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Embedding tensors on Lie triple systems

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Abstract. In this paper, first we introduce the notion of embedding tensors on Lie triple systems and show that embedding tensors induce naturally 3-Leibniz algebras. Next, we construct a Lie 3-algebra whose Maurer-Cartan elements are embedding tensors. Then, we have the twisted L_∞-algebra that governs deformations of embedding tensors. Following this, we establish the cohomology of an embedding tensor on a Lie triple system and realize it as the cohomology of the induced 3-Leibniz algebra with coefficients in a suitable representation. As applications, we consider infinitesimal and finite order deformations of an embedding tensor on a Lie triple system from a cohomological viewpoint.

1. Introduction

Lie triple systems, a class of mathematical structures, occupy a central role across various fields of mathematics and theoretical physics. This is because a Lie triple system can be considered a subspace of a Lie algebra that is closed under the ternary product. The notion of Lie triple systems first appeared in Cartan's work [4] on totally geodesic submanifolds and symmetric spaces. These were subsequently explored from an algebraic perspective by Jacobson in [17, 18]. Notably, the tangent space of a symmetric space forms a Lie triple system. These systems have found significant applications in physics, particularly in the realms of elementary particle theory and quantum mechanics, as well as in the numerical analysis of differential equations [26]. Due to the importance of Lie triple systems, Lister constructed a structure theory of Lie triple systems in [23]. The representation and cohomology theory of Lie triple systems were established in [42], and a description of the free Lie triple system was presented in [27]. See also [7, 8, 14, 15, 21, 22, 34, 38, 40, 41, 43] for some interesting related about Lie triple systems.

The concept of embedding tensors has been extensively researched and applied within the realms of mathematics and physics. The advent of embedding tensors can be traced back to the study of supergravity theories and higher gauge theories [28]. In [2], both the N = 8 supersymmetric gauge theories and the Bagger-Lambert theory of multiple M2-branes were examined using embedding tensors. Kotov and Strobl [20] constructed tensor hierarchies that demonstrate the potential mathematical properties of embedding tensors from a physics perspective. In particular, an embedding tensor on a Lie algebra can give rise to a Leibniz algebra structure. From a mathematical point of view, embedding tensors are also known as averaging operators [1, 31]. Averaging operators are intimately connected with operad theory and double

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algebras theory [29, 30]. In the last century, averaging operators have been mostly studied on various function spaces and Banach algebras [24, 25]. See [3, 9–11] for some further research on the averaging operators. Recently, the controlling algebras of embedding tensors on Lie algebras, their cohomology and deformations were investigated in [32]. Embedding tensors on Lie_∞-algebras with respect to Lie_∞-actions have also been developed in [6]. Besides, the results of [32] have been extended to embedding tensors on Hom-Lie algebras [12], 3-Lie algebras [16] and 3-Hom-Lie algebras [37].

Due to the importance of embedding tensors and Lie triple systems, the aim of this paper is to extend the mathematical study of embedding tensors to the context of Lie triple systems. More precisely, in Section 2, we introduce the notion of embedding tensors on Lie triple systems, which naturally induce 3-Leibniz algebra structures (See Proposition 2.9). We show that embedding tensors can be characterized by graphs of the hemisemidirect product 3-Leibniz algebra. In Section 3, we construct an L_∞-algebra whose Maurer-Cartan elements are embedding tensors on Lie triple systems (See Theorem 3.11). We also construct the L_∞-algebra that governs deformations of embedding tensors (See Theorem 3.12). Following this, we establish a cohomology theory of embedding tensors on Lie triple systems and realize it as the cohomology of the induced 3-Leibniz algebra with coefficients in a suitable representation. In Section 4, we study infinitesimal deformations and extendability of order n deformations of an embedding tensor using the established cohomology theory.

2. Embedding tensors on Lie triple systems

In this section, we introduce the notion of embedding tensors on Lie triple systems. We show that a linear map $E : V \to \mathbb{G}$ is an embedding tensor if and only if the graph of E is a subalgebra of the hemisemidirect product 3-Leibniz algebra. Consequently, an embedding tensor E induces a 3-Leibniz algebra structure on V. Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All algebras and vector spaces are considered over $\mathbb{K}$.

Definition 2.1. *[17] (i) A Lie triple system (LTS in short) is a vector space $\mathbb{G}$ together with a trilinear operation $[-,-,-]$ on $\mathbb{G}$ satisfying the following equations:*

$$[x,y,z] + [y,x,z] = 0, \tag{2.1}$$

$$[x,y,z] + [z,x,y] + [y,z,x] = 0, \tag{2.2}$$

$$[a,b,[x,y,z]] = [[a,b,x],y,z] + [x,[a,b,y],z] + [x,y,[a,b,z]], \tag{2.3}$$

where $x,y,z,a,b \in \mathbb{G}$. Denote it by $(\mathbb{G},[-,-,-])$.

(ii) A homomorphism between two LTSs $(\mathbb{G}_1,[-,-,-]_1)$ and $(\mathbb{G}_2,[-,-,-]_2)$ is a linear map $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ satisfying

$$\varphi([x,y,z]_1) = [\varphi(x),\varphi(y),\varphi(z)]_2, \ \ \forall\, x,y,z \in \mathbb{G}_1.$$

Remark 2.2. *[42] Let $(\mathbb{G},[-,-])$ be a Lie algebra. Then $(\mathbb{G},[-,-,-])$ is a LTS, where $[x,y,z] = [[x,y],z]$, for all $x,y,z \in \mathbb{G}$. Conversely, it was shown in [43] that if $(\mathbb{G},[-,-,-])$ is a LTS, then $L(\mathbb{G}) = \mathfrak{L}(\mathbb{G},\mathbb{G}) \oplus \mathbb{G}$ with the product*

$$[(\mathfrak{L}(x,y),z),(\mathfrak{L}(a,b),c)] = \Big(\mathfrak{L}([a,b,y],x) - \mathfrak{L}([a,b,x],y) + \mathfrak{L}(z,c), \mathfrak{L}(x,y)c - \mathfrak{L}(a,b)z\Big)$$

is a Lie algebra, where $\mathfrak{L}(\mathbb{G},\mathbb{G}) = \mathrm{span}\{\mathfrak{L}(x,y) \mid x,y \in \mathbb{G}\}$ and $\mathfrak{L}(x,y) = [x,y,-]$.

Definition 2.3. *[42] A representation of a LTS $(\mathbb{G},[-,-,-])$ on a vector space V is a bilinear map $\theta : \mathbb{G}\times\mathbb{G} \to \mathrm{End}(V)$ such that for all $x,y,a,b \in \mathbb{G}$,*

$$\theta(a,b)\theta(x,y) - \theta(y,b)\theta(x,a) - \theta(x,[y,a,b]) + D_\theta(y,a)\theta(x,b) = 0, \tag{2.4}$$

$$\theta(a,b)D_\theta(x,y) - D_\theta(x,y)\theta(a,b) + \theta([x,y,a],b) + \theta(a,[x,y,b]) = 0, \tag{2.5}$$

where the bilinear map $D_\theta : \mathbb{G}\times\mathbb{G} \to \mathrm{End}(V)$ is given by

$$D_\theta(x,y) = \theta(y,x) - \theta(x,y). \tag{2.6}$$

From Eqs. (2.5) *and* (2.6) *we get*

$$D_\theta(a,b)D_\theta(x,y) - D_\theta(x,y)D_\theta(a,b) + D_\theta([x,y,a],b) + D_\theta(a,[x,y,b]) = 0. \tag{2.7}$$

We denote a representation of $\mathbb{G}$ *on* V *by* $(V;\theta)$*.*

For example, given a LTS $(\mathbb{G},[-,-,-])$, there is a natural adjoint representation on itself. The corresponding maps θ and D_θ are given by

$$\theta(x,y)z := [z,x,y] \text{ and}$$

$$D_\theta(x,y)z = \theta(y,x)z - \theta(x,y)z = [x,y,z], \text{ for } x,y,z \in \mathbb{G}.$$

Definition 2.4. *[5, 19] A 3-Leibniz algebra is a vector space* $\mathfrak{g}$ *endowed with a multilinear map* $[-,-,-]_\mathfrak{g} : \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ *satisfying*

$$[a,b,[x,y,z]_\mathfrak{g}]_\mathfrak{g} = [[a,b,x]_\mathfrak{g},y,z]_\mathfrak{g} + [x,[a,b,y]_\mathfrak{g},z]_\mathfrak{g} + [x,y,[a,b,z]_\mathfrak{g}]_\mathfrak{g}, \tag{2.8}$$

where $x,y,z,a,b \in \mathfrak{g}$*.*

Proposition 2.5. *Let* $(V;\theta)$ *be a representation of a LTS* $(\mathbb{G},[-,-,-])$*. Then* $(\mathbb{G}\oplus V,[-,-,-]_\ltimes)$ *is a 3-Leibniz algebra, where*

$$[(a,u),(b,v),(c,w)]_\ltimes = ([a,b,c], D_\theta(a,b)w),$$

for any $(a,u),(b,v),(c,w) \in \mathbb{G}\oplus V$*. This 3-Leibniz algebra is called the hemisemidirect product 3-Leibniz algebra and denoted by* $\mathbb{G}\ltimes V$*.*

Proof. For any $(x,u),(y,v),(z,w),(a,s),(b,t) \in \mathbb{G}\oplus V$, by Eqs. (2.3) and (2.7), we have

$$\begin{aligned}
&[[(a,s),(b,t),(x,u)]_\ltimes,(y,v),(z,w)]_\ltimes + [(x,u),[(a,s),(b,t),(y,v)]_\ltimes,(z,w)]_\ltimes\\
&+ [(x,u),(y,v),[(a,s),(b,t),(z,w)]_\ltimes]_\ltimes - [(a,s),(b,t),[(x,u),(y,v),(z,w)]_\ltimes]_\ltimes\\
&= \Big([[a,b,x],y,z], D_\theta([a,b,x],y)w\Big) + \Big([x,[a,b,y],z], D_\theta(x,[a,b,y])w\Big)\\
&\quad + \Big([x,y,[a,b,z]], D_\theta(x,y)D_\theta(a,b)w\Big) - \Big([a,b,[x,y,z]], D_\theta(a,b)D_\theta(x,y)w\Big)\\
&= \Big([[a,b,x],y,z] + [x,[a,b,y],z] + [x,y,[a,b,z]] - [a,b,[x,y,z]],\\
&\quad D_\theta([a,b,x],y)w + D_\theta(x,[a,b,y])w + D_\theta(x,y)D_\theta(a,b)w - D_\theta(a,b)D_\theta(x,y)w\Big)\\
&= 0.
\end{aligned}$$

Therefore, $\mathbb{G}\ltimes V$ is a 3-Leibniz algebra. □

Definition 2.6. *(i) Let* $(V;\theta)$ *be a representation of a LTS* $(\mathbb{G},[-,-,-])$*. A linear map* $E : V \to \mathbb{G}$ *is called an embedding tensor on the LTS* $(\mathbb{G},[-,-,-])$ *with respect to the representation* $(V;\theta)$ *if* E *satisfies*

$$[Eu,Ev,Ew] = ED_\theta(Eu,Ev)w, \tag{2.9}$$

for $u,v,w \in V$*.*

(ii) A Lie 3-Leibniz triple is a triple $(\mathbb{G},V,E)$*, where* $(\mathbb{G},[-,-,-])$ *is a LTS,* $(V;\theta)$ *is a representation of* $(\mathbb{G},[-,-,-])$ *and* $E : V \to \mathbb{G}$ *is an embedding tensor on a LTS* $(\mathbb{G},[-,-,-])$ *with respect to the representation* $(V;\theta)$*.*

Remark 2.7. *Given the notion of Lie 3-Leibniz triples, it is natural to establish a cohomology and controlling algebra of Lie 3-Leibniz triples, and then deformations and extensions can be explored consequently. We will examine this problem in the future and we are also looking forward to new studies in this direction.*

Proposition 2.8. *A linear map $E : V \to \mathbb{G}$ is an embedding tensor on a LTS $(\mathbb{G}, [-,-,-])$ with respect to the representation $(V;\theta)$ if and only if the graph $Gr(E) = \{(Eu,u) \mid u \in V\}$ is a subalgebra of the hemisemidirect product 3-Leibniz algebra $\mathbb{G} \ltimes V$.*

Proof. Let $E : V \to \mathbb{G}$ be a linear map. For any $u, v, w \in V$, we have

$$[(Eu,u),(Ev,v),(Ew,w)]_{\ltimes} = \Big([Eu,Ev,Ew], D_\theta(Eu,Ev)w\Big).$$

Therefore, the graph $Gr(E) = \{(Eu,u) \mid u \in V\}$ is a subalgebra of the hemisemidirect product 3-Leibniz algebra $\mathbb{G} \ltimes V$ if and only if E satisfies Eq. (2.9), which implies that E is an embedding tensor. □

The algebraic structure underlying an embedding tensor on the LTS $(\mathbb{G}, [-,-,-])$ is a 3-Leibniz algebra.

Proposition 2.9. *Let $E : V \to \mathbb{G}$ be an embedding tensor on the LTS $(\mathbb{G}, [-,-,-])$ with respect to the representation $(V;\theta)$. Define a linear map $[-,-,-]_E : V \times V \times V \to V$ by*

$$[u,v,w]_E = D_\theta(Eu,Ev)w, \tag{2.10}$$

for $u, v, w \in V$. Then $(V, [-,-,-]_E)$ is a 3-Leibniz algebra. Moreover, E is a homomorphism from the 3-Leibniz algebra $(V, [-,-,-]_E)$ to the Lie triple system $(\mathbb{G}, [-,-,-])$.

Proof. For any $u, v, w, s, t \in V$, by Eqs. (2.7) and (2.9), we have

$$\begin{aligned}
&[s,t,[u,v,w]_E]_E - [[s,t,u]_E,v,w]_E - [u,[s,t,v]_E,w]_E - [u,v,[s,t,w]_E]_E\\
&= D_\theta(Es,Et)D_\theta(Eu,Ev)w - D_\theta(ED_\theta(Es,Et)u,Ev)w - D_\theta(Eu,ED_\theta(Es,Et)v)w\\
&\quad - D_\theta(Eu,Ev)D_\theta(Es,Et)w\\
&= D_\theta(Es,Et)D_\theta(Eu,Ev)w - D_\theta([Es,Et,Eu],Ev)w - D_\theta(Eu,[Es,Et,Ev])w\\
&\quad - D_\theta(Eu,Ev)D_\theta(Es,Et)w\\
&= 0.
\end{aligned}$$

Therefore, $(V, [-,-,-]_E)$ is a 3-Leibniz algebra. By Eq. (2.9), E is a homomorphism from the 3-Leibniz algebra $(V, [-,-,-]_E)$ to the Lie triple system $(\mathbb{G}, [-,-,-])$. □

Remark 2.10. *Proposition 2.9 establishes a significant relationship between LTSs and 3-Leibniz algebras via the framework of embedding tensors. Despite the distinct concepts and representations characterizing LTSs and 3-Lie algebras, their algebraic structures associated with the embedded tensors are indeed congruent, as both are classified as 3-Leibniz algebras. Refer to [16] about embedding tensors on 3-Lie algebras. Therefore, the findings of this study have the potential to enhance understanding of the structural intricacies of LTSs.*

In the sequel, we give some examples of embedding tensors on LTSs.

Example 2.11. *The identity map* id : $\mathbb{G} \to \mathbb{G}$ *is an embedding tensor on the LTS $(\mathbb{G}, [-,-,-])$ with respect to the adjoint representation $(\mathbb{G};\theta)$.*

Example 2.12. *Let $\mathbb{G}$ be a two-dimensional LTS with a basis $\{\varepsilon_1, \varepsilon_2\}$ and the nonzero multiplication is defined by*

$$[\varepsilon_1,\varepsilon_2,\varepsilon_2] = -[\varepsilon_2,\varepsilon_1,\varepsilon_2] = \varepsilon_1.$$

Then, for $k, k_1 \in \mathbb{K}$, the operator $E = \begin{pmatrix} k & k_1 \\ 0 & k \end{pmatrix}$ is an embedding tensor on the LTS $(\mathbb{G}, [-,-,-])$ with respect to the adjoint representation $(\mathbb{G};\theta)$.

Example 2.13. *Recall that a differential d on a LTS $(\mathbb{G}, [-,-,-])$ is a linear map $d : \mathbb{G} \to \mathbb{G}$ satisfying*

$$d[x,y,z] = [dx,y,z] + [x,dy,z] + [x,y,dz] \ \text{ and } \ d^2 = 0, \text{ for any } \ x,y,z \in \mathbb{G}.$$

Then d is an embedding tensor on the LTS $(\mathbb{G}, [-,-,-])$ with respect to the adjoint representation $(\mathbb{G};\theta)$.

Example 2.14. *Let $(V;\theta)$ be a representation of a LTS $(\mathbb{G},[-,-,-])$. If a linear map $f:V\to\mathbb{G}$ satisfies:*

$$f(\theta(x,f(u))v)=[f(v),x,f(u)],$$

for $x\in\mathbb{G},u,v\in V$, then f is an embedding tensors on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$.

The notion of a strict Lie triple 2-system was introduced in [41].

Example 2.15. *(strict Lie triple 2-systems) A strict Lie triple 2-system is a 2-term graded vector spaces $\mathbb{G}=\mathbb{G}_0\oplus\mathbb{G}_1$ equipped with a linear map $h:\mathbb{G}_1\to\mathbb{G}_0$ and trilinear maps $l_3:\mathbb{G}_i\times\mathbb{G}_j\times\mathbb{G}_k\longrightarrow\mathbb{G}_{i+j+k}$ $(0\le i+j+k\le 1)$, such that for all $x,y,z,x_i\in\mathbb{G}_0$ $(i=1,\cdots,7),u,v\in\mathbb{G}_1$, the following equations are satisfied:*

$$l_3(x,x,y)=0,\ \ l_3(x,x,u)=0,\ \ l_3(u,x,y)+l_3(x,u,y)=0, \tag{2.11}$$
$$hl_3(u,y,z)=l_3(h(u),y,z), \tag{2.12}$$
$$l_3(h(u),v,x)=l_3(u,h(v),x),\ \ l_3(h(u),x,v)=l_3(u,x,h(v)),\ \ l_3(x,h(u),v)=l_3(x,u,h(v)), \tag{2.13}$$
$$l_3(x,y,z)+l_3(y,z,x)+l_3(z,x,y)=0,\ \ l_3(x,y,u)+l_3(y,u,x)+l_3(u,x,y)=0, \tag{2.14}$$
$$l_3(x_1,x_2,l_3(x_3,x_4,x_5))=l_3(x_3,l_3(x_1,x_2,x_4),x_5)+l_3(l_3(x_1,x_2,x_3),x_4,x_5)+l_3(x_3,x_4,l_3(x_1,x_2,x_5)), \tag{2.15}$$
$$l_3(u,x_2,l_3(x_3,x_4,x_5))=l_3(l_3(u,x_2,x_3),x_4,x_5)+l_3(x_3,l_3(u,x_2,x_4),x_5)+l_3(x_3,x_4,l_3(u,x_2,x_5)), \tag{2.16}$$
$$l_3(x_1,u,l_3(x_3,x_4,x_5))=l_3(l_3(x_1,u,x_3),x_4,x_5)+l_3(x_3,l_3(x_1,u,x_4),x_5)+l_3(x_3,x_4,l_3(x_1,u,x_5)), \tag{2.17}$$
$$l_3(x_1,x_2,l_3(u,x_4,x_5))=l_3(l_3(x_1,x_2,u),x_4,x_5)+l_3(u,l_3(x_1,x_2,x_4),x_5)+l_3(u,x_4,l_3(x_1,x_2,x_5)), \tag{2.18}$$
$$l_3(x_1,x_2,l_3(x_3,u,x_5))=l_3(l_3(x_1,x_2,x_3),u,x_5)+l_3(x_3,l_3(x_1,x_2,u),x_5)+l_3(x_3,u,l_3(x_1,x_2,x_5)), \tag{2.19}$$
$$l_3(x_1,x_2,l_3(x_3,x_4,u))=l_3(l_3(x_1,x_2,x_3),x_4,u)+l_3(x_3,l_3(x_1,x_2,x_4),u)+l_3(x_3,x_4,l_3(x_1,x_2,u)). \tag{2.20}$$

Define a linear map $\vartheta:\mathbb{G}_0\times\mathbb{G}_0\to\mathbb{G}_1$ by

$$\vartheta(x,y)u=l_3(u,x,y),$$

then by Eqs. (2.16) *and* (2.18), *$(\mathbb{G}_1;\vartheta)$ is a representation of the LTS $(\mathbb{G}_0,l_3)$. By Eqs.* (2.12) *and* (2.13), *h is an embedding tensor on the LTS $(\mathbb{G}_0,l_3)$ with respect to the representation $(\mathbb{G}_1;\vartheta)$.*

Additionally, as strict Lie triple 2-systems are equivalent to crossed modules of LTSs [41], we naturally have the following example.

Example 2.16. *(crossed module of LTSs) A crossed module of LTSs is a quadruple $\Big((\mathbb{G}_0,[-,-,-]_{\mathbb{G}_0}),(\mathbb{G}_1,[-,-,-]_{\mathbb{G}_1}),h,\theta\Big)$, where $(\mathbb{G}_0,[-,-,-]_{\mathbb{G}_0})$ and $(\mathbb{G}_1,[-,-,-]_{\mathbb{G}_1})$ are LTSs, $h:\mathbb{G}_1\to\mathbb{G}_0$ is a homomorphism of LTSs, and $(\mathbb{G}_1;\theta)$ is a representation of $(\mathbb{G}_0,[-,-,-]_{\mathbb{G}_0})$, such that for all $x,y\in\mathbb{G}_0,u,v,w\in\mathbb{G}_1$, the following equations hold:*

$$\begin{aligned}h(\theta(x,y)u)&=[h(u),x,y]_{\mathbb{G}_0},\\ \theta(h(u),h(v))w&=[w,u,v]_{\mathbb{G}_1},\\ h(\theta(x,h(v))w)&=[h(w),x,h(v)]_{\mathbb{G}_0},\\ h(\theta(h(u),y)w)&=[h(w),h(u),y]_{\mathbb{G}_0}.\end{aligned}$$

Then h is an embedding tensor on the LTS $(\mathbb{G}_0,[-,-,-]_{\mathbb{G}_0})$ with respect to the representation $(\mathbb{G}_1;\theta)$.

3. Maurer-Cartan characterization and cohomology of embedding tensors on Lie triple systems

In this section, we construct a suitable L_∞-algebra, which characterize embedding tensors on LTSs as Maurer-Cartan elements. Then we construct a twisted L_∞-algebra that controls deformations of embedding tensors. Following this, we establish the cohomology of embedding tensors on LTSs and realize it as the cohomology of the induced 3-Leibniz algebra with coefficients in a suitable representation.

Definition 3.1. *[33] An L_∞-algebra is a $\mathbb{Z}$-graded vector space $g = \oplus_{k\in\mathbb{Z}} g^k$ equipped with a collection $(k \geq 1)$ of linear maps $l_k : \otimes^k g \to g$ of degree 1 with the property that, for any homogeneous elements $x_1, x_2, \cdots, x_n \in g$, we have*
(i) (graded symmetry) for every $\sigma \in S_n$,

$$l_n(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = \varepsilon(\sigma) l_n(x_1, x_2, \ldots, x_n).$$

(ii) (generalized Jacobi Identity) for all $n \geq 1$,

$$\sum_{i=1}^{n} \sum_{\sigma\in S_n} \varepsilon(\sigma) l_{n-i+1}(l_n(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(i)}), x_{i+1}, \ldots, x_n) = 0.$$

The notion of a Lie 3-algebra is a special case of L_∞-algebra, in which only the 3-ary bracket is nonzero.

Definition 3.2. *[13, 36] A Lie 3-algebra is a $\mathbb{Z}$-graded vector space $g = \oplus_{k\in\mathbb{Z}} g^k$ equipped with a trilinear bracket $[\![-,-,-]\!]_g : \otimes^3 g \to g$ of degree 1 satisfying the following.*
(i) (graded symmetry) For all homogeneous elements $x_1, x_2, x_3 \in g$,

$$[\![x_1, x_2, x_3]\!]_g = (-1)^{x_1x_2}[\![x_2, x_1, x_3]\!]_g = (-1)^{x_2x_3}[\![x_1, x_3, x_2]\!]_g.$$

(ii) (generalized Jacobi Identity) For all homogeneous elements $x_1, x_2, \cdots, x_5 \in g$,

$$\sum_{\sigma\in S_5} \varepsilon(\sigma)[\![[\![x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]\!]_g, x_4, x_5]\!]_g = 0.$$

Definition 3.3. *[13] (i) A Maurer-Cartan element of an L_∞-algebra $(g = \oplus_{k\in\mathbb{Z}} g^k, \{l_i\}_{i=1}^{\infty})$ is an element $\mathfrak{m} \in g^0$ satisfying the Maurer-Cartan equation,*

$$\sum_{n=1}^{\infty} \frac{1}{n!} l_n(\mathfrak{m}, \ldots, \mathfrak{m}) = 0.$$

(ii) A Maurer-Cartan element of a Lie 3-algebra $(g = \oplus_{k\in\mathbb{Z}} g^k, [\![-,-,-]\!]_g)$ is an element $\mathfrak{m} \in g^0$ satisfying the Maurer-Cartan equation,

$$\frac{1}{3!}[\![\mathfrak{m}, \mathfrak{m}, \mathfrak{m}]\!]_g = 0.$$

Let $\mathfrak{m}$ be a Maurer-Cartan element of a Lie 3-algebra $(g = \oplus_{k\in\mathbb{Z}} g^k, [\![-,-,-]\!]_g)$. For all $k \geq 1$ and $x_1, x_2, x_3 \in g$, define $l_k^{\mathfrak{m}} : \otimes^k g \to g$ of degree 1 by

$$\begin{aligned} l_1^{\mathfrak{m}}(x_1) &= \frac{1}{2}[\![\mathfrak{m}, \mathfrak{m}, x_1]\!]_g, \\ l_2^{\mathfrak{m}}(x_1, x_2) &= [\![\mathfrak{m}, x_1, x_2]\!]_g, \\ l_3^{\mathfrak{m}}(x_1, x_2, x_3) &= [\![x_1, x_2, x_3]\!]_g, \\ l_k^{\mathfrak{m}} &= 0, k \geq 4. \end{aligned}$$

Proposition 3.4. *[13] With the above notations, $(g, l_1^{\mathfrak{m}}, l_2^{\mathfrak{m}}, l_3^{\mathfrak{m}})$ is an L_∞-algebra, obtained from the Lie 3-algebra $(g, [\![-,-,-]\!]_g)$ by twisting with the Maurer-Cartan element $\mathfrak{m}$. Moreover, $\mathfrak{m} + \mathfrak{m}'$ is a Maurer-Cartan element of $(g, [\![-,-,-]\!]_g)$ if and only if $\mathfrak{m}'$ is a Maurer-Cartan element of the twisted L_∞-algebra $(g, l_1^{\mathfrak{m}}, l_2^{\mathfrak{m}}, l_3^{\mathfrak{m}})$.*

In the sequel, we recall Voronov's derived brackets theory [39], which is a useful tool to construct explicit L_∞-algebras.

Definition 3.5. *[39] A V-data consists of a quadruple $(\mathbb{L}, \mathbb{F}, \mathcal{P}, \Delta)$, where*
- *$(\mathbb{L}, [-,-])$ is a graded Lie algebra;*
- *$\mathbb{F}$ is an abelian graded Lie subalgebra of $(\mathbb{L}, [-,-])$;*
- *$\mathcal{P} : \mathbb{L} \to \mathbb{L}$ is a projection, that is, $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$, whose image is $\mathbb{F}$ and kernel is a graded Lie subalgebra of $(\mathbb{L}, [-,-])$;*
- *Δ is an element in $\ker(\mathcal{P})^1$ such that $[\Delta, \Delta] = 0$.*

Proposition 3.6. *[39] Let $(\mathbb{L},\mathbb{F},\mathcal{P},\Delta)$ be a V-data. Then $(\mathbb{F},\{l_i\}_{i=1}^{\infty})$ is an L_∞-algebra, where*

$$l_i(x_1,x_2,\dots,x_i)=\mathcal{P}\underbrace{[\cdots[[}_{i}\Delta,x_1],x_2],\dots,x_i], \tag{3.1}$$

for homogeneous $x_1,x_2,\dots,x_i\in\mathbb{F}$. We call $\{l_i\}_{i=1}^{\infty}$ the higher derived brackets of the V-data $(\mathbb{L},\mathbb{F},\mathcal{P},\Delta)$.

Let g be a vector space. We consider the graded vector space $\mathfrak{C}^*_{\mathrm{NA}}(g,g)=\oplus_{n\geq -1}\mathfrak{C}^{n+1}_{\mathrm{NA}}(g,g)$, where $\mathfrak{C}^0_{\mathrm{NA}}(g,g)=\wedge^2 g$ and $\mathfrak{C}^{n+1}_{\mathrm{NA}}(g,g)$ is the set of linear maps $P\in\mathrm{Hom}\Big(\underbrace{(\wedge^2 g)\otimes\cdots\otimes(\wedge^2 g)}_{n}\otimes g,\,g\Big)$.

The degree of elements in $\mathfrak{C}^{n+1}_{\mathrm{NA}}(g,g)$ are defined to be n. Define

$$[P,Q]_{\mathrm{NA}}=P\circ Q-(-1)^{pq}Q\circ P,\forall P\in\mathfrak{C}^{p+1}_{\mathrm{NA}}(g,g),Q\in\mathfrak{C}^{q+1}_{\mathrm{NA}}(g,g),$$

where $P\circ Q\in\mathfrak{C}^{p+q+1}_{\mathrm{NA}}(g,g)$ is defined by

$$\begin{aligned}
&(P\circ Q)(\mathcal{X}_1,\dots,\mathcal{X}_{p+q},x)\\
=&\sum_{k=1}^{p}(-1)^{(k-1)q}\sum_{\sigma\in S(k-1,q)}(-1)^{\sigma}P\Big(\mathcal{X}_{\sigma(1)},\dots,\mathcal{X}_{\sigma(k-1)},Q\Big(\mathcal{X}_{\sigma(k)},\dots,\mathcal{X}_{\sigma(k+q-1)},x_{(k+q)}\Big)\otimes y_{k+q},\\
&\mathcal{X}_{k+q+1},\dots,\mathcal{X}_{p+q},x\Big)+\sum_{k=1}^{p}(-1)^{(k-1)q}\sum_{\sigma\in S(k-1,q)}(-1)^{\sigma}P\Big(\mathcal{X}_{\sigma(1)},\dots,\mathcal{X}_{\sigma(k-1)},\\
&x_{k+q}\otimes Q\Big(\mathcal{X}_{\sigma(k)},\dots,\mathcal{X}_{\sigma(k+q-1)},y_{k+q}\Big),\mathcal{X}_{k+q+1},\dots,\mathcal{X}_{p+q},x\Big)\\
&+\sum_{\sigma\in S(p,q)}(-1)^{pq}(-1)^{\sigma}P\Big(\mathcal{X}_{\sigma(1)},\dots,\mathcal{X}_{\sigma(p)},Q\Big(\mathcal{X}_{\sigma(p+1)},\dots,\mathcal{X}_{\sigma(p+q)},x\Big)\Big),
\end{aligned}$$

for $\mathcal{X}_i=x_i\wedge y_i\in\wedge^2 g, i=1,2,\dots,p+q$ and $x\in g$.

Proposition 3.7. *[39] With the above notations, the pair $(\mathfrak{C}^*_{\mathrm{NA}}(g,g),[-,-]_{\mathrm{NA}})$ is a graded Lie algebra.*

Lemma 3.8. *For $\pi\in\mathfrak{C}^2_{\mathrm{NA}}(g,g)$, we have*

$$\begin{aligned}
&[\pi,\pi]_{\mathrm{NA}}(\mathcal{X}_1,\mathcal{X}_2,x)=2(\pi\circ\pi)(\mathcal{X}_1,\mathcal{X}_2,x)\\
=&2\Big(\pi(\pi(x_1,y_1,x_2),y_2,x)+\pi(x_2,\pi(x_1,y_1,y_2),x)+\pi(x_2,y_2,\pi(x_1,y_1,x))-\pi(x_1,y_1,\pi(x_2,y_2,x))\Big).
\end{aligned}$$

Thus, π is a 3-Leibniz algebras structure if and only if $[\pi,\pi]_{\mathrm{NA}}=0$, i.e. π is a Maurer-Cartan element of the graded Lie algebra $(\mathfrak{C}^*_{\mathrm{NA}}(g,g),[-,-]_{\mathrm{NA}})$.

Let $(V;\theta)$ be a representation of a LTS $(\mathbb{G},[-,-,-])$. For convenience, we use ϖ to indicate the LTS bracket $[-,-,-]$. Then, $\varpi\boxtimes D_\theta$ corresponds to the hemisemidirect product 3-Leibniz algebra structure on $\mathbb{G}\oplus V$ given by

$$\varpi\boxtimes D_\theta\Big((a,u),(b,v),(c,w)\Big)=([a,b,c],D_\theta(a,b)w),$$

for $(a,u),(b,v),(c,w)\in\mathbb{G}\oplus V$. Therefore, according to Lemma 3.8, we have

$$[\varpi\boxtimes D_\theta,\varpi\boxtimes D_\theta]_{\mathrm{NA}}=0.$$

Lemma 3.9. *Let $E : V \to \mathbb{G}$ be an embedding tensor on a LTS $(\mathbb{G}, [-,-,-])$ with respect to the representation $(V;\theta)$. Then, we have*

$$
\begin{aligned}
&[\varpi \boxtimes D_\theta, E]_{\mathrm{NA}}\Big((a,u),(b,v),(c,w)\Big)\\
=&\Big([Eu,b,c]+[a,b,Ew]+[a,Ev,c]-ED_\theta(a,b)w, D_\theta(a,Ev)w+D_\theta(Eu,b)w\Big),\\
&[[\varpi \boxtimes D_\theta, E]_{\mathrm{NA}}, E]_{\mathrm{NA}}\Big((a,u),(b,v),(c,w)\Big)\\
=&2\Big([Eu,Ev,c]+[Eu,b,Ew]+[a,Ev,Ew]-E(D_\theta(Eu,b)w+D_\theta(a,Ev)w), D_\theta(Eu,Ev)w\Big),
\end{aligned}
$$

for $(a,u),(b,v),(c,w) \in \mathbb{G} \oplus V$.

Proof. For any $(a,u),(b,v),(c,w) \in \mathbb{G} \oplus V$, we have

$$
\begin{aligned}
&[\varpi \boxtimes D_\theta, E]_{\mathrm{NA}}\Big((a,u),(b,v),(c,w)\Big) = ((\varpi \boxtimes D_\theta)\circ E - E \circ (\varpi \boxtimes D_\theta))\Big((a,u),(b,v),(c,w)\Big)\\
=&\varpi \boxtimes D_\theta\Big(E(a,u),(b,v),(c,w)\Big)+\varpi \boxtimes D_\theta\Big((a,u),E(b,v),(c,w)\Big)+\varpi \boxtimes D_\theta\Big((a,u),(b,v),E(c,w)\Big)\\
&-E([a,b,c],D_\theta(a,b)w)\\
=&\Big([Eu,b,c]+[a,b,Ew]+[a,Ev,c]-ED_\theta(a,b)w, D_\theta(a,Ev)w+D_\theta(Eu,b)w\Big),\\
&[[\varpi \boxtimes D_\theta, E]_{\mathrm{NA}}, E]_{\mathrm{NA}}\Big((a,u),(b,v),(c,w)\Big)\\
=&[\varpi \boxtimes D_\theta, E]_{\mathrm{NA}}\Big(E(a,u),(b,v),(c,w)\Big)+[\varpi \boxtimes D_\theta, E]_{\mathrm{NA}}\Big((a,u),E(b,v),(c,w)\Big)\\
&+[\varpi \boxtimes D_\theta, E]_{\mathrm{NA}}\Big((a,u),(b,v),E(c,w)\Big)-E[\varpi \boxtimes D_\theta, E]_{\mathrm{NA}}\Big((a,u),(b,v),(c,w)\Big)\\
=&2\Big([Eu,Ev,c]+[Eu,b,Ew]+[a,Ev,Ew]-E(D_\theta(Eu,b)w+D_\theta(a,Ev)w), D(Eu,Ev)w\Big).
\end{aligned}
$$

The proof is finished. □

Proposition 3.10. *Let $(V;\theta)$ be a representation of a LTS $(\mathbb{G}, [-,-,-])$. Then we have a V-data $(\mathbb{L}, \mathbb{F}, \mathcal{P}, \Delta)$ as follows:*

- *the graded Lie algebra $(\mathbb{L}, [-,-])$ is given by $(\mathfrak{C}^*_{\mathrm{NA}}(\mathbb{G}\oplus V, \mathbb{G}\oplus V), [-,-]_{\mathrm{NA}})$;*
- *the abelian graded Lie subalgebra $\mathbb{F}$ is given by*

$$\mathbb{F} = \mathfrak{C}^*(V,\mathbb{G}) = \oplus_{n\geq 0}\mathfrak{C}^{n+1}(V,\mathbb{G}) = \oplus_{n\geq 0}\mathrm{Hom}\Big(\underbrace{(\wedge^2 V)\otimes\cdots\otimes(\wedge^2 V)}_{n\geq 0}\otimes V, \mathbb{G}\Big);$$

- *$\mathcal{P} : \mathbb{L} \to \mathbb{L}$ is the projection onto the subspace $\mathbb{F}$;*
- *$\Delta = \varpi \boxtimes D_\theta$.*

Consequently, we obtain a Lie 3-algebra $(\mathfrak{C}^*(V,\mathbb{G}), [\![-,-,-]\!])$, where

$$[\![P,Q,R]\!] = [[[\varpi \boxtimes D_\theta, P]_{\mathrm{NA}}, Q]_{\mathrm{NA}}, R]_{\mathrm{NA}} \in \mathfrak{C}^{m+n+k+2}(V,\mathbb{G}), \tag{3.2}$$

for $P \in \mathfrak{C}^{m+1}(V,\mathbb{G})$, $Q \in \mathfrak{C}^{n+1}(V,\mathbb{G})$ and $R \in \mathfrak{C}^{k+1}(V,\mathbb{G})$.

Proof. It is obvious that $\Delta = \varpi \boxtimes D_\theta \in \ker(\mathcal{P})^1$. Hence by Definition 3.5, $(\mathbb{L}, \mathbb{F}, \mathcal{P}, \Delta)$ is a V-data. According to Proposition 3.6, $(\mathbb{F}, \{l_i\}_{i=1}^\infty)$ is an L_∞-algebra, where l_i is given by Eq. (3.1). Furthermore, for any $P \in \mathfrak{C}^{m+1}(V,\mathbb{G})$ and $Q \in \mathfrak{C}^{n+1}(V,\mathbb{G})$, according to Lemma 3.9, we have

$$[\varpi \boxtimes D_\theta, P]_{\mathrm{NA}} \in \ker(\mathcal{P}),$$

$$[[\varpi \boxtimes D_\theta, P]_{\mathrm{NA}}, Q]_{\mathrm{NA}} \in \ker(\mathcal{P}),$$

which implies that $l_1 = 0$ and $l_2 = 0$. Similarly, we have $l_k = 0$, for $k \geq 4$. Therefore, the pair $(\mathfrak{C}^*(V,\mathbb{G}), [\![-,-,-]\!])$ is a Lie 3-algebra. □

Theorem 3.11. *Let $(V;\theta)$ be a representation of a LTS $(\mathbb{G},[-,-,-])$. Then Maurer-Cartan elements of the Lie 3-algebra $(\mathfrak{C}^*(V,\mathbb{G}),[\![-,-,-]\!])$ are precisely embedding tensors on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$.*

Proof. For any $E\in\mathfrak{C}^1(V,\mathbb{G})$ and $u,v,w\in V$. By Eq. (3.2), we can deduce that

$$
\begin{aligned}
&[\![E,E,E]\!](u,v,w)\\
&=[[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}},E]_{\mathrm{NA}}(u,v,w)\\
&=[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(Eu,v,w)+[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(u,Ev,w)\\
&\quad+[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(u,v,Ew)-E[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(u,v,w)\\
&=[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(Eu,Ev,w)+[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(Eu,v,Ew)-E[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(Eu,v,w)\\
&\quad+[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(Eu,Ev,w)+[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(u,Ev,Ew)-E[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(u,Ev,w)\\
&\quad+[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(Eu,v,Ew)+[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(u,Ev,Ew)-E[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(u,v,Ew)\\
&\quad-E[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(Eu,v,w)-E[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(u,Ev,w)-E[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}}(u,v,Ew)\\
&=6\Big([Eu,Ev,Ew]-ED_\theta(Eu,Ev)w\Big),
\end{aligned}
$$

which implies that E is a Maurer-Cartan element of the Lie 3-algebra $(\mathfrak{C}^*(V,\mathbb{G}),[\![-,-,-]\!])$ if and only if E is an embedding tensor. □

Let E be an embedding tensor on a LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Since E is a Maurer-Cartan element of the Lie 3-algebra $(\mathfrak{C}^*(V,\mathbb{G}),[\![-,-,-]\!])$, then by Proposition 3.4, $\mathfrak{C}^*(V,\mathbb{G})$ carries a twisted L_∞-algebra structure as following:

$$
\begin{aligned}
l_1^E(P)=&\frac{1}{2}[\![E,E,P]\!],\\
l_2^E(P,Q)=&[\![E,P,Q]\!],\\
l_3^E(P,Q,R)=&[\![P,Q,R]\!],\\
l_k^E=&0,k\geq 4,
\end{aligned}
$$

where $P\in\mathfrak{C}^p(V,\mathbb{G})$, $Q\in\mathfrak{C}^q(V,\mathbb{G})$ and $R\in\mathfrak{C}^r(V,\mathbb{G})$.

Theorem 3.12. *Let E be an embedding tensor on a LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Then for a linear map $\widetilde{E}:V\to\mathbb{G}$, $E+\widetilde{E}$ is an embedding tensor if and only if $\widetilde{E}$ is a Maurer-Cartan element of the twisted L_∞-algebra $(\mathfrak{C}^*(V,\mathbb{G}),l_1^E,l_2^E,l_3^E)$, that is, $\widetilde{E}$ satisfies the Maurer-Cartan equation:*

$$l_1^E(\widetilde{E})+\frac{1}{2!}l_2^E(\widetilde{E},\widetilde{E})+\frac{1}{3!}l_3^E(\widetilde{E},\widetilde{E},\widetilde{E})=0.$$

Proof. By Theorem 3.11, $E+\widetilde{E}$ is an embedding tensor if and only if

$$\frac{1}{3!}[\![E+\widetilde{E},E+\widetilde{E},E+\widetilde{E}]\!]=0.$$

Applying $\frac{1}{3!}[\![E,E,E]\!]=0$, the above equation is equivalent to

$$\frac{1}{2}[\![E,E,\widetilde{E}]\!]+\frac{1}{2}[\![E,\widetilde{E},\widetilde{E}]\!]+\frac{1}{6}[\![\widetilde{E},\widetilde{E},\widetilde{E}]\!]=0,$$

that is, $l_1^E(\widetilde{E})+\frac{1}{2!}l_2^E(\widetilde{E},\widetilde{E})+\frac{1}{3!}l_3^E(\widetilde{E},\widetilde{E},\widetilde{E})=0$, which implies that $\widetilde{E}$ is a Maurer-Cartan element of the twisted L_∞-algebra $(\mathfrak{C}^*(V,\mathbb{G}),l_1^E,l_2^E,l_3^E)$. □

The above characterization of an embedding tensor E allows us to define a cohomology associated with E. Therefore, we consider the cochain complex: $(\mathfrak{C}^\bullet_E(V,\mathbb{G}),\mathrm{d}^\bullet_E)$.

More precisely,

$$\mathfrak{C}^n_E(V,\mathbb{G}) := \begin{cases}\mathfrak{C}^n(V,\mathbb{G}), & n\geq 1,\\ \wedge^2\mathbb{G}, & n=0\end{cases}$$

and its differential $\mathrm{d}^n_E:\mathfrak{C}^n_E(V,\mathbb{G})\to\mathfrak{C}^{n+1}_E(V,\mathbb{G})$,

$$\mathrm{d}^n_E(f)=l^E_1(f)=\frac{1}{2}[\![E,E,f]\!],\ \ \forall\, f\in\mathfrak{C}^n_E(V,\mathbb{G}),\ \text{for}\ \ n\geq 1; \tag{3.3}$$

if $n=0$, define $\mathrm{d}^0_E:\mathfrak{C}^0_E(V,\mathbb{G})\to\mathfrak{C}^1_E(V,\mathbb{G})$ by

$$\mathrm{d}^0_E(a,b)u=E(D_\theta(a,b)u)-[a,b,Eu],\forall(a,b)\in\mathfrak{C}^0_E(V,\mathbb{G}),u\in V. \tag{3.4}$$

The corresponding cohomology group is

$$\mathcal{H}^n_E(V,\mathbb{G})=\frac{\mathcal{Z}^n_E(V,\mathbb{G})}{\mathcal{B}^n_E(V,\mathbb{G})}=\frac{\{f\in\mathfrak{C}^n_E(V,\mathbb{G})\mid \mathrm{d}^n_E(f)=0\}}{\{\mathrm{d}^n_E(f)\mid f\in\mathfrak{C}^{n-1}_E(V,\mathbb{G})\}}.$$

Next, we will introduce the cohomology of embedding tensors on LTSs from another angle. First we recall some basic results about representations and cohomologies of 3-Leibniz algebra $(\mathfrak{g},[-,-,-]_\mathfrak{g})$.

Definition 3.13. *[5] A representation of a 3-Leibniz algebra $(\mathfrak{g},[-,-,-]_\mathfrak{g})$ is a vector space $\mathcal{V}$ equipped with 3 actions*

$$\rho^l:\mathfrak{g}\otimes\mathfrak{g}\otimes\mathcal{V}\to\mathcal{V},$$
$$\rho^m:\mathfrak{g}\otimes\mathcal{V}\otimes\mathfrak{g}\to\mathcal{V},$$
$$\rho^r:\mathcal{V}\otimes\mathfrak{g}\otimes\mathfrak{g}\to\mathcal{V},$$

such that for any $x,y,z,a,b\in\mathfrak{g}$ and $u\in\mathcal{V}$,

$$\rho^l(a,b,\rho^l(x,y,u))=\rho^l([a,b,x]_\mathfrak{g},y,u)+\rho^l(x,[a,b,y]_\mathfrak{g},u)+\rho^l(x,y,\rho^l(a,b,u)), \tag{3.5}$$
$$\rho^l(a,b,\rho^m(x,u,z))=\rho^m([a,b,x]_\mathfrak{g},u,z)+\rho^m(x,\rho^l(a,b,u),z)+\rho^m(x,u,[a,b,z]_\mathfrak{g}), \tag{3.6}$$
$$\rho^l(a,b,\rho^r(u,y,z))=\rho^r(\rho^l(a,b,u),y,z)+\rho^r(u,[a,b,y]_\mathfrak{g},z)+\rho^r(u,y,[a,b,z]_\mathfrak{g}), \tag{3.7}$$
$$\rho^m(a,u,[x,y,z]_\mathfrak{g})=\rho^r(\rho^m(a,u,x),y,z)+\rho^m(x,\rho^m(a,u,y),z)+\rho^l(x,y,\rho^m(a,u,z)), \tag{3.8}$$
$$\rho^r(u,b,[x,y,z]_\mathfrak{g})=\rho^r(\rho^r(u,b,x),y,z)+\rho^m(x,\rho^r(u,b,y),z)+\rho^l(x,y,\rho^r(u,b,z)). \tag{3.9}$$

An n-cochain on a 3-Leibniz algebra $(\mathfrak{g},[-,-,-]_\mathfrak{g})$ with coefficients in a representation $(\mathcal{V};\rho^l,\rho^m,\rho^r)$ is a linear map

$$f:\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n-1}\otimes\mathfrak{g}\to\mathcal{V},n\geq 1.$$

The space generated by n-cochains is denoted as $C^n_{\text{3Leib}}(\mathfrak{g},\mathcal{V})$. The coboundary map $\delta^n:C^n_{\text{3Leib}}(\mathfrak{g},\mathcal{V})\to C^{n+1}_{\text{3Leib}}(\mathfrak{g},\mathcal{V})$, $f\mapsto(\delta^n f)$, for $\mathfrak{X}_i=x_i\wedge y_i\in\wedge^2\mathfrak{g}(1\leq i\leq n)$ and $z\in\mathfrak{g}$, as

$$\begin{aligned}&(\delta^n f)(\mathfrak{X}_1,\mathfrak{X}_2,\ldots,\mathfrak{X}_n,z)\\ =&\sum_{1\leq j<k\leq n}(-1)^j f(\mathfrak{X}_1,\ldots,\widehat{\mathfrak{X}_j},\ldots,\mathfrak{X}_{k-1},x_k\wedge[x_j,y_j,y_k]_\mathfrak{g}+[x_j,y_j,x_k]_\mathfrak{g}\wedge y_k,\ldots,\mathfrak{X}_n,z)\\ &+\sum_{j=1}^n(-1)^j f(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}_j},\ldots,\mathfrak{X}_n,[x_j,y_j,z]_\mathfrak{g})+\sum_{j=1}^n(-1)^{j+1}\rho^l(\mathfrak{X}_j,f(\mathfrak{X}_1,\ldots,\widehat{\mathfrak{X}_j},\ldots,\mathfrak{X}_n,z))\\ &+(-1)^{n+1}(\rho^m(x_n,f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n),z)+\rho^r(f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n),y_n,z)).\end{aligned}$$

It was proved in [5, 35] that $\delta^{n+1}\circ\delta^n = 0$. Thus, $(\oplus_{n=1}^{+\infty}C^n_{3\mathrm{Leib}}(\mathfrak{g},\mathcal{V}),\delta^\bullet)$ is a cochain complex. The corresponding n-th cohomology group by $\mathcal{H}^n_{3\mathrm{Leib}}(\mathfrak{g},\mathcal{V}) = \mathcal{Z}^n_{3\mathrm{Leib}}(\mathfrak{g},\mathcal{V})/\mathcal{B}^n_{3\mathrm{Leib}}(\mathfrak{g},\mathcal{V})$.

Lemma 3.14. *Let $E : V \to \mathbb{G}$ be an embedding tensor on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Define actions*

$$\rho^l_E : V\otimes V\otimes \mathbb{G}\to\mathbb{G},\quad \rho^m_E : V\otimes\mathbb{G}\otimes V\to\mathbb{G},\quad \rho^r_E : \mathbb{G}\otimes V\otimes V\to\mathbb{G},$$

by

$$\rho^l_E(u,v,x) = [Eu,Ev,x], \tag{3.10}$$
$$\rho^m_E(u,x,v) = [Eu,x,Ev] - ED_\theta(Eu,x)v, \tag{3.11}$$
$$\rho^r_E(x,u,v) = [x,Eu,Ev] - ED_\theta(x,Eu)v, \tag{3.12}$$

for $u,v\in V$ and $x\in\mathbb{G}$. Then $(\mathbb{G};\rho^l_E,\rho^m_E,\rho^r_E)$ is a representation of the 3-Leibniz algebra $(V,[-,-,-]_E)$ given in Proposition 2.9.

Proof. For any $u,v,s,t\in V$ and $x\in\mathbb{G}$, by Eqs. (2.3), (2.9) and (2.10), we have

$$\begin{aligned}
&\rho^l_E(u,v,\rho^l_E(s,t,x)) - \rho^l_E([u,v,s]_E,t,x) - \rho^l_E(s,[u,v,t]_E,x) - \rho^l_E(s,t,\rho^l_E(u,v,x))\\
=&[Eu,Ev,[Es,Et,x]] - [E[u,v,s]_E,Et,x] - [Es,E[u,v,t]_E,x] - [Es,Et,[Eu,Ev,x]]\\
=&[Eu,Ev,[Es,Et,x]] - [[Eu,Ev,Es],Et,x] - [Es,[Eu,Ev,Et],x] - [Es,Et,[Eu,Ev,x]]\\
=&0,
\end{aligned}$$

which indicates that Eq. (3.5) follows.

Further, by Eqs. (2.3), (2.7), (2.9) and (2.10), we have

$$\begin{aligned}
&\rho^l_E(u,v,\rho^m_E(s,x,t)) - \rho^m_E([u,v,s]_E,x,t) - \rho^m_E(s,\rho^l_E(u,v,x),t) - \rho^m_E(s,x,[u,v,t]_E)\\
=&[Eu,Ev,[Es,x,Et]] - [Eu,Ev,ED_\theta(Es,x)t] - [E[u,v,s]_E,x,Et] + ED_\theta(E[u,v,s]_E,x)t\\
&- [Es,[Eu,Ev,x],Et] + ED_\theta(Es,[Eu,Ev,x])t - [Es,x,E[u,v,t]_E] + ED_\theta(Es,x)[u,v,t]_E\\
=&[Eu,Ev,[Es,x,Et]] - [Eu,Ev,ED_\theta(Es,x)t] - [[Eu,Ev,Es],x,Et] + ED_\theta([Eu,Ev,Es],x)t\\
&- [Es,[Eu,Ev,x],Et] + ED_\theta(Es,[Eu,Ev,x])t - [Es,x,[Eu,Ev,Et]] + ED_\theta(Es,x)[u,v,t]_E\\
=&E\Big(-D_\theta(Eu,Ev)D_\theta(Es,x)t + D_\theta([Eu,Ev,Es],x)t + D_\theta(Es,[Eu,Ev,x])t + D_\theta(Es,x)D_\theta(Eu,Ev)t\Big)\\
=&0,\\
&\rho^l_E(u,v,\rho^r_E(x,s,t)) - \rho^r_E(\rho^l_E(u,v,x),s,t) - \rho^r_E(x,[u,v,s]_E,t) - \rho^r_E(x,s,[u,v,t]_E)\\
=&[Eu,Ev,[x,Es,Et]] - [Eu,Ev,ED_\theta(x,Es)t] - [[Eu,Ev,x],Es,Et] + ED_\theta([Eu,Ev,x],Es)t\\
&- [x,E[u,v,s]_E,Et] + ED_\theta(x,E[u,v,s]_E)t - [x,Es,E[u,v,t]_E] + ED_\theta(x,Es)[u,v,t]_E\\
=&E\Big(-D_\theta(Eu,Ev)D_\theta(x,Es)t + D_\theta([Eu,Ev,x],Es)t + D_\theta(x,[Eu,Ev,Es])t + D_\theta(x,Es)D_\theta(Eu,Ev)t\Big)\\
=&0,
\end{aligned}$$

which imply that Eqs. (3.6) and (3.7) also follow. Similarly, we can prove that Eqs.(3.8) and (3.9) hold. Thus, $(\mathbb{G};\rho^l_E,\rho^m_E,\rho^r_E)$ is a representation of the 3-Leibniz algebra $(V,[-,-,-]_E)$. □

Let $E : V \to \mathbb{G}$ be an embedding tensor on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Recall that Proposition 2.9 and Lemma 3.14 give a new 3-Leibniz algebra $(V,[-,-,-]_E)$ and a new representation $(\mathbb{G};\rho^l_E,\rho^m_E,\rho^r_E)$ of $(V,[-,-,-]_E)$. Consider the cochain complex of $(V,[-,-,-]_E)$ with coefficients in $(\mathbb{G};\rho^l_E,\rho^m_E,\rho^r_E)$:

$$(C^\bullet_{3\mathrm{Leib}}(V,\mathbb{G}),\partial^\bullet_E) = (\oplus_{n=1}^{\infty}C^n_{3\mathrm{Leib}}(V,\mathbb{G}),\partial^\bullet_E).$$

More precisely,

$$C^n_{3\mathrm{Leib}}(V,\mathbb{G}) = \mathrm{Hom}(\underbrace{\wedge^2 V\otimes\cdots\otimes\wedge^2 V}_{n-1}\otimes V,\mathbb{G})$$

and its coboundary operator $\partial^n_E : C^n_{3\mathrm{Leib}}(V,\mathbb{G}) \to C^{n+1}_{3\mathrm{Leib}}(V,\mathbb{G}), f \mapsto (\partial^n_E f)$ is given as follows:

$$\begin{aligned}
&(\partial^n_E f)(\mathfrak{U}_1,\mathfrak{U}_2,\ldots,\mathfrak{U}_n,w)\\
&=\sum_{1\le j<k\le n}(-1)^j f(\mathfrak{U}_1,\ldots,\widehat{\mathfrak{U}_j},\ldots,\mathfrak{U}_{k-1},u_k\wedge[u_j,v_j,v_k]_E+[u_j,v_j,u_k]_E\wedge v_k,\ldots,\mathfrak{U}_n,w)\\
&\quad+\sum_{j=1}^n(-1)^j f(\mathfrak{U}_1,\ldots,\widehat{\mathfrak{U}_j},\ldots,\mathfrak{U}_n,[u_j,v_j,w]_E)+\sum_{j=1}^n(-1)^{j+1}\rho^l_E(\mathfrak{U}_j,f(\mathfrak{U}_1,\ldots,\widehat{\mathfrak{U}_j},\ldots,\mathfrak{U}_n,w))\\
&\quad+(-1)^{n+1}(\rho^m_E(u_n,f(\mathfrak{U}_1,\cdots,\mathfrak{U}_{n-1},v_n),w)+\rho^r_E(f(\mathfrak{U}_1,\ldots,\mathfrak{U}_{n-1},u_n),v_n,w)),
\end{aligned}$$

for $\mathfrak{U}_i = u_i\wedge v_i \in \wedge^2 V, 1\le i\le n$ and $w\in V$. For example, when $n=1$, the coboundary map $\partial^1_E : C^1_{3\mathrm{Leib}}(V,\mathbb{G}) \to C^2_{3\mathrm{Leib}}(V,\mathbb{G}), f\mapsto(\partial^1_E f)$ by:

$$\begin{aligned}
&(\partial^1_E f)(u_1,v_1,w)\\
&=-f([u_1,v_1,w]_E)+\rho^l_E(u_1,v_1,f(w))+\rho^m_E(u_1,f(v_1),w)+\rho^r_E(f(u_1),v_1,w)\\
&=-f(D_\theta(Eu_1,Ev_1)w)+[Eu_1,Ev_1,f(w)]+[Eu_1,f(v_1),Ew]-ED_\theta(Eu_1,f(v_1))w\\
&\quad+[f(u_1),Ev_1,Ew]-ED_\theta(f(u_1),Ev_1)w.
\end{aligned}$$

When $n=0$, for any $(a,b)\in C^0_{3\mathrm{Leib}}(V,\mathbb{G}) := \wedge^2\mathbb{G}$, we define $\partial^0_E : C^0_{3\mathrm{Leib}}(V,\mathbb{G}) \to C^1_{3\mathrm{Leib}}(V,\mathbb{G}), (a,b)\mapsto \mathfrak{I}(a,b)$ by

$$\mathfrak{I}(a,b)u = ED_\theta(a,b)u-[a,b,Eu],\quad \forall\, u\in V.$$

Proposition 3.15. *Let $E : V\to\mathbb{G}$ be an embedding tensor on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Then $\partial^1_E\mathfrak{I}(a,b)=0$, that is the composition $C^0_{3\mathrm{Leib}}(V,\mathbb{G}) \xrightarrow{\partial^0_E} C^1_{3\mathrm{Leib}}(V,\mathbb{G}) \xrightarrow{\partial^1_E} C^2_{3\mathrm{Leib}}(V,\mathbb{G})$ is the zero map.*

Proof. For any $u,v,w\in V$, by Eqs. (2.3), (2.7), (2.9) and (2.10), we have

$$\begin{aligned}
&(\partial^1_E\mathfrak{I}(a,b))(u,v,w)\\
&=-\mathfrak{I}(a,b)D_\theta(Eu,Ev)w+[Eu,Ev,\mathfrak{I}(a,b)w]+[Eu,\mathfrak{I}(a,b)v,Ew]-ED_\theta(Eu,\mathfrak{I}(a,b)v)w\\
&\quad+[\mathfrak{I}(a,b)u,Ev,Ew]-ED_\theta(\mathfrak{I}(a,b)u,Ev)w\\
&=-E(D_\theta(a,b)D_\theta(Eu,Ev)w)+[a,b,ED_\theta(Eu,Ev)w]+[Eu,Ev,ED_\theta(a,b)w]-[Eu,Ev,[a,b,Ew]]\\
&\quad+[Eu,ED_\theta(a,b)v,Ew]-[Eu,[a,b,Ev],Ew]-ED_\theta(Eu,ED_\theta(a,b)v)w+ED_\theta(Eu,[a,b,Ev])w\\
&\quad+[ED_\theta(a,b)u,Ev,Ew]-[[a,b,Eu],Ev,Ew]-ED_\theta(ED_\theta(a,b)u,Ev)w+ED_\theta([a,b,Eu],Ev)w\\
&=-E(D_\theta(a,b)D_\theta(Eu,Ev)w)+E(D_\theta(Eu,Ev)D_\theta(a,b)w)+[Eu,ED_\theta(a,b)v,Ew]-ED_\theta(Eu,ED_\theta(a,b)v)w\\
&\quad+ED_\theta(Eu,[a,b,Ev])w+[ED_\theta(a,b)u,Ev,Ew]-ED_\theta(ED_\theta(a,b)u,Ev)w+ED_\theta([a,b,Eu],Ev)w\\
&=0.
\end{aligned}$$

Thus, $\partial^1_E\mathfrak{I}(a,b)=0$. □

Next we define the cohomology theory of an embedding tensor E on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. For $n\ge 0$, define the set of n-cochains of E by

$$C^n_E(V,\mathbb{G}) = C^n_{3\mathrm{Leib}}(V,\mathbb{G}).$$

Then, by Proposition 3.15, $(\oplus_{n=0}^{\infty}C_E^n(V,\mathbb{G}),\partial_E^{\bullet})$ is a cochain complex. For $n\geq 1$, we denote the set of n-cocycles by $\mathcal{Z}_E^n(V,\mathbb{G})$, the set of n-coboundaries by $\mathcal{B}_E^n(V,\mathbb{G})$ and the n-th cohomology group of the embedding tensor E by $\mathcal{H}_E^n(V,\mathbb{G})=\mathcal{Z}_E^n(V,\mathbb{G})/\mathcal{B}_E^n(V,\mathbb{G})$.

Furthermore, comparing the coboundary operator ∂_E^n given above with the differential d_E^n given in Eq. (3.3), we obtain the following result.

Theorem 3.16. *Let $E: V\to\mathbb{G}$ be an embedding tensor on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Then we have $\partial_E^n f=(-1)^{n-1}\mathrm{d}_E^n(f)$ for $f\in C_E^n(V,\mathbb{G}), n=1,2,\ldots$.*

Proof. For any $\mathfrak{U}_i=u_i\wedge v_i\in\wedge^2V, 1\leq i\leq n$ and $w\in V$, according to Lemma 3.9, we have

$$\begin{aligned}
&\mathrm{d}_E^n(f)(\mathfrak{U}_1,\mathfrak{U}_2,\ldots,\mathfrak{U}_n,w)=l_1^E(f)(\mathfrak{U}_1,\mathfrak{U}_2,\ldots,\mathfrak{U}_n,w)\\
=&\frac{1}{2}[\![E,E,f]\!](\mathfrak{U}_1,\mathfrak{U}_2,\ldots,\mathfrak{U}_n,w)\\
=&\frac{1}{2}[[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}},f]_{\mathrm{NA}}(\mathfrak{U}_1,\mathfrak{U}_2,\ldots,\mathfrak{U}_n,w)\\
=&\frac{1}{2}\Big([[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}\circ f-(-1)^{n-1}f\circ[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}\Big)(\mathfrak{U}_1,\mathfrak{U}_2,\ldots,\mathfrak{U}_n,w)\\
=&\frac{1}{2}\Big([[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(u_n\wedge f(\mathfrak{U}_1,\mathfrak{U}_2,\ldots,\mathfrak{U}_{n-1},v_n),w)\\
&+[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(f(\mathfrak{U}_1,\mathfrak{U}_2,\ldots,\mathfrak{U}_{n-1},u_n)\wedge v_n,w)\\
&+\sum_{i=1}^{n}(-1)^{n-1}(-1)^{i-1}[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(\mathfrak{U}_i,f(\mathfrak{U}_1,\ldots,\hat{\mathfrak{U}}_i,\ldots,\mathfrak{U}_n,w)\\
&-(-1)^{n-1}\sum_{j=1}^{n-1}\sum_{i=1}^{j}(-1)^{i+1}f(\mathfrak{U}_i,f(\mathfrak{U}_1,\ldots,\hat{\mathfrak{U}}_i,\ldots,\mathfrak{U}_j,[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(\mathfrak{U}_i,u_{j+1})\wedge v_{j+1},\mathfrak{U}_{j+2},\ldots,\mathfrak{U}_n,w)\\
&-(-1)^{n-1}\sum_{j=1}^{n-1}\sum_{i=1}^{j}(-1)^{i+1}f(\mathfrak{U}_i,f(\mathfrak{U}_1,\ldots,\hat{\mathfrak{U}}_i,\ldots,\mathfrak{U}_j,u_{j+1}\wedge[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(\mathfrak{U}_i,v_{j+1}),\mathfrak{U}_{j+2},\ldots,\mathfrak{U}_n,w)\\
&-(-1)^{n-1}\sum_{i=1}^{n}(-1)^{i+1}f(\mathfrak{U}_i,f(\mathfrak{U}_1,\ldots,\hat{\mathfrak{U}}_i,\ldots,\mathfrak{U}_n,[[\varpi\boxtimes D_\theta,E]_{\mathrm{NA}},E]_{\mathrm{NA}}(\mathfrak{U}_i,w))\Big)\\
=&(-1)^{n-1}\partial_E^n f,
\end{aligned}$$

which implies that $\partial_E^n f=(-1)^{n-1}\mathrm{d}_E^n(f)$. The proof is completed. □

4. Deformations of embedding tensors on Lie triple systems

In this section, we study infinitesimal deformations of embedding tensors E on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Moreover, we also discuss the extendability of an order n deformation to an order $(n+1)$ deformation of embedding tensors E.

Let $(\mathbb{G},[-,-,-])$ be a LTS over $\mathbb{K}$ and $\mathbb{K}[[t]]$ be the polynomial ring in one variable t. Then $\mathbb{K}[[t]]/(t^2)\otimes\mathbb{G}$ is an $\mathbb{K}[[t]]/(t^2)$-module. Moreover, $\mathbb{K}[[t]]/(t^2)\otimes\mathbb{G}$ is a LTS over $\mathbb{K}[[t]]/(t^2)$, where the LTS structure is defined by

$$[f(t)\otimes x,g(t)\otimes y,h(t)\otimes z]=f(t)g(t)h(t)\otimes[x,y,z],$$

for $f(t),g(t),h(t)\in\mathbb{K}[[t]]/(t^2)$ and $x,y,z\in\mathbb{G}$. In the sequel, all the vector spaces are finite dimensional vector spaces over $\mathbb{K}$ and we denote $f(t)\otimes x$ by $f(t)x$.

Definition 4.1. *Let E be an embedding tensor on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. A parametrized sum $E_t=E+tE_1$, for some $E_1\in C_E^1(V,\mathbb{G})$, is called an infinitesimal deformation of E if E_t is an embedding tensor on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$ modulo t^2 for all values of parameter t. In this case, we say that E_1 generates an infinitesimal deformation of the embedding tensor E.*

If E_1 generates an infinitesimal deformation of E, then we have

$$[E_t u, E_t v, E_t w] = E_t D_\theta(E_t u, E_t v)w,$$

for $u, v, w \in V$. This is equivalent to the following equation

$$\begin{aligned}&[Eu, Ev, E_1 w] + [Eu, E_1 v, Ew] + [E_1 u, Ev, Ew] \\ &= E_1 D_\theta(Eu, Ev)w + ED_\theta(E_1 u, Ev)w + ED_\theta(Eu, E_1 v)w.\end{aligned} \tag{4.1}$$

Then we have the following result.

Proposition 4.2. *Let $E_t = E + tE_1$ be an infinitesimal deformation of an embedding tensor on the LTS $(\mathbb{G}, [-,-,-])$ with respect to the representation $(V;\theta)$. Then E_1 is a 1-cocycle for the embedding tensor E, that is $\partial^1_E E_1 = 0$. Moreover, the 1-cocycle E_1 is called the infinitesimal of the infinitesimal deformation E_t of E.*

Proof. By Eq. (4.1), we have $\partial^1_E E_1 = 0$. □

Definition 4.3. *Let E and E' be two embedding tensors on the LTS $(\mathbb{G}, [-,-,-])$ with respect to the representation $(V;\theta)$. A homomorphism from E' to E is a pair $(\phi^{\mathbb{G}}, \phi^V)$, where $\phi^{\mathbb{G}} : \mathbb{G} \to \mathbb{G}$ is a LTS homomorphism and $\phi^V : V \to V$ is a linear map satisfying*

$$E\phi^V(u) = \phi^{\mathbb{G}}(E'u), \tag{4.2}$$

$$\phi^V(\theta(x, y)u) = \theta(\phi^{\mathbb{G}}(x), \phi^{\mathbb{G}}(y))\phi^V(u), \tag{4.3}$$

for $x, y \in \mathbb{G}$ and $u \in V$. From Eq. (2.6) *and* (4.3) *we obtain*

$$\phi^V(D_\theta(x, y)u) = D_\theta(\phi^{\mathbb{G}}(x), \phi^{\mathbb{G}}(y))\phi^V(u), \tag{4.4}$$

In particular, if both $\phi^{\mathbb{G}}$ and ϕ^V are invertible, $(\phi^{\mathbb{G}}, \phi^V)$ is called an isomorphism from E' to E.

Definition 4.4. *Two infinitesimal deformations $E_t = E + tE_1$ and $E'_t = E + tE'_1$ of an embedding tensor E on the LTS $(\mathbb{G}, [-,-,-])$ with respect to the representation $(V;\theta)$ are said to be equivalent if there exists an element $(a, b) \in \wedge^2\mathbb{G}$ such that the pair*

$$(\phi^{\mathbb{G}}_t = \mathrm{id}_{\mathbb{G}} + t[a, b, -], \phi^V_t = \mathrm{id}_V + tD_\theta(a, b)-)$$

defines a homomorphism of embedding tensors modulo t^2 from E'_t to E_t.

Let $(\phi^{\mathbb{G}}_t = \mathrm{id}_{\mathbb{G}} + t[a, b, -], \phi^V_t = \mathrm{id}_V + tD_\theta(a, b)-)$ be a homomorphism from E'_t to E_t. Then by Eq. (4.2), we get

$$E_t(u + tD_\theta(a, b)u) = E'_t u + t[a, b, E'_t u],$$

which implies

$$E'_1 u - E_1 u = E(D_\theta(a, b)u) - [a, b, Eu] = \partial^0_E(a, b)u. \tag{4.5}$$

Now we are ready to give the main result in this section.

Theorem 4.5. *Let E be an embedding tensor on the LTS $(\mathbb{G}, [-,-,-])$ with respect to the representation $(V;\theta)$. If $E_t = E + tE_1$ and $E'_t = E + tE'_1$ are two equivalent infinitesimal deformations of an embedding tensor E, then E_1 and E'_1 define the same cohomology class in $\mathcal{H}^1_E(V, \mathbb{G})$.*

Proof. By Eq. (4.5), we have $E'_1 u = E_1 u + \partial^0_E(a, b)u$, that is $[E'_1] = [E_1] \in \mathcal{H}^1_E(V, \mathbb{G})$. □

Next, we introduce a special obstruction class associated to an order n deformation of an embedding tensor, and show that an order n deformation of an embedding tensor is extendable if and only if this obstruction class vanishes in the second cohomology group.

Let $(\mathbb{G},[-,-,-])$ be a LTS over $\mathbb{K}$. For any positive integer n, consider the ring $\mathbb{K}[[t]]$, then $\mathbb{G}[[t]]/(t^{n+1})$ is an $\mathbb{K}[[t]]/(t^{n+1})$-module.

Definition 4.6. *Let E be an embedding tensor on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. If $E_t=\sum_{i=0}^{n}E_it$, where $E_0=E$ and $E_i\in C_E^1(V,\mathbb{G}), i=1,2,\dots,n$, defines a $\mathbb{K}[[t]]/(t^{n+1})$-module map from $V[[t]]/(t^{n+1})$ to the LTS $\mathbb{G}[[t]]/(t^{n+1})$ and satisfies*

$$[E_tu,E_tv,E_tw]=E_tD_\theta(E_tu,E_tv)w, \tag{4.6}$$

for $u,v,w\in V$, then E_t is called an order n deformation of the embedding tensor E.

Definition 4.7. *Let $E_t=\sum_{i=0}^{n}E_it$ be an order n deformation of an embedding tensor on a LTS $(\mathbb{G},[-,-,-])$ with respect to a representation $(V;\theta)$. If there exists a 1-cochain $E_{n+1}\in C_E^1(V,\mathbb{G})$ such that $E_t'=E_t+tE_{n+1}$ is an order $n+1$ deformation of the embedding tensor E, then E_t is called extendable.*

Let $E_t=\sum_{i=0}^{n}E_it$ be an order n deformation of an embedding tensor E on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Define $Ob_E^n\in C_E^2(V,\mathbb{G})$ by

$$Obs_E^n(u_1,u_2,u_3)=\sum_{\substack{i+j+k=n+1\\0\le i,j,k\le n}}\Big([E_iu_1,E_ju_2,E_ku_3]-E_iD_\theta(E_ju_1,E_ku_2)u_3\Big). \tag{4.7}$$

Then we have the following result.

Proposition 4.8. *With the above notations, Obs_E^n is a 2-cocycle, that is $\partial_E^2Obs_E^n=0$.*

Definition 4.9. *Let $E_t=\sum_{i=0}^{n}E_it$ be an order n deformation of an embedding tensor on a LTS $(\mathbb{G},[-,-,-])$ with respect to a representation $(V;\theta)$. The cohomology class $[Obs_E^n]\in\mathcal{H}_E^2(V,\mathbb{G})$ is called the obstruction class of E_t being extendable.*

Theorem 4.10. *Let E be an embedding tensor on the LTS $(\mathbb{G},[-,-,-])$ with respect to the representation $(V;\theta)$. Then E_t is extendable if and only if the obstruction class $[Obs_E^n]\in\mathcal{H}_E^2(V,\mathbb{G})$ vanishes.*

Proof. Let $E_t'=E_t+t^{n+1}E_{n+1}$ be the extension of E_t, then for all $u_1,u_2,u_3\in V$

$$[E_t'u_1,E_t'u_2,E_t'u_3]=E_t'D_\theta(E_t'u_1,E_t'u_2)u_3. \tag{4.8}$$

Expanding the Eq. (4.8) and comparing the coefficients of t^{n+1} yields that:

$$\sum_{\substack{i+j+k=n+1\\0\le i,j,k\le n+1}}\Big([E_iu_1,E_ju_2,E_ku_3]-E_iD_\theta(E_ju_1,E_ku_2)u_3\Big)=0,$$

which is equivalent to

$$\begin{aligned}&\sum_{\substack{i+j+k=n+1\\0\le i,j,k\le n}}\Big([E_iu_1,E_ju_2,E_ku_3]-E_iD_\theta(E_ju_1,E_ku_2)u_3\Big)+[E_{n+1}u_1,Eu_2,Eu_3]\\&+[Eu_1,E_{n+1}u_2,Eu_3]+[Eu_1,Eu_2,E_{n+1}u_3]-E_{n+1}D_\theta(Eu_1,Eu_2)u_3\\&-ED_\theta(E_{n+1}u_1,Eu_2)u_3-ED_\theta(Eu_1,E_{n+1}u_2)u_3=0,\end{aligned}$$

that is $Obs_E^n(u_1,u_2,u_3)+(\partial^1E_{n+1})(u_1,u_2,u_3)=0$. Hence $Obs_E^n=-\partial_E^1E_{n+1}$, further $\partial_E^2Obs_E^n=-\partial_E^2\circ\partial_E^1E_{n+1}=0$, which implies that the obstruction class $[Obs_E^n]\in\mathcal{H}_E^2(V,\mathbb{G})$ vanishes.

Conversely, suppose that the obstruction class $[Obs^n_E]$ vanishes, then there exists a 1-cochain $E_{n+1} \in C^1_E(V, \mathbb{G})$ such that $Obs^n_E = -\partial^1_E E_{n+1}$. Set $E'_t = E_t + t^{n+1}E_{n+1}$. Then E'_t satisfies

$$\sum_{i+j+k=s} \Big([E_i u_1, E_j u_2, E_k u_3] - E_i D_\theta(E_j u_1, E_k u_2)u_3\Big) = 0, \quad 0 \le s \le n+1,$$

which implies that Eq. (4.8) holds, that is, E'_t is an order $(n+1)$ deformation of E. So it is an extension of E_t. □

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