, \ \phi(e_2) = e_1, \ \phi(e_3) = 0$$

So, using the lineartiy of ϕ and g, we have

$$\eta(e_3) = 1,$$
$$\phi^2 Z = -Z + \eta(Z)\, e_3,$$
$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any Z, W $\in \chi(M)$.
Thus for $\xi = e_3$, the structure $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.
Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of g. Then we have,

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1, \ [e_2, e_3] = -e_2.$$

The Riemannian connection of the metric g which is known as Koszul's formula is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y) \tag{65}$$

From (65) we can have,

$$\begin{aligned} &\nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = e_3, \quad \nabla_{e_1} e_3 = -e_1 \\ &\nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_3 = -e_2 \\ &\nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_3 = 0 \end{aligned} \tag{66}$$

From (66), we can easily verify that the structure (ϕ, ξ, η, g) satisfies the relation (8). Hence M(ϕ, ξ, η, g) is a three dimensional Sasakian manifold. It is known that,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{67}$$

Using (67) and (66) the components of Riemann curvature tensor are given by,

$$\begin{aligned} &R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1 \\ &R(e_1, e_2)e_2 = -e_1, \quad R(e_2, e_3)e_2 = e_2, \quad R(e_1, e_3)e_2 = 0 \\ &R(e_1, e_2)e_1 = e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = e_3 \end{aligned} \tag{68}$$

From the definition of the Ricci tensor in three dimensional manifold we get,

$$S(X,Y) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i) \tag{69}$$

From the components of the curvature tensor and (69), we obtain the following results,

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2 \ \ and \ \ S(e_i, e_j) = 0 \ \ for\ i \neq j$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6$$

Hence, we obtain the scalar curvature is constant.

It is well known from Watanabe's result [24] that a three dimensional Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant.

Hence the manifold is locally ϕ - symmetric. Thus Theorem 6.2 is verified.

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