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Pointwise hemi bi-slant submersions

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Abstract. In this study, we introduce a new notion of pointwise hemi bi-slant Riemannian submersions, a profound generalization encompassing various established submersion types like anti-invariant, slant, semi-slant, pointwise slant, pointwise semi-slant, and bi-slant submersions, all within the framework of almost product manifolds. After presenting a unique illustrative example, our investigation delves into the integrability conditions and geodesics governing this novel submersion concept. Furthermore, we unravel the complexities of $\phi-$pluriharmonicity and $\phi-$invariance within this context, revealing the subtle interplay between pointwise hemi bi-slant submersions' fibers and their classification as either geodesic or mixed geodesic. This in-depth analysis provides valuable insights into the intricate geometric properties of these fascinating mappings, offering a comprehensive understanding of their underlying principles and paving the way for future research and application.

1. Introduction

The concept of submanifolds in Differential Geometry proves highly valuable, extending the notions of curves and surfaces into higher dimensions. This framework plays a pivotal role in representing configuration spaces of physical systems, enabling the efficient depiction of intricate shapes and motion paths in robotics and computer science. In essence, submanifolds offer a versatile approach to comprehending complex geometries and their inherent characteristics. By delving into the structure of spaces, they provide profound insights and applications spanning diverse fields, establishing them as a fundamental concept in contemporary mathematics and its myriad applications. Recognizing the pivotal role of submanifolds, geometers have dedicated efforts to define and scrutinize specific instances of these mathematical structures. One approach to establishing a submanifold involves working with submersions, with the Riemannian Submersion being the most extensively studied. The concept of Riemannian submersion was initially introduced by O'Neill [11] and carried profound implications in physics, particularly in exploring gauge and field theories. In the context of fiber bundles, Riemannian submersions frequently come into play when projecting a higher-dimensional physical space onto a lower-dimensional base manifold. This projection preserves crucial geometric and metric properties, rendering it an invaluable tool for precise modeling and understanding of physical phenomena, including gauge field theories and the broader geometry of spacetime in general relativity. Moreover, Riemannian submersions find practical applications in optimal control theory, offering valuable insights into the dynamics and symmetries of physical systems.

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In this context, we present a groundbreaking submersion concept *"pointwise hemi bi-slant Riemannian."* This innovative approach serves as a substantial expansion of previously explored submersions in the existing literature, offering valuable contributions and paving the way for new avenues of exploration in various contexts. This newly introduced submersion not only encapsulates but also extends the principles outlined in earlier studies, including pointwise hemi-slant [2], pointwise slant [22], bi-slant [17], slant [19] and other related submersions, enriching the field with its comprehensive scope. The structure can be outlined as follows:

The Preliminaries section covers the basics of Riemannian submersion and almost product Riemannian manifolds. In Section 3, our primary focus is on *pointwise hemi bi-slant Riemannian submersion*. This section begins by establishing the framework for these submersions, presenting an original example, and highlighting the significance of our work. This importance is substantiated through Table 1, various decompositions, and specific results essential throughout the paper. In the initial subsection, efforts are directed toward finding solutions for the integrability conditions of the distributions outlined in the definition of these submersions. The following subsection focuses on totally geodesic structures and concludes by expressing the fibers as local products of the integral manifolds of the distributions. In the third subsection, a novel approach named $\phi-$pluriharmonicity of the submersion π is explored, especially in cases where the fibers exhibit either totally geodesic or mixed geodesic properties. This innovative approach provides a pathway for exploring the harmonicity of a submersion. The final subsection is dedicated to investigating the $\phi-$invariance of the submersion, establishing specific conditions under which pointwise slant angles are implicated.

2. Preliminaries

2.1. Riemannian submersions

This section is devoted to the basics of Riemannian submersions.

Let (M, g) and $(N, \bar{g})$ be Riemannian manifolds. A surjective mapping $\pi : (M, g) \to (N, \bar{g})$ is called a *Riemannian submersion* [11] if

i) π has maximal rank;

ii) the restriction of the differential map π_* on $(\ker \pi_*)^\perp$ is a linear isometry.

In this case, we recall the following observations and concepts;

- For each $q \in N$, $\pi^{-1}(q)$ is a k-dimensional submanifold of M and called a *fiber*, where $k = \dim(M) - \dim(N)$.
- A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers.
- We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $\ker \pi_*$ and the horizontal distribution $\ker \pi_*^\perp$, respectively.
- The manifold (M, g) is called *total manifold* and the manifold $(N, \bar{g})$ is called *base manifold* of the submersion $\pi : (M, g) \to (N, \bar{g})$.
- A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e.,

$$\pi_* X_p = X_{*\pi(p)}, \ \forall p \in M.$$

The last fact given above yields the following Lemma [11], which explains the preservation of brackets, inner products, and covariant derivatives;

Lemma 2.1. *Let $\pi : (M, g) \to (N, \bar{g})$ be a Riemannian submersion between Riemannian manifolds. If X and Y are basic vector fields, then*

- $g(X, Y) = \bar{g}(X_*, Y_*) \circ \pi$,
- *the horizontal part* $\mathcal{H}[X, Y]$ *of* $[X, Y]$ *is a basic vector field corresponding to* $[X_*, Y_*]$,
- *the horizontal part* $\mathcal{H}(\nabla^M_X Y)$ *of* $\nabla^M_X Y$ *is the basic vector field corresponding to* $\nabla^N_{X_*} Y_*$,
- $[U, X]$ *is vertical for any vector field* U *of* $\ker \pi_*$.

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_E G = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}G + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}G, \tag{1}$$

$$\mathcal{A}_E G = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}G + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}G \tag{2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection of g. One can see that a Riemannian submersion π has totally geodesic fibers if and only if $\mathcal{T}$ vanishes. On the other side, $\mathcal{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. Moreover, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions.
Now we give the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$.

Let V, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{4}$$

On the other hand, from (1) and (2), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{5}$$

$$\nabla_V X = \mathrm{T}_V X + \mathcal{H}\nabla_V X, \tag{6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{8}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic

$$\mathcal{H}\nabla_V X = \mathcal{A}_X V.$$

Remark 2.2. *In this paper, we will assume all horizontal vector fields as basic vector fields.*

For more details, we refer to O'Neill's paper [11] and the book [7].

Let Ψ be a C^∞-map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). The second fundamental form of Ψ is given by

$$(\nabla\Psi_*)(X, Y) = \nabla^\Psi_X \Psi_* Y - \Psi_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM_1), \tag{9}$$

where ∇^Ψ is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_1 and g_2, [3].

If $(\nabla\Psi_*)(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$, Ψ is called a *totally geodesic map*. In particular, if $(\nabla\Psi_*)(X, Y) = 0$, $X, Y \in \Gamma(D)$ for any subset D of TM, Ψ is called a $D-$*totally geodesic map*, [3].

2.2. Almost product Riemannian and locally product Riemannian manifolds

An m-dimensional manifold M is called *almost product manifold* with *almost product structure* ϕ which is a tensor field of type (1,1) satisfying

$$\phi^2 = id, (\phi \neq \pm id) \ , \tag{10}$$

denoted by (M, ϕ). Also for $E, G \in \Gamma(TM)$, (M, ϕ) admits a Riemannian metric g satisfying

$$g(\phi E, \phi G) = g(E, G), \tag{11}$$

then M is said to be an *almost product Riemannian manifold*.
Let ∇ be the Riemannian connection with respect to the metric g on M. Then M is called a *locally product Riemannian manifold* (briefly, *l.p.R.*) if ϕ is parallel with respect to the connection, i.e. [28]

$$\nabla\phi = 0. \tag{12}$$

3. Pointwise hemi bi-slant submersions

This section is the main part of our work. We define and study pointwise hemi bi-slant Riemannian submersion.

Definition 3.1. *Let (M, g, ϕ) be an almost product Riemannian manifold and $(N, \bar{g})$ be a Riemannian manifold. A Riemannian submersion $\pi : (M, g, \phi) \to (N, \bar{g})$ is called a pointwise hemi bi-slant Riemannian submersion if the vertical distribution $ker\pi_*$ of ϕ decomposes into three orthogonal complementary anti-invariant distribution $\mathcal{D}^\perp$, (pointwise slant) distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$.*
In this case, we have the decomposition

$$ker\pi_* = \mathcal{D}^\perp \oplus \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}, \tag{13}$$

where $\mathcal{D}^\perp$ is an anti-invariant distribution, $\mathcal{D}^{\theta_i}$ is a pointwise slant distribution and the angle θ_i between ϕU and the space $(D^{\theta_i})_q$, $(\forall q \in M)$, is independent of the choice of nonzero vector $U \in \Gamma(\mathcal{D}^{\theta_i})_q$, which is called slant function of the pointwise bi-slant Riemannian submersion, for $i = 1, 2$.

Remark 3.2. *Given concept pointwise hemi bi-slant submersion in this work is a special version of generalized skew-semi-invariant submersions, [18]. The goal of this work is obtaining results in case of the existence of only anti-invariant distribution and two pointwise distributions.*

We give an example to show the existence of such submersions.

Example 3.3. *Consider the standard Euclidean space $\mathbb{R}^{10}$ with the standard metric g. One can see that*

$$\phi_1(x_1, x_2, ..., x_8, x_9, x_{10}) = (-x_3, x_4, -x_1, x_2, -x_7, x_8, -x_5, x_6, x_9, x_{10})$$

and

$$\phi_2(x_1, x_2, ..., x_8, x_9, x_{10}) = (x_2, x_1, x_4, x_3, x_6, x_5, x_8, x_7, x_9, x_{10})$$

are almost product Riemannian structures on $\mathbb{R}^8$, where $\phi_1\phi_2 = -\phi_2\phi_1$. For any smooth function $\pi : \mathbb{R}^{10} \to \mathbb{R}^5$, we can define a new almost product Riemannian structure such that

$$\phi_{1,2} = f\phi_1 + g\phi_2,$$

where f and g defined by

$$\begin{aligned} f : \mathbb{R}^{10} - \{-1\} &\to \mathbb{R} \\ f(x_1, x_2, ..., x_8, x_9, x_{10}) &= -\frac{x_1}{\sqrt{(x_1)^2 + 1}} \end{aligned}$$

$$\begin{aligned} g : \mathbb{R}^{10} &\rightarrow \mathbb{R} \\ g(x_1, x_2, ..., x_8, x_9, x_{10}) &= \frac{1}{\sqrt{(x_1)^2 + 1}}. \end{aligned}$$

Therefore, $(\mathbb{R}^{10}, \phi_{1,2}, g)$ *is an almost product Riemannian manifold.*

Now, let π *be a map between* $\mathbb{R}^{10}$ *and* $\mathbb{R}^5$ *defined by*

$$\pi(x_1, x_2, ..., x_8, x_9, x_{10}) = \left(\frac{x_1 - x_3}{\sqrt{2}}, \frac{x_2 - x_4}{\sqrt{2}}, \frac{x_5 + x_8}{\sqrt{2}}, \frac{-x_6 + x_7}{\sqrt{2}}, x_9\right).$$

The following decomposition of $\ker \pi_*$

$$\ker \pi_* = \mathcal{D}^\perp \oplus D^{\theta_1} \oplus D^{\theta_2},$$

where

$$\begin{aligned} D^\perp &= span\left\{\frac{\partial}{\partial x_{10}}\right\} \\ D^{\theta_1} &= span\left\{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\right\}, \\ D^{\theta_2} &= span\left\{\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_8}, -\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}\right\} \end{aligned}$$

shows that π *is a pointwise bi-slant submersion with the slant functions*

$$\theta_1 = \cos^{-1}(g), \quad \text{and} \quad \theta_2 = \cos^{-1}(-f).$$

Remark 3.4. *The following table, which gives some exceptional cases of our work, explains the significance of it and how generic concept it is.*

Table 1: Subclasses of a pointwise hemi bi-slant Riemannian submersion

$\dim D^\perp$	$\dim D^{\theta_1}$	$\dim D^{\theta_2}$	**Submersion**	**Reference**
$\neq 0$	0	0	anti-invariant	[9],[14]
0	$\neq 0$ (θ_1 constant)	$\neq 0$ (θ_2 constant)	bi-slant	[17]
0	$\neq 0$ ($\theta_1 = 0$)	$\neq 0$ ($\theta_2 > 0$ function)	pointwise semi-slant	[16]
$\neq 0$	$\neq 0$ ($\theta_1 > 0$ constant)	0	hemi-slant	[23]
$\neq 0$	$\neq 0$ ($\theta_1 > 0$ function)	0	pointwise hemi-slant	[2]
0	$\neq 0$ ($\theta_1 > 0$ constant)	0	slant	[15]
0	$\neq 0$ ($\theta_1 > 0$ function)	0	pointwise slant	[10]

Let π be a pointwise hemi bi-slant Riemannian submersion from a locally product Riemannian (l.p.R.) manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. We may set the following projections:

- For any $U \in \Gamma(ker\pi_*)$,

$$\phi U = PU + NU, \tag{14}$$

  where $PU \in \Gamma(ker\pi_*)$, $NU \in \Gamma(ker\pi_*^\perp)$.

- For any $\xi \in \Gamma(ker\pi_*^\perp)$,

$$\phi\xi = \delta\xi + \omega\xi, \tag{15}$$

  where $\delta\xi \in \Gamma(ker\pi_*)$, $\omega\xi \in \Gamma(ker\pi_*^\perp)$.

One can see that the pointwise slant distributions given in Definition 3.1 are P-invariant, i.e.

$$PD^{\theta_1} \subseteq D^{\theta_1},\ PD^{\theta_2} \subseteq D^{\theta_2}.$$

In this case, the horizontal distribution $ker\pi_*^{\perp}$ can be decomposed as

$$ker\pi_*^{\perp} = \phi D^{\perp} \oplus ND^{\theta_1} \oplus ND^{\theta_2} \oplus \mu, \tag{16}$$

where μ is the orthogonal complementary distribution of $\phi D^{\perp} \oplus ND^{\theta_1} \oplus ND^{\theta_2}$ in $ker\pi_*^{\perp}$. Moreover, μ is ϕ-invariant. If $\mu = \{0\}$, we will call the submersion *Lagrangian pointwise bi-slant submersion.*
The decompositions given above yield us the following facts;

$$(P^2 + \delta N)U = U, \quad NPU = -\omega NU,\ \forall U \in \Gamma(ker\pi_*), \tag{17}$$

$$(P\delta + \delta\omega)\xi = 0, \quad (N\delta + \omega^2)\xi = \xi,\ \forall \xi \in \Gamma(ker\pi_*^{\perp}). \tag{18}$$

If we consider (13) with (17), we have

$$\delta NU = U,\ \forall U \in \Gamma(D^{\perp}), \tag{19}$$

$$P^2U = \cos^2\theta_i\, U, \forall U \in \Gamma(D^{\theta_i}),\ i = 1, 2, \tag{20}$$

$$\delta NU = \sin^2\theta_i\, U, \forall U \in \Gamma(D^{\theta_i}),\ i = 1, 2. \tag{21}$$

In this paper, we will work in the case of the endomorphism ϕ is parallel, in other words, with locally product Riemannian manifolds. Thus we have the following relations

Lemma 3.5. *Let π be a pointwise hemi bi-slant Riemannian submersion from a locally product Riemannian (l.p.R.) manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, for any $X, Y \in \Gamma(\ker\pi_*)$ and $\xi, \beta \in \Gamma(\ker\pi_*^{\perp})$, we have the followings*

$$\begin{aligned}
T_XPY + A_{NY}X &= \omega T_XY + N\hat{\nabla}_XY, && (22)\\
\hat{\nabla}_XPY + T_XNY &= \delta T_XY + P\hat{\nabla}_XY, && (23)\\
T_X\delta\xi + A_{\omega\xi}X &= NT_X\xi + \omega A_\xi X, && (24)\\
\hat{\nabla}_X\delta\xi + T_X\omega\xi &= PT_X\xi + \delta A_\xi X, && (25)\\
A_\xi PX + \mathcal{H}\nabla_\xi NX &= \omega A_\xi X + N\mathcal{V}\nabla_\xi X, && (26)\\
\mathcal{V}\nabla_\xi PX + A_\xi NX &= \delta A_\xi X + P\mathcal{V}\nabla_\xi X, && (27)\\
A_\xi\delta\beta + \mathcal{H}\nabla_\xi\omega\beta &= \omega\mathcal{H}\nabla_\xi\beta + NA_\xi\beta && (28)\\
\mathcal{V}\nabla_\xi\delta\beta + A_\xi\omega\beta &= \delta\mathcal{H}\nabla_\xi\beta + PA_\xi\beta && (29)
\end{aligned}$$

Proof. The fact $\nabla\phi = 0$ completes the proof. □

Lemma 3.6. *Let π be a pointwise hemi bi-slant Riemannian submersion from a locally product Riemannian (l.p.R.) manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, we have, for any $X, Y \in \Gamma(D^{\perp})$, $U, V \in \Gamma(D^{\theta_i})$, and $Z \in \Gamma(D^j)$, $i \neq j$, $i = 1, 2$,*

i) *$g(\nabla_XY, U)$ is equivalent to the followings;*

$$-\csc^2\theta_i\, g(T_YNPU + A_{NY}NU, X), \tag{30}$$

$$\sec^2\theta_i\, g(T_{PU}NY + T_YNPU, X). \tag{31}$$

ii) *$g(\nabla_UV, X)$ is equivalent to the followings;*

$$\csc^2\theta_i\, g(T_XNPV + A_{NX}NV, U), \tag{32}$$

$$\sec^2\theta_i\, g(T_{PV}NX - T_XNPV, U). \tag{33}$$

iii) *$g(\nabla_U V, Z)$ is equivalent to the followings;*

$$\csc^2\theta_j\Big(g(\mathcal{T}_U V, NPZ) + g(\mathcal{T}_U PV + \mathcal{A}_{NV}U, NZ)\Big), \tag{34}$$

$$\sec^2\theta_j\Big(g(\hat{\nabla}_U PV + \mathcal{T}_U NV, PZ) + g(\mathcal{T}_U V, \omega NZ)\Big). \tag{35}$$

Proof. Let $X, Y \in \Gamma(D^\perp)$ and $U, V \in \Gamma(D^{\theta_i})$, and $Z \in \Gamma(D^j)$, $i \neq j$, $i = 1, 2$.
i) The identities (11), (12), and (14) yield us

$$g(\nabla_X Y, U) = g(\nabla_X \phi Y, PU) + g(\nabla_X \phi Y, NU). \tag{36}$$

The first expression on the right side of (36), with the help of (5), (17), (20) and the symmetry of ϕ on the Riemannian metric takes form

$$g(\nabla_X \phi Y, PU) = \cos^2\theta_i g(\nabla_X Y, U) + g(\mathcal{T}_X Y, NPU),$$

which makes (36) with (6)

$$g(\nabla_X Y, U) = -\csc^2\theta_i\, g(\mathcal{T}_Y NPU + \mathcal{A}_{NY} NU, X).$$

On the other side, the second expression on the right side of (36) with the help of (5), (17), (21) and the symmetry of ϕ on the Riemannian metric takes form

$$g(\nabla_X \phi Y, NU) = \sin^2\theta_i g(\nabla_X Y, U) + g(\nabla_X Y, \omega NU),$$

which makes (36) with (6)

$$g(\nabla_X Y, U) = \sec^2\theta_i\, g(\mathcal{T}_{PV} NX + \mathcal{T}_X NPV, U),$$

completes the proof of **i)**.
ii) The identities (11), (12), and (14) gives us

$$g(\nabla_U V, X) = g(\nabla_U PV, NX) + g(\nabla_U NV, NX). \tag{37}$$

The first expression on the right side of (37), with the help of (5), (17), (20) and the symmetry of ϕ on the Riemannian metric takes form

$$g(\nabla_U PV, NX) = \cos^2\theta_i g(\nabla_U V, X) + g(\mathcal{T}_X NPV, U),$$

which makes (37) with (6)

$$g(\nabla_U V, X) = -\csc^2\theta_i\, g(\mathcal{T}_Y NPU + \mathcal{A}_{NY} NU, X).$$

On the other side, the second expression on the right side of (37) with the help of (5), (18), (21) and the symmetry of ϕ on the Riemannian metric takes form

$$g(\nabla_U NV, NX) = -\sin^2\theta_i g(\nabla_U X, V) + g(\mathcal{T}_X \omega NV, U),$$

which makes (37) with (5)

$$g(\nabla_U V, X) = \sec^2\theta_i\, g(\mathcal{T}_{PV} NX - \mathcal{T}_X NPV, U)$$

completes the proof of **ii)**.
iii) The equations (11) and (14) give us

$$g(\nabla_U V, Z) = g(\phi\nabla_U V, PZ) + g(\phi\nabla_U V, NZ). \tag{38}$$

The first expression on the right side of (38), with the help of (12), (14), (20), and the symmetry of ϕ on the Riemannian metric takes form

$$g(\phi\nabla_U V, PZ) = \cos^2\theta_j\, g(\nabla_U V, Z) + g(\nabla_U V, NPZ),$$

which makes (38) with (12), (5), (6), and the symmetry of ϕ on the Riemannian metric

$$g(\nabla_U V, Z) \;=\; \csc^2\theta_j \left(g(T_U V, NPZ) + g(T_U PV + A_{NV}U, NZ)\right).$$

On the other hand, the second expression on the right side of (38) with the help of (5), (11), (14), (15), (21), and the symmetry of ϕ takes form

$$g(\phi\nabla_U V, NZ) = \sin^2\theta_j\, g(\nabla_U V, Z) + g(T_U V, \omega NZ),$$

which makes (38) with (5)

$$g(\nabla_U V, Z) \csc^2\theta_j \Big(g(T_U V, NPZ) + g(T_U PV + A_{NV}U, NZ)\Big),$$

which completes the proof. □

3.1. Integrability

Since certain distributions appear in our work, a natural question is *the integrability of the given distributions*, i.e. *under what conditions we can find an integral manifolds for the corresponding distributions?* In this section, we investigate the integrability of the given distributions in the definition of a pointwise hemi bi-slant submersion.

Before we discuss the conditions for the integrability, we present a useful lemma.

Lemma 3.7. *Let π be a pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, we have, for any $X, Y \in \Gamma(D^\perp)$ and $U \in \Gamma(D^{\theta_1} \oplus D^{\theta_2})$,*

$$g(A_{NX}NU, Y) = -g(A_{NY}NU, X), \tag{39}$$

$$g(T_{PU}NY, X) = -g(T_{PU}NX, Y). \tag{40}$$

Proof. Let $X, Y \in \Gamma(D^\perp)$ and $U \in \Gamma(D^{\theta_1} \oplus D^{\theta_2})$. By the symmetry of ϕ, (8), and the skew-symmetry of A, we obtain

$$\begin{aligned} g(A_{NX}NU, Y) &= -g(A_{NU}NX, Y) = -g(\nabla_{NU}NX, Y) = -g(\nabla_{NU}\phi X, Y) \\ &= g(\nabla_{NU}\phi Y, X) = g(A_{NU}NY, X) = -g(A_{NY}NU, X), \end{aligned}$$

which proves (39). On the other side, by the symmetry of ϕ, (6), and the skew-symmetry of T, we obtain

$$\begin{aligned} g(T_{PU}NY, X) &= -g(T_{PU}X, NY) = -g(\nabla_{PU}NY, X) \\ &= -g(\nabla_{PU}\phi X, Y) = -g(T_{PU}NX, Y), \end{aligned}$$

which completes the proof. □

Now, we start with the integrability of the anti-invariant distribution $D^\perp$.

Theorem 3.8. *Let π be a pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the followings are equivalent to each other;*

i) *The anti-invariant distribution $D^\perp$ is integrable,*

ii) $g(A_{\phi Y}NU, X) = 0$,

iii) $g(T_{\phi U}NY, X) = 0$,

where $X, Y \in \Gamma(D^{\perp})$, $U \in \Gamma(D^{\theta_1} \oplus D^{\theta_2})$.

Proof. Let $X, Y \in \Gamma(D^{\perp})$, and $U \in \Gamma(D^{\theta_i})$, $i = 1, 2$. From the properties of O'Neill tensor T, (30), and Lemma 3.7, we have

$$\begin{aligned} g([X, Y], U) &= -\csc^2\theta_i\, g(T_Y NPU + A_{NY}NU, X) \\ &\quad + \csc^2\theta_i\, g(T_X NPU + A_{NX}NU, Y) \\ &= 2\csc^2\theta_i\, g(A_{NX}NU, Y). \end{aligned}$$

On the other side, from the properties of O'Neill tensor T, (31) and Lemma 3.7, we have

$$\begin{aligned} g([X, Y], U) &= \sec^2\theta_i\, g(T_{PU}NY + T_Y NPU, X) \\ &\quad - \sec^2\theta_i\, g(T_{PU}NX + T_X NPU, Y) \\ &= 2\sec^2\theta_i\, g(T_{PU}NY, X), \end{aligned}$$

which completes the proof. □

The next theorem is related to the integrability of the pointwise slant distributions D^{θ_1} and D^{θ_2}.

Theorem 3.9. *Let* π *be a pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold* (M, ϕ, g) *onto a Riemannian manifold* $(N, \bar{g})$. *Then, the followings are equivalent to each other;*

i) *The pointwise distribution* D^{θ_i} *is integrable,*

ii)

$$g(T_V NPU - T_U NPV, X) = g(A_{NU}V - A_{NV}U, NX),$$

and

$$g(T_U PV - T_V PU, NZ) = g(A_{NU}V - A_{NV}U, NZ)$$

iii)

$$g(T_{PU}V - T_{PV}U, NX) = g(T_V NPU - T_U NPV, X),$$

and

$$g(\hat{\nabla}_U PV - \hat{\nabla}_V PU, Z) = g(T_V NU - T_U NV, Z),$$

where $X \in \Gamma(D^{\perp})$, $U, V \in \Gamma(D^{\theta_i})$, $Z \in \Gamma(D^{\theta_j})$, $i \neq j$, $i, j = 1, 2$.

Proof. Let $X \in \Gamma(D^{\perp})$, $U, V \in \Gamma(D^{\theta_i})$, and $Z \in \Gamma(D^{\theta_j})$, $i \neq j$, $i, j = 1, 2$. The pointwise distribution D^{θ_i} is integrable if and only if $[U, V] \in D^{\theta_i}$ i.e. $[U, V] \perp D^{\perp}$ and $[U, V] \perp D^{\theta_j}$. With the help of (32), we have

$$\begin{aligned} g([U, V], X) &= \csc^2\theta_i\, g(T_X NPV + A_{NX}NV, U) \\ &\quad - \csc^2\theta_i\, g(T_X NPU + A_{NX}NU, V), \end{aligned}$$

which proves $g(T_V NPU - T_U NPV, X) = g(A_{NU}V - A_{NV}U, NX)$.

On the other side, by (34), we get

$$\begin{aligned} g([U, V], Z) &= \csc^2\theta_j \Big(g(T_U V, NPZ) + g(T_U PV + A_{NV}U, NZ) \Big) \\ &\quad - \csc^2\theta_j \Big(g(T_V U, NPZ) + g(T_V PU + A_{NU}V, NZ) \Big), \end{aligned}$$

which proves $g(T_U PV - T_V PU, NZ) = g(A_{NU}V - A_{NV}U, NZ)$ and shows **i)** $\Leftrightarrow$ **ii)**.

To show **i)** $\Leftrightarrow$ **iii)**, the equation (33) leads us to see

$$\begin{aligned} g([U, V], X) &= \sec^2\theta_i\, g(T_{PV}NX - T_X NPV, U) \\ &\quad - \sec^2\theta_i\, g(T_{PU}NX - T_X NPU, V), \end{aligned}$$

which proves $g(T_{PU}V - T_{PV}U, NX) = g(T_V NPU - T_U NPV, X)$.

On the other hand, (35) leads us to get

$$g([U,V],Z) = \sec^2\theta_j\Big(g(\hat{\nabla}_U PV + T_U NV, PZ) + g(T_U V, \omega NZ)\Big) - \sec^2\theta_j\Big(g(\hat{\nabla}_V PU + T_V NVU, PZ) + g(T_V U, \omega NZ)\Big),$$

which proves $g(\hat{\nabla}_U PV - \hat{\nabla}_V PU, Z) = g(T_V NU - T_U NV, Z)$, and completes the proof. □

3.2. Totally Geodesic Foliations

Another key feature for the submersions is geodesics. In this direction, the current section is devoted to the geodesics of the distributions mentioned in the definition of a pointwise hemi bi-slant submersion.

Now, we give some conditions for the geodesics of the anti-invariant distribution $D^\perp$.

Theorem 3.10. *Let π be a pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the followings are equivalent to each other;*

i) *The distribution $\mathcal{D}^\perp$ defines totally geodesic foliations on M,*

ii) $g(T_{PU}NY - A_{NY}NU, X) = 0$,

iii) $g(T_X NY, PU) = \bar{g}((\nabla\pi_*)(X, \phi Y), \pi_*(NU))$,

where $X, Y \in \Gamma(D^\perp)$ and $U \in \Gamma(D^{\theta_1} \oplus D^{\theta_2})$.

Proof. Let $X, Y \in \Gamma(D^\perp)$ and $U \in \Gamma(D^{\theta_1} \oplus D^{\theta_2})$. To prove the distribution $\mathcal{D}^\perp$ defines totally geodesic foliations on M, we need to show that $\hat{\nabla}_X Y \in D^\perp$, i.e. $\hat{\nabla}_X Y \perp D^{\theta_1} \oplus D^{\theta_2}$. By (30), we have

$$\begin{aligned} g(\hat{\nabla}_X Y, U) &= g(\nabla_X Y, U) \\ &= -\csc^2\theta_i\, g(T_Y NPU + A_{NY} NU, X). \end{aligned}$$

On the other hand, by (31), we get

$$\begin{aligned} g(\hat{\nabla}_X Y, U) &= g(\nabla_X Y, U) \\ &= \sec^2\theta_i\, g(T_{PU} NY + T_Y NPU, X). \end{aligned}$$

With the last two expression of $g(\hat{\nabla}_X Y, U)$, we prove **i)** $\Leftrightarrow$ **ii)**.

Using (11), (5), (14), and Lemma 2.1, we have

$$\begin{aligned} g(\hat{\nabla}_X Y, U) &= g(\nabla_X Y, U) \\ &= g(\nabla_X NY, PU) + g(\nabla_X NY, NU) \\ &= g(T_X NY, PU) + g(\mathcal{H}\nabla_X NY, NU) \\ &= g(T_X NY, PU) - \bar{g}((\nabla\pi_*)(X, \phi Y), \pi_*(NU)), \end{aligned}$$

which gives **i)** $\Leftrightarrow$ **iii)**, and completes the proof. □

The following theorem gives some conditions for the geodesics of the pointwise slant distributions D^{θ_1} and D^{θ_2}.

Theorem 3.11. *Let π be a pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the followings are equivalent to each other;*

i) *The distribution $\mathcal{D}^{\theta_i}$ defines totally geodesic foliations on M,*

ii)

$$g(T_X NPV + A_{NX} NV, U) = 0$$

and

$$g(T_U V, NPZ) + g(T_U PV + A_{NV} U, NZ) = 0.$$

iii)

$$g(T_{PV} NX - T_X NPV, U) = 0,$$

and

$$g(\hat{\nabla}_U PV + T_U NV, PZ) + g(T_U V, \omega NZ) = 0.$$

iv)

$$g(A_{NV} U, NX) = \bar{g}((\nabla \pi_*)(U, PV), NX)$$

and

$$\begin{aligned} g(\hat{\nabla}_U PV, PZ) + g(A_{NV} U, NZ) &= \bar{g}((\nabla \pi_*)(U, PV), \pi_*(NZ)) \\ &\quad - \bar{g}((\nabla \pi_*)(U, PZ), \pi_*(NV)), \end{aligned}$$

where $X \in \Gamma(D^\perp)$, $U, V \in \Gamma(D^{\theta_i})$, $Z \in \Gamma(D^{\theta_j})$, $i \neq j$, $i, j = 1, 2$.

Proof. Let $U, V \in \Gamma(D^{\theta_i})$. The distribution $\mathcal{D}^{\theta_i}$ defines totally geodesic foliations on M if and only if $\hat{\nabla}_U V \in D^{\theta_i}$, i.e. $\hat{\nabla}_U V \perp D^\perp \oplus D^{\theta_j}$, $i \neq j$, $i, j = 1, 2$.

The relation **i)** $\Leftrightarrow$ **ii)** can be proved by (32) and (34). Moreover, we can prove the relation **i)** $\Leftrightarrow$ **iii)** by using (33) and (35).

Now, let $X \in \Gamma(D^\perp)$, $U, V \in \Gamma(D^{\theta_i})$, $Z \in \Gamma(D^{\theta_j})$, $i \neq j$, $i, j = 1, 2$. Then, by using (11), (5), (6), (14), and (15), we have

$$\begin{aligned} g(\hat{\nabla}_U V, X) &= g(\nabla_U V, X) \\ &= g(\nabla_U PV, NX) + g(\nabla_U NV, NX) \\ &= -\bar{g}((\nabla \pi_*)(U, PV), NX) + g(A_{NV} U, NX) \end{aligned}$$

and

$$\begin{aligned} g(\hat{\nabla}_U V, Z) &= g(\nabla_U V, Z) \\ &= g(\phi \nabla_U V, PZ) + g(\phi \nabla_U V, NZ) \\ &= g(\hat{\nabla}_U V, PZ) - g(\nabla_U PZ, NV) - \bar{g}((\nabla \pi_*)(U, PV), \pi_*(NZ)) \\ &\quad + g(A_{NV} U, NZ) \\ &= g(\hat{\nabla}_U V, PZ) + g(A_{NV} U, NZ) - \bar{g}((\nabla \pi_*)(U, PV), \pi_*(NZ)) \\ &\quad + \bar{g}((\nabla \pi_*)(U, PZ), \pi_*(NV)), \end{aligned}$$

which proves **i)** $\Leftrightarrow$ **iv)**, and completes the proof. □

Corollary 3.12. *Let π be a pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the integral manifold of $\ker\pi_*$ is a locally product*

$$M_{D^\perp} \times M_{D^{\theta_1}} \times M_{D^{\theta_2}}$$

if and only if at least one of the conditions in Theorem 3.10 and one in Theorem 3.11 are satisfied, where $M_{D^\perp}$, $M_{D^{\theta_1}}$, and $M_{D^{\theta_2}}$ are integral manifolds of the distributions $D^\perp$, D^{θ_1}, and D^{θ_2}, respectively.

3.3. $\phi-$pluriharmonicity of π

This section is devoted to some interesting results for the $\phi-$pluriharmonicity of π. First, we give the definition of this concept.

Definition 3.13. *[2] Let π be a pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$.*
π is called

- *$D^{\perp} - \phi-$pluriharmonic if for any $X, Y \in \Gamma(D^{\perp})$,*
- *$D^{\theta_i} - \phi-$pluriharmonic if for any $X, Y \in \Gamma(D^{\theta_i})$, $i = 1, 2$,*
- *$(D^{\perp} - D^{\theta_i}) - \phi-$pluriharmonic if for any $X \in \Gamma(D^{\perp})$, $Y \in \Gamma(D^{\theta_i})$, $i = 1, 2$,*
- *$(ker\pi_* - ker\pi_*^{\perp}) - \phi-$pluriharmonic if for any $X \in \Gamma(ker\pi_*)$, $Y \in \Gamma(ker\pi_*^{\perp})$,*

$$(\nabla\pi_*)(X, Y) + (\nabla\pi_*)(\phi X, \phi Y) = 0. \tag{41}$$

The following theorem gives a relation for the totally geodesics of the fibers of the submersion π under the $D^{\perp} - \phi-$pluriharmonicity assumption.

Theorem 3.14. *Let π be a $D^{\perp} - \phi-$pluriharmonic pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the map π is a $ND^{\perp}-$geodesic map if and only if the fibres define totally geodesic foliations in M.*

Proof. Since π is $D^{\perp} - \phi-$pluriharmonic, for any $X, Y \in \Gamma(D^{\perp})$, we have

$$0 = (\nabla\pi_*)(X, Y) + (\nabla\pi_*)(\phi X, \phi Y),$$

which gives with (5) and (14)

$$0 = -\pi_*(T_X Y) + (\nabla\pi_*)(NX, NY).$$

The last expression shows that π is a $ND^{\perp}-$geodesic map if and only if $T = 0$, which completes the proof. □

The next theorem gives a result under the $D^{\theta_i} - \phi-$pluriharmonicity assumption, for $i = 1, 2$.

Theorem 3.15. *Let π be a $D^{\theta_i} - \phi-$pluriharmonic pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$, $(i = 1, 2)$. Then, the map π is a $ND^{\theta_i}-$geodesic map if and only if*

$$T_{PU}PV + A_{NV}PU + A_{NU}PV + T_U V = 0,$$

where $U, V \in \Gamma(D^{\theta_i})$, $i = 1, 2$.

Proof. Since π is $D^{\theta_i} - \phi-$pluriharmonic, for any $U, V \in \Gamma(D^{\theta_i})$, $i = 1, 2$, we have

$$0 = (\nabla\pi_*)(U, V) + (\nabla\pi_*)(\phi U, \phi V),$$

which gives with (5), (9), and (14)

$$(\nabla\pi_*)(NU, NV) = \pi_*(T_{PU}PV + A_{NV}PU + A_{NU}PV + T_U V)$$

completes the proof. □

We recall the definition of *mixed geodesic.* For given two distributions D^1 and D^2 defined on the fibers of a Riemannian submersion π, the fibers are called $D^1 - D^2-$ *mixed geodesic* if $T_{D^1} D^2 = 0$.

$(D^{\perp} - D^{\theta_i}) - \phi-$pluriharmonicity of the pointwise hemi bi-slant submersion gives the following result.

Theorem 3.16. *Let ϕ be a $(D^{\perp} - D^{\theta_i}) - \phi-$pluriharmonic pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$, $(i = 1, 2)$. Then, the fibers are $(D^{\perp} - D^{\theta_i})-$ mixed geodesic if and only if*

$$\nabla^N_{\pi_*(NX)}\pi_*(NU) = \pi_*(\omega A_{NX}U + N\mathcal{V}\nabla_{NX}U),$$

where $X \in \Gamma(D^{\perp})$, $U \in \Gamma(D^{\theta_i})$.

Proof. The assumption π is $(D^{\perp} - D^{\theta_i}) - \phi-$pluriharmonic, with the help of (5), (9), (14), and (15), for any $X \in \Gamma(D^{\perp})$, $U \in \Gamma(D^{\theta_i})$, we have

$$\begin{aligned} 0 &= (\nabla\pi_*)(X, U) + (\nabla\pi_*)(\phi X, \phi U) \\ &= -\pi_*(\nabla_X U) + \nabla^{\pi}_{\phi X}\pi_*(\phi U) - \pi_*(\nabla_{\phi X}\phi U) \\ &= -\pi_*(\nabla_X U) + \nabla^N_{\phi_*(NX)}\pi_*(NU) - \pi_*(\phi\nabla_{NX}U) \\ &= -\pi_*(\nabla_X U) + \nabla^N_{\phi_*(NX)}\pi_*(NU) \\ &\quad -\pi_*(\omega A_{NX}U + N\mathcal{V}\nabla_{NX}U), \end{aligned}$$

which completes the proof. □

Now, we consider the case π is $(ker\pi_* - ker\pi^{\perp}) - \phi-$pluriharmonic. The following theorem gives a condition for the fibers being mixed geodesics.

Theorem 3.17. *Let π be a $(ker\pi_* - ker\pi_*{}^{\perp}) - \phi-$ pluriharmonic pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$, $(i = 1, 2)$. Then, the fibers are $(ker\pi_* - ker\pi^{\perp})$-mixed geodesic if and only if*

$$(\nabla\pi_*)(\omega\xi, NX) = \pi_*(T_{\delta\xi}PX + A_{NX}\delta\xi + A_{\omega\xi}PX),$$

where $X \in \Gamma(ker\pi_*)$, $\xi \in \Gamma(ker\pi_*^{\perp})$.

Proof. The assumption π is $(ker\pi_* - ker\pi_*{}^{\perp}) - \phi-$ pluriharmonic, with the help of (5), (6), (9), (14), and (15), yields

$$\begin{aligned} 0 &= (\nabla\pi_*)(\xi, X) + (\nabla\pi_*)(\phi\xi, \phi X) \\ &= -\pi_*(\nabla_\xi X) + (\nabla\pi_*)(\delta\xi, PX) + (\nabla\pi_*)(\delta\xi, NX) \\ &\quad +(\nabla\pi_*)(\omega\xi, PX) + (\nabla\pi_*)(\omega\xi, NX) \\ &= -\pi_*(\nabla_\xi X) - \pi_*(\nabla_{\delta\xi}PX + \nabla_{\delta\xi}NX + \nabla_{\omega\xi}PX) \\ &\quad +(\nabla\pi_*)(\omega\xi, NX) \\ &= -\pi_*(\nabla_\xi X) - \pi_*(T_{\delta\xi}PX + A_{NX}\delta\xi + A_{\omega\xi}PX) \\ &\quad +(\nabla\pi_*)(\omega\xi, NX), \end{aligned}$$

which completes the proof. □

3.4. $\phi-$invariant and totally geodesics

In this section, we find some conditions for the map to be the $\phi-$invariant of the distributions on the total space.

We give the following concept, which helps to provide new conditions for some other concepts studied before.

Definition 3.18. *[2] Let π be a pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$.*
π is called

- *$D^{\perp} - \phi-$invariant if for any $X, Y \in \Gamma(D^{\perp})$,*

- $D^{\theta_i} - \phi-$*invariant if for any* $X, Y \in \Gamma(D^{\theta_i})$,
- $(D^{\perp} - D^{\theta_i}) - \phi-$*invariant if for any* $X \in \Gamma(D^{\perp})$, $Y \in \Gamma(D^{\theta_i})$,
- $(ker\pi_* - ker\pi_*^{\perp}) - \phi-$*invariant if for any* $X \in \Gamma(ker\pi_*)$, $Y \in \Gamma(ker\pi_*^{\perp})$,

$$(\nabla\pi_*)(X, Y) = (\nabla\pi_*)(\phi X, \phi Y). \tag{42}$$

The following theorem gives a condition anti-invariant distribution $D^{\perp}$ to define totally geodesic foliations in M.

Theorem 3.19. *Let* π *be a* $D^{\perp} - \phi-$*invariant pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold* (M, ϕ, g) *onto a Riemannian manifold* $(N, \bar{g})$. *Then, the anti-invariant distribution* $D^{\perp}$ *defines totally geodesic foliations on M if and only if*

$$\nabla^N_{\pi_*(\phi X)}\pi_*(\phi Y) = \pi_*(\omega A_{NX} + N\mathcal{V}\nabla_X V),$$

where $X, Y \in \Gamma(D^{\perp})$.

Proof. Let π be a $D^{\perp} - \phi-$invariant, i.e.

$$(\nabla\pi_*)(X, Y) = (\nabla\pi_*)(\phi X, \phi Y), \ \forall X, Y \in \Gamma(D^{\perp}),$$

which gives with (7), (8), and (9),

$$-\pi_*(\nabla_X Y) = \nabla^N_{\pi_*(\phi X)}\pi_*(\phi Y) - \pi_*(\omega A_{NX} + N\mathcal{V}\nabla_X V),$$

completes the proof. □

The following theorem gives a condition for D^{θ_i}, $i = 1, 2$, to define totally geodesic foliations in M.

Theorem 3.20. *Let* π *be a* $D^{\theta_i} - \phi-$*invariant pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold* (M, ϕ, g) *onto a Riemannian manifold* $(N, \bar{g})$, *for* $i = 1, 2$. *Then, the pointwise slant distributions* D^{θ_i} *define totally geodesic foliations on M if and only if*

$$\begin{aligned}(\nabla\pi_*)(NU, NV) &= N\Big(- \sin(2\theta_i)NU(\theta_i)V + \cos^2(\theta_i)\mathcal{V}\nabla_{NU}V + A_{NU}NPV \\ &+ \sin(2\theta_i)PU(\theta_i)V + \sin^2(\theta_i)\hat{\nabla}_{PU}V + T_{PU}\omega NV\Big) \\ &+ \omega\Big(\cos^2(\theta_i)A_{NU}V + \mathcal{H}\nabla_{NU}NPV + \sin^2(\theta_i)T_{PU}V \\ &+ A_{\omega NV}PU\Big),\end{aligned}$$

where $U, V \in \Gamma(D^{\theta_i})$, $i = 1, 2$.

Proof. Since π is $D^{\theta_i} - \phi-$invariant, for any $U, V \in \Gamma(D^{\theta_i})$, $i = 1, 2$, we have

$$(\nabla\pi_*)(U, V) = (\nabla\pi_*)(\phi U, \phi V),$$

which gives with (9)

$$\begin{aligned}-\pi_*(\nabla_U V) &= (\nabla\pi_*)(NU, NV) + (\nabla\pi_*)(NU, PV) + (\nabla\pi_*)(PU, NV) \\ &= (\nabla\pi_*)(NU, NV) - \pi_*(\nabla_{NU}PV) - \pi_*(\nabla_{PU}NV).\end{aligned} \tag{43}$$

Now, by using (5) ∼ (8),(9), (14), and (15), we compute the second term on the right side of (43);

$$\begin{aligned}-\pi_*(\nabla_{NU}PV) &= -\pi_*(\phi\nabla_{NU}P^2V) - \pi_*(\phi\nabla_{NU}NPV) \\ &= -\pi_*(\phi\nabla_{NU}(\cos^2\theta_i)V) - \pi_*(\phi\nabla_{NU}NPV) \\ &= -\pi_*\Big(- \sin(2\theta_i)NU(\theta_i)NV + \cos^2(\theta_i)\omega A_{NU}V \\ &\quad + \cos^2(\theta_i)N\mathcal{V}\nabla_{NU}V + \omega\mathcal{H}\nabla_{NU}NPV + NA_{NU}NPV\Big),\end{aligned}$$

the third term on the right side of (43);

$$\begin{aligned}-\pi_*(\nabla_{PU}NV) &= -\pi_*(\phi\nabla_{PU}\delta NV) - \pi_*(\phi\nabla_{PU}\omega NV)\\ &= -\pi_*(\phi\nabla_{PU}(\sin^2(\theta_i))V) - \pi_*(\phi\nabla_{PU}\omega NV)\\ &= -\pi_*\Big(\sin(2\theta_i)PU(\theta_i)NV + \sin^2(\theta_i)\omega T_{PU}V\\ &\quad \sin^2(\theta_i)N\hat{\nabla}_{PU}V + NT_{PU}\omega NV + \omega A_{\omega NV}PU\Big).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}-\pi_*(\nabla_U V) &= \pi_*\Big(-N\Big(-\sin(2\theta_i)NU(\theta_i)V + \cos^2(\theta_i)\mathcal{V}\nabla_{NU}V + A_{NU}NPV\\ &+ \sin(2\theta_i)PU(\theta_i)V + \sin^2(\theta_i)\hat{\nabla}_{PU}V + T_{PU}\omega NV\Big)\\ &- \omega\Big(\cos^2(\theta_i)A_{NU}V + \mathcal{H}\nabla_{NU}NPV + \sin^2(\theta_i)T_{PU}V\\ &+ A_{\omega NV}PU\Big)\Big) + (\nabla\pi_*)(NU, NV),\end{aligned}$$

which completes the proof. □

We give a relation between $(D^\perp - D^{\theta_i}) - \phi$–invariance of the submersion and mixed geodesics of the fibers.

Theorem 3.21. *Let π be a $(D^\perp - D^{\theta_i}) - \phi$–invariant pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$, for $i = 1, 2$. Then, the fibers are $(D^\perp - D^{\theta_i})$– mixed geodesic if and only if*

$$\nabla^N_{\pi_*(\phi X)}\pi_*(NU) = \pi_*(\omega A_{\phi X}U + N\mathcal{V}\nabla_{\phi X}U),$$

where $X \in \Gamma(D^\perp)$ and $U \in \Gamma(D^{\theta_i})$, $i = 1, 2$.

Proof. By the assumption, for any $X \in \Gamma(D^\perp)$, $U \in \Gamma(D^{\theta_i})$, $i = 1, 2$, we have

$$(\nabla\pi_*)(X, U) = (\nabla\pi_*)(\phi X, \phi U), \tag{44}$$

which is with the help of (6), (9), (14) and (15)

$$-\pi_*(\nabla_X, U) = \nabla^N_{\pi_*(\phi X)}\pi_*(NU) - \pi_*(\omega A_{\phi X}U + N\mathcal{V}\nabla_{\phi X}U)$$

completes the proof. □

Now, we give a result for the relation between $(ker\pi_* - ker\pi_*^\perp)$– mixed geodesics of the fiber and $(ker\pi_* - ker\pi_*^\perp) - \phi$– invariance of the submersion.

Theorem 3.22. *Let π be a $(ker\pi_* - ker\pi_*^\perp) - \phi$– invariant pointwise hemi bi-slant Riemannian submersion from an l.p.R. manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the fibers are $(ker\pi_* - ker\pi_*^\perp)$– mixed geodesics if and only if for any $X \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^\perp)$,*

$$(\nabla\pi_*)(NX, \omega\xi) = \pi_*(\omega(T_{PX}\omega\xi + A_{NX}\delta\xi) + N(A_{\omega\xi}PX + \mathcal{V}\nabla_{NX}\delta\xi)).$$

Proof. The assumption $(ker\pi_* - ker\pi_*^\perp) - \phi$– invariant yields

$$(\nabla\pi_*)(X, \xi) = (\nabla\pi_*)(\phi X, \phi\xi),$$

which gives with (6), (7), (9), (14), and (15)

$$\begin{aligned}-\pi_*(\nabla_X\xi) &= (\nabla\pi_*)(NX, \omega\xi) - \pi_*(\omega(T_{PX}\omega\xi + A_{NX}\delta\xi)\\ &\quad +N(A_{\omega\xi}PX + \mathcal{V}\nabla_{NX}\delta\xi)),\end{aligned}$$

which completes the proof. □

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