Filomat 39:7 (2025), 2171–2177
https://doi.org/10.2298/FIL2507171A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Chain graphs are not spectrally determined

**Milica Anđelić[a,*], Fernando C. Tura[b]**

[a]*Department of Mathematics, College of Science, Kuwait University, Al-Shadadiyah, Kuwait*
[b]*Departamento de Matemática, Universidade Federal de Santa Maria, Santa Maria, Rio Grande do Sul, Brazil*

**Abstract.** Two non isomorphic graphs $G$ and $H$ are said to be cospectral, if they share the same adjacency eigenvalues. In this paper, we use a recursive procedure for computing the characteristic polynomial of chain graphs in order to obtain infinitely many pairs of connected cospectral chain graphs. This result disproves a conjecture posed in [Czechoslovak Math. J. 70 (4) (2020), 1125–1138]. In addition we construct infinite families of bipartite graphs sharing the same spectrum.

## 1. Introduction

Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$, without loops or multiple edges. For $v \in V, N(v)$ denotes the open neighborhood of $v$, that is, $\{w|\{v, w\} \in E\}$. If $|V| = n$, the adjacency matrix $A = [a_{ij}]$ is an $n \times n$ matrix whose rows and columns are indexed by the vertices of $G$, and is defined to have entry $a_{ij} = 1$ if and only if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. The characteristic polynomial of $G$, denoted by $p(x, G)$, is $p(x, G) = \det(A - xI)$. Its roots are called the eigenvalues of $G$, and they comprise the spectrum of $G$.

This paper is concerned with chain graphs, also called double nested graphs. This class of graphs plays an important role in *spectral graph theory* [5]. In fact, among all connected bipartite graphs with fixed order and size, the graphs with maximal index (largest eigenvalue of adjacency matrix) are chain graphs [4].

A chain graph has several alternative characterizations. For example, a chain graph is a $\{2K_2, C_3, C_5\}$-free graph [1], that is, a graph which does not contain $2K_2, C_3$, and $C_5$ as induced subgraphs. In addition, a chain graph can be defined recursively through its generating binary sequence (more details will be given in the next section). Linear time algorithms for recognizing this class of graphs are given in [12].

We say that a graph $G$ is determined by its spectrum (for short, DS) if any other graph non-isomorphic to $G$ possesses a different spectrum. This notion is originally defined for the adjacency matrix of a graph $G$, but it has been also extended to other graph matrices. Common problem related to the spectral determination is to characterize families of graphs that are determined by their spectrum with respect to particular graph matrix. On the other hand, providing families of non-DS graphs, that is, non-isomorphic graphs sharing

2020 *Mathematics Subject Classification*. Primary 15A18; Secondary 05C50.
*Keywords*. chain graphs, adjacency matrix, cospectral graphs.
Received: 11 September 2024; Revised: 28 October 2024; Accepted: 31 October 2024
Communicated by Ljubiša D. R. Kočinac
This research was supported by CNPq Grant 200716/2022-0.
* Corresponding author: Milica Anđelić
*Email addresses:* milica.andelic@ku.edu.kw (Milica Anđelić), fernando.tura@ufsm.br (Fernando C. Tura)
ORCID iDs: https://orcid.org/0000-0002-3348-1141 (Milica Anđelić), https://ordid.org/0000-0001-5423-7121 (Fernando C. Tura)

the same spectrum (also known as *cospectral graphs*) is related counter problem. We mention that in the literature one can find many constructions of families of cospectral graphs, and among them famous Godsil and McKay switching [7]. Haemers and Spende [11] enumerated all cospectral graphs on up to 11 vertices for several matrices. An alternative construction due to Schwenk [14] leads to the probabilistic result that almost all trees have a cospectral mate. Schwenk's result was originally shown for the adjacency matrix, but soon after Godsil and McKay [8] and Mckay [13] proved that almost all trees are non-DS with respect to several other matrices, including the Laplacian and, hence, because trees are bipartite graphs, the signless Laplacian. Recently, a family of $2^{n-4}$ threshold graphs on $n \geq 4$ vertices, having the same signless Laplacian eigenvalues was constructed in [6].

In [3] the authors provided an explicit formula for computing the characteristic polynomial of a chain graph deduced from its generating binary sequence. The same reference also contains the following conjecture.

**Conjecture 1.1.** *Let G and H be two connected chain graphs of the same order, generated by* $(0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\cdots(0^{t_h}1^{s_h})$ *and* $(0^{p_1}1^{q_1})(0^{p_2}1^{q_2})\cdots(0^{p_h}1^{q_h})$*, respectively. If G and H are cospectral then* $(t_i, s_i) = (p_i, q_i)$ *for* $i = 1, 2, \dots, h$.

In this article, we use a recursive procedure for computing the characteristic polynomial of chain graph to provide pairs of connected cospectral chain graphs. As a by-product we disprove Conjecture 1.1.

An outline of the remainder of this paper reads the following. In Section 2, we present several definitions and the recursive procedure that will be used as a main tool for our constructions. In Section 3, we give two different ways for obtaining infinitely many pairs of connected, cospectral chain graphs.

## 2. Preliminaries

We first recall the structure of chain graphs. A bipartite graph $G$ with bipartition $(U, V)$ is a *chain graph* if there exist partitions $U = U_1 \cup U_2 \cup \cdots \cup U_h$ and $V = V_1 \cup V_2 \cup \cdots \cup V_h$, such that $U_i$ and $V_i$ are non-empty sets, and the neighborhood of each vertex in $U_i$ is $V_i \cup V_2 \cup \cdots \cup V_h$ for $1 \leq i \leq h$. The $2h$ non empty cells $V_i$ and $U_i$ for $i = 1, 2, \dots, h$, are called *white* and *black* cells, respectively.

Alternatively a chain graph can be constructed via binary sequences. Namely, for a given binary sequence $b = b_1 b_2 \dots b_n$, where $b_i \in \{0, 1\}$, the associated chain graph $G(b)$ is constructed as follows:

- for $i = 1$, $G_1 = G(b_1) = K_1$, i.e. a single (say white) vertex;
- for $i = 2, \dots, n$, with $G_{i-1} = G(b_1 \dots b_{i-1})$ already constructed, $G_i = G(b_1 \dots b_{i-1} b_i)$ is formed by adding to $G_{i-1}$ an isolated white vertex if $b_i = 0$, or a black vertex which dominates all existing white vertices if $b_i = 1$.

Therefore, a chain graph can be created by generating binary sequence of the following form:

$$b = (0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\cdots(0^{t_h}1^{s_h}), \qquad (t_i, s_i > 0),$$

where $t_i$ and $s_i$ are the lengths of maximal runs of consecutive zeros and ones, respectively. The Figure 1 shows the representation of a chain graph from its generating binary sequence.

We denote by $p(x, G)$ the characteristic polynomial of a graph $G$ in the indeterminate $x$ of its adjacency matrix $A$. The characteristic polynomial of a chain graph $G(b)$ generated by $b = (0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\cdots(0^{t_h}1^{s_h})$ was studied in [2]. There, a fast algorithm for its computation was presented. Here we focus on a direct computation via determinant of certain tridiagonal matrix as indirectly emphasized in [3, Theorem 3.1].

**Theorem 2.1.** *The characteristic polynomial* $p(x, G)$ *of the chain graph* $G(b)$ *generated by* $b = (0^{t_1}1^{s_1})\cdots(0^{t_h}1^{s_h})$*, where* $t_i, s_i > 0$*, is given by the formula*

$$p(x, G) = x^{\sum_{i=1}^{h}(t_i+s_i-2)}(-1)^h \det M_x^h(G) \tag{1}$$

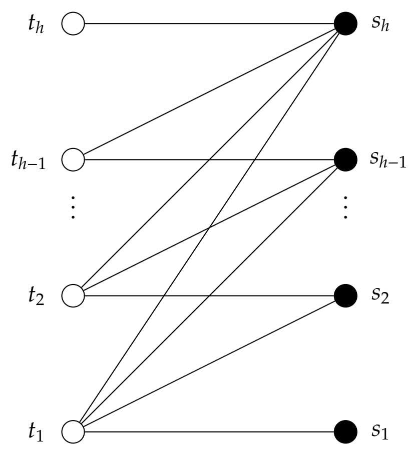


Figure 1: The structure of chain graph $G(b)$.

*where $M_x^h(G)$ is the following tridiagonal matrix*

$$M_x^h(G) = \begin{bmatrix} t_1 & x & & & & \\ x & s_1 & x & & & \\ & x & t_2 & x & & \\ & & \ddots & \ddots & \ddots & \\ & & & x & t_h & x \\ & & & & x & s_h \end{bmatrix}_{2h}. \tag{2}$$

*Proof.* Let $U_i$ (resp. $V_i$) denote the set of vertices corresponding to $0^{t_i}$ (resp. $1^{s_i}$). Then $V(G) = U_1 \cup V_1 \cup \cdots \cup U_h \cup V_h$ induces an equitable partition (see [9]) with the corresponding divisor matrix of the form:

$$D(G) = \begin{bmatrix} 0 & s_1 & 0 & s_2 & \dots & 0 & s_h \\ t_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & s_2 & \dots & 0 & s_h \\ t_1 & 0 & t_2 & 0 & \dots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & & \dots & 0 & s_h \\ t_1 & 0 & t_2 & 0 & \dots & t_h & 0 \end{bmatrix}_{2h}.$$

According to [2], $p(x, G) = x^{\sum_{i=1}^{h}(t_i+s_i-2)} \det(D(G) - xI)$. To compute $\det(D(G) - xI)$, i.e., the determinant of

$$D(G) - xI = \begin{bmatrix} -x & s_1 & 0 & s_2 & \dots & 0 & s_h \\ t_1 & -x & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -x & s_2 & \dots & 0 & s_h \\ t_1 & 0 & t_2 & -x & \dots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & & \dots & -x & s_h \\ t_1 & 0 & t_2 & 0 & \dots & t_h & -x \end{bmatrix}_{2h}.$$

we apply the following elementary row operations: $R_{2i-1} \leftarrow R_{2i-1} - R_{2i+1}$, for $i = 1, \dots, h-1$; $R_{2i} \leftarrow R_{2i} - R_{2i-2}$, for $i = 2, \dots, h$; and $R_{2i-1} \leftrightarrow R_{2i}$, for $i = 1, \dots, h$. Consequently, $\det(D(G) - xI) = (-1)^h \det M_x^h(G)$, for $M_x^h(G)$ given in (2). This completes the proof. □

Next, we present an alternative way to compute the characteristic polynomial of a chain graph.

**Theorem 2.2.** *The characteristic polynomial $p(x, G)$ of the chain graph $G(b)$ generated by $b = (0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\cdots(0^{t_h}1^{s_h})$, where $t_i, s_i > 0$, is given by the equation* (1). *In addition,* $\det M_x^h(G) = \Theta_h(x)$ *satisfies the recurrence relation*

$$\Theta_h(x) = \left(\frac{s_{h-1} + s_h}{s_{h-1}} x^2 - t_h s_h\right)\Theta_{h-1}(x) - \frac{s_h}{s_{h-1}} x^4 \Theta_{h-2}(x), \tag{3}$$

*where* $\Theta_0(x) = 1, \Theta_1(x) = \det\begin{bmatrix} t_1 & x \\ x & s_1 \end{bmatrix}$.

*Proof.* Let $p(x, G)$ be the characteristic polynomial of the chain graph $G(b)$ given by (1), that is, $p(x, G) = x^{\sum_{i=1}^{h}(t_i+s_i-2)}(-1)^h \det M_x^h(G)$. By setting, $\Theta_h(x) = \det M_x^h(G)$, we have that

$$\Theta_h(x) = s_h \Theta'_h(x) - x^2 \Theta_{h-1}(x) \tag{4}$$

and

$$\Theta'_h(x) = t_h \Theta_{h-1}(x) - x^2 \Theta'_{h-1}(x). \tag{5}$$

Since

$$\Theta_{h-1}(x) = s_{h-1}\Theta'_{h-1}(x) - x^2\Theta_{h-2}(x),$$

then

$$\Theta'_{h-1}(x) = \frac{\Theta_{h-1}(x) + x^2\Theta_{h-2}(x)}{s_{h-1}}. \tag{6}$$

By replacing the equation (6) in (5), we arrive at

$$\Theta'_h(x) = t_h\Theta_{h-1}(x) - x^2\frac{\Theta_{h-1}(x) + x^2\Theta_{h-2}(x)}{s_{h-1}} = t_h\Theta_{h-1}(x) - \frac{x^2}{s_{h-1}}\Theta_{h-1}(x) - \frac{x^4}{s_{h-1}}\Theta_{h-2}(x)$$

which then replaced in (4) results in

$$\begin{aligned}\Theta_h(x) &= s_h\left(t_h\Theta_{h-1}(x) - \frac{x^2}{s_{h-1}}\Theta_{h-1}(x) - \frac{x^4}{s_{h-1}}\Theta_{h-2}(x)\right) - x^2\Theta_{h-1}(x)\\ &= \left(s_h t_h - \frac{s_h}{s_{h-1}}x^2 - x^2\right)\Theta_{h-1}(x) - \frac{s_h}{s_{h-1}}x^4\Theta_{h-2}(x)\\ &= \left(\frac{s_{h-1} + s_h}{s_{h-1}}x^2 - s_h t_h\right)\Theta_{h-1}(x) - \frac{s_h}{s_{h-1}}x^4\Theta_{h-2}(x)\end{aligned}$$

as desired. □

## 3. Cospectral chain graphs

In this section, we use the recurrence relation given in (3) to present two different ways for constructing pairs of connected chain graphs which are cospectral with respect to adjacency matrix.

**Theorem 3.1.** *Let $G = G(b)$ be the chain graph generated by $b = (0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\cdots(0^{t_h}1^{s_h})$ and $H = H(b')$ be the chain graph generated by $b' = (0^{p_1}1^{q_1})(0^{p_2}1^{q_2})\cdots(0^{p_h}1^{q_h})$, both of the same order $n$. If*

$$\begin{cases} t_i \cdot s_i = p_i \cdot q_i & \text{for } i = 1, 2, \ldots, h. \\ t_i(s_i + s_{i+1}) = p_i(q_i + q_{i+1}) & \text{for } i = 1, 2, \ldots, h-1, \end{cases}$$

*then $G(b)$ and $H(b')$ are cospectral graphs.*

*Proof.* If $G$ and $H$ are isomorphic then the result is obvious. Now, assume that $G$ and $H$ are non isomorphic graphs. Let $p(x, G)$ and $p(x, H)$ be the characteristic polynomials of $G(b)$ and $H(b')$, respectively. We will show that under given assumptions $p(x, G) = p(x, H)$.

We have that

$$p(x, G) = x^{\sum_{i=1}^{h}(s_i+t_i-2)}(-1)^h \det M_x^h(G) \text{ as well as } p(x, H) = x^{\sum_{i=1}^{h}(p_i+q_i-2)}(-1)^h \det M_x^h(H).$$

Since $\sum_{i=1}^{h}(s_i+t_i-2) = \sum_{i=1}^{h}(p_i+q_i-2)$ it is sufficient to show that $\det M_x^h(G) = \det M_x^h(H)$, that is, $\Theta_h^G(x) = \Theta_h^H(x)$.

For $h = 0$ and $h = 1$ we have that $\Theta_0^G(x) = 1 = \Theta_0^H(x)$ and

$$\Theta_1^G(x) = \det\begin{bmatrix} t_1 & x \\ x & s_1 \end{bmatrix} = t_1 s_1 - x^2 = p_1 q_1 - x^2 = \Theta_1^H(x).$$

Now, for $k = 2, \ldots, h-1$, we assume that $\Theta_k^G(x) = \Theta_k^H(x)$. Taking into account that

$$\frac{s_{h-1} + s_h}{s_{h-1}} = \frac{t_{h-1}(s_{h-1} + s_h)}{t_{h-1}s_{h-1}} = \frac{p_{h-1}(q_{h-1} + q_h)}{p_{h-1}q_{h-1}} = \frac{q_{h-1} + q_h}{q_{h-1}} \tag{7}$$

and

$$p_{h-1}(q_{h-1} + q_h) = t_{h-1}(s_{h-1} + s_h)$$

it follows that

$$p_{h-1}q_h = t_{h-1}s_h$$

and thus

$$\frac{s_h}{s_{h-1}} = \frac{s_h t_{h-1}}{s_{h-1}t_{h-1}} = \frac{p_{h-1}q_h}{p_{h-1}q_{h-1}} = \frac{q_h}{q_{h-1}}. \tag{8}$$

By replacing (7) and (8) in the recurrence relation (3), we obtain

$$\begin{aligned} \Theta_h^G(x) &= \left(\frac{s_{h-1} + s_h}{s_{h-1}}x^2 - t_h s_h\right)\Theta_{h-1}^G(x) - \frac{s_h}{s_{h-1}}x^4\Theta_{h-2}^G(x) \\ &= \left(\frac{q_{h-1} + q_h}{q_{h-1}}x^2 - p_h q_h\right)\Theta_{h-1}^H(x) - \frac{q_h}{q_{h-1}}x^4\Theta_{h-2}^H(x) \\ &= \Theta_h^H(x) \end{aligned}$$

This completes the proof. □

Let $G(b) = (0^{t_1}1^{s_1})(0^{t_2}1^{s_2})\cdots(0^{t_h}1^{s_h})$ be a connected chain graph of order $n$ such that

$$\sum_{i=1}^{h} t_i = \sum_{i=1}^{h} s_i$$

and $k \geq 2$ a positive integer.

Assume that $G(b')$ and $H(b'')$ are the chain graphs of the same order generated by $b' = (0^{t_1}1^{ks_1})(0^{t_2}1^{ks_2})\cdots(0^{t_h}1^{ks_h})$ and $b'' = (0^{ks_1}1^{t_1})(0^{ks_2}1^{t_2})\cdots(0^{ks_h}1^{t_h})$ respectively. It is easy to verify that these graphs satisfy the conditions given by Theorem 3.1. That is,

$$t_i(ks_i) = (ks_i)t_i \qquad i = 1,\dots,h$$

and

$$t_i(ks_i + ks_{i+1}) = kt_i(s_i + s_{i+1}) \qquad i = 1,\dots,h-1$$

As a consequence, we obtain the following.

**Corollary 3.2.** *For any positive integer $k \geq 2$ chain graphs $G(b')$ and $H(b'')$ generated by $b' = (0^{t_1}1^{ks_1})(0^{t_2}1^{ks_2})\cdots(0^{t_h}1^{ks_h})$ and $b'' = (0^{ks_1}1^{t_1})(0^{ks_2}1^{t_2})\cdots(0^{ks_h}1^{t_h})$ respectively, are cospectral graphs.*

**Example 3.3.** *Let G and H be chain graphs generated by $(0^11^3)(0^21^6)(0^31^9)$ and $(0^31^1)(0^61^2)(0^91^3)$. These are cospectral chain graphs. In fact, their characteristic polynomials are both given by*

$$p(x,G) = x^{18}(x^6 - 75x^4 + 657x^2 - 972) = p(x,H).$$

*Clearly, they are non-isomorphic, since their degree sequences are not equal.*

The result given in Corollary 3.2 was also presented in the paper [10] using a different approach. In fact, there, chain graphs appear as minimizers for the coefficient next to $x^4$ in the characteristic polynomial in the class of connected bipartite graphs of fixed order and size.

The following example shows that the conditions given in Theorem 3.1 are not necessary.

**Example 3.4.** *Let G and H be chain graphs generated by $(0^21^4)(0^31^6)$ and $(0^41^6)(0^31^2)$ as shown in Figure 2. We have that $t_1 \cdot s_1 = 2\cdot 4 \neq 4\cdot 6 = p_1 \cdot q_1$, $t_2 \cdot s_2 = 3\cdot 6 \neq 3\cdot 2 = p_2\cdot q_2$ and $t_1(s_1+s_2) = 2(4+6) \neq 4(6+2) = p_1(q_1+q_2)$. However, the graphs are cospectral, since their characteristic polynomials are*

$$p(x,G) = x^{11}(x^4 - 38x^2 + 144) = p(x,H).$$

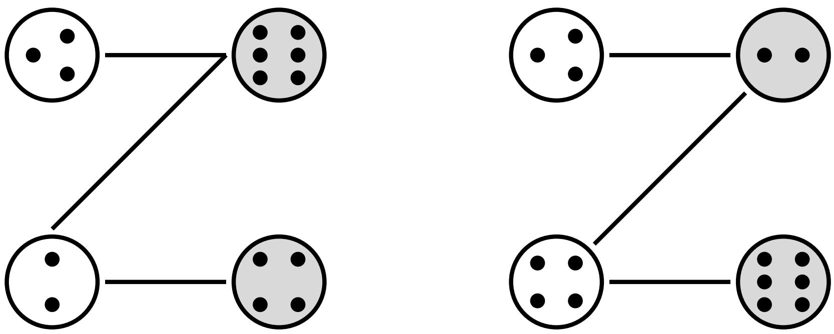

Figure 2: Cospectral chain graphs

The result in the sequel provides an additional way to construct infinitely many pairs of connected cospectral chain graphs.

**Theorem 3.5.** *Let $k, t_1, t_2, s_1$ be positive integers, $k \geq 2$. If*

$$k(s_1 - t_1) - t_2(k-1) = 0,$$

*then connected chain graphs $G, H$ of order $n$ generated respectively by $(0^{t_1}1^{s_1})(0^{t_2}1^{k\cdot t_1})$ and $(0^{s_1}1^{k\cdot t_1})(0^{t_2}1^{t_1})$ are cospectral.*

*Proof.* It is clear that $G$ and $H$ are not isomorphic graphs since they have distinct degree sequences. Let $p(x, G)$ and $p(x, H)$ be the characteristic polynomials of $G$ and $H$, respectively. We will show that $p(x, G) = p(x, H)$.

Applying the recurrence relation (3) for $G$ we obtain

$$p(x,G) = x^{s_1+t_2+(k+1)t_1-2}\left(\left(\frac{s_1+s_2}{s_1}x^2 - t_2s_2\right)\Theta_1^G(x) - \frac{s_2}{s_1}x^4\Theta_0^G(x)\right)$$

that is

$$p(x,G) = x^{s_1+t_2+(k+1)t_1-2}\left(x^4 - (t_1s_1 + kt_1^2 + kt_1t_2)x^2 + kt_1^2t_2s_1\right)$$

By a similar computation for $H$ we have that

$$\begin{aligned} p(x,H) &= x^{s_1+t_2+(k+1)t_1-2}\left(\left(\frac{kt_1+t_1}{kt_1}x^2 - t_1t_2\right)\Theta_1^H(x) - \frac{t_1}{kt_1}x^4\Theta_0^H(x)\right) \\ &= x^{s_1+t_2+(k+1)t_1-2}\left(x^4 - (kt_1s_1 + t_1s_1 + t_1t_2)x^2 + kt_1^2t_2s_1\right) \end{aligned}$$

By equating the corresponding coefficients in $p(x, G)$ and $p(x, H)$ it follows that

$$s_1(t_1 + kt_1) + t_1t_2 = t_1s_1 + kt_1^2 + kt_1t_2$$

which is equivalent to $ks_1 - kt_1 - t_2(k-1) = 0$. This completes the proof. □

## 4. Acknowledgements

This research started while Fernando Tura was doing his Post-Doctoral studies at Georgia State University, on leave from UFSM.

## References

[1] A. Alazemi, M. Anđelić, S.K. Simić, *Eigenvalue location for chain graphs*, Linear Algebra Appl. **505** (2016), 194–210.
[2] M. Anđelić, S.K. Simić, D. Živković, E. Dolićanin, *Fast algorithms for computing the characteristic polynomial of threshold and chain graphs*, Appl. Math. Comput. **332** (2018), 329–337.
[3] M. Anđelić, Z. Du, C.M. da Fonseca, S.K. Simić, *Tridiagonal matrices and spectral properties of some graph classes*, Czechoslovak Math. J. 70 **4** (2020), 1125–1138.
[4] A. Bhattacharya, S. Friesland, U.N. Peled, *On the first eigenvalue of bipartite graphs*, Electron. J.Combin., **15** (2008), 144
[5] A. E. Brouwer, W. H. Haemers, *Spectra of Graphs*, Springer, Berlin, 2012.
[6] J. Carvalho, B. S. Souza, V. Trevisan, F. C. Tura, *Exponentially many graphs have a Q-cospectral mate*, Discrete Math. **340** (2017), 2079–2085.
[7] C. D. Godsil, B. D. McKay, *Constructing cospectral graphs*, Aequationes Math. **25** (1982), 257–268.
[8] C. Godsil, B. McKay, *Some computational results on the spectra of graphs*, in: Combinatorial Mathematics, IV (Proc. Fourth Australian Conf., Univ. Adelaide, Adelaide, 1975), in: Lecture Notes in Math., **560**, Springer, Berlin, 1976, pp. 73–92.
[9] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.
[10] S. C. Gong, L. P. Zhang, S. W. Sun, *On Bipartite Graphs Having Minimum Fourth Adjacency Coefficient*, Graphs Combin. **38**, 60 (2022)
[11] E. R. van Dam, W. H. Haemers, *Which graphs are determined by their spectrum?*, Linear Algebra Appl. **373** (2003), 241–272.
[12] P. Heggernes, D. Kratsch, *Linear-time certifying recognition algorithms and forbidden induced subgraphs* , Nordic. J.Comput. **14** (2007), 87–108.
[13] B. D. McKay, *On the spectral characterisation of trees*, Ars Combin. (ISSN: 0381-7032) **3** (1977), 219–232.
[14] A. J. Schwenk, *Almost all trees are cospectral*, in: New Directions in the Theory of Graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971), Academic Press, New York, 1973, pp. 275–307.