Filomat 39:7 (2025), 2209–2219
https://doi.org/10.2298/FIL2507209C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

General solution and stability analysis of a quadratic-cubic functional equation

Apurba Chutia[a], Hemen Dutta[a]

[a]*Department of Mathematics, Gauhati University, Guwahati - 781 014, India*

Abstract. In this paper, we derive the general solution of a quadratic-cubic functional equation $3f(x+3y) - 3f(x-3y) + 36f(2y) = 9f(x+y) - 9f(x-y) + 16f(3y)$ and investigate the Hyers-Ulam stability of the equation.

1. Introduction

The stability problem was first introduced by Ulam [22] in 1940 where he proposed the following question concerning the stability of homomorphisms:

Let $(G_1, \star)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, then there exists a $\delta(\epsilon) > 0$ such that is a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x \star y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then is there a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? In other words, if a mapping is almost homomorphism, then is there a true homomorphism near it with small error as much as possible? If the answer is affirmative, we would call that the equation $H(x \star y) = H(x) \diamond H(y)$ of homomorphism is stable.

In 1941, Hyers [10] was the first mathematician to present the result concerning the stability of functional equations. He answered the question of Ulam for the cases where G and H are assumed to be Banach spaces. Later Hyers' theorem was generalized by Aoki [2] for approximate additive mappings and by Rassias [20] for approximate linear mappings. Moreover, a generalization of the Rassias' theorem was obtained by Găvruţă [8]. Subsequently, numerous authors have thoroughly explored the stability problems of several

2020 *Mathematics Subject Classification*. Primary 39B82; Secondary 39B52, 39B05.
Keywords. Quadratic functional equatio; Cubic functional equation; Hyers-Ulam stability.
Received: 22 April 2024; Accepted: 08 January 2025
Communicated by Calogero Vetro
Supported by NBHM, DAE, India (Grant No. 02011/19/2023/NBHM(R.P)/R&D-II/5949).
Email addresses: apurbachutia03@gmail.com (Apurba Chutia), hemen_dutta08@rediffmail.com (Hemen Dutta)
ORCID iDs: https://orcid.org/0009-0006-8927-2927 (Apurba Chutia), https://orcid.org/0000-0003-2765-2386 (Hemen Dutta)

functional equations [3, 5, 7, 9, 11–14, 21]. Specifically, one of the significant functional equations studied is the following [1, 6, 16–19]:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

This functional equation is called quadratic because $f(x) = ax^2$ is a solution of the above functional equation. Another well-known functional equation is the cubic functional equation which is represented by

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.2}$$

The cubic function $f(x) = ax^3$ satisfies the above functional equation. This equation was solved by Jun and Kim [15]. They also proved that a functional equation $f : X \to Y$ where X and Y are real vector spaces, is a solution of (1.2) if and only if there exists a function $F : X \times X \times X \to Y$ such that $f(x) = F(x, x, x)$ for all $x \in X$, where F is symmetric for each fixed one variable and is additive for fixed two variables and is given $F(x, y, z) = \frac{1}{24} f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)$, for all $x, y, z \in X$. Furthermore, they investigated the Hyers–Ulam stability for Eq. (1.2).

In 2003, Chang and Jung [4] introduced the following mixed type functional equation

$$6f(x+y) - 6f(x-y) + 4f(3y) = 3f(x+2y) - 3f(x-2y) + 9f(2y). \tag{1.3}$$

The function $f(x) = ax^2 + bx^3$ is a solution of the functional equation (1.3). They obtained the general solution of the functional equation (1.3) and investigated the Hyers-Ulam stability of the functional equation (1.3) in normed spaces.

In this paper we deal with the functional equation

$$3f(x+3y) - 3f(x-3y) + 36f(2y) = 9f(x+y) - 9f(x-y) + 16f(3y). \tag{1.4}$$

It is easy to verify that the function $f(x) = ax^2 + bx^3$ is a solution of the functional equation (1.4).

The main purpose of this paper is to investigate the general solution of (1.4) as well as its Hyers-Ulam stability.

2. Preliminaries

In this section, we consider X and Y as real vector spaces.

Lemma 2.1. *If $f : X \to Y$ is an even mapping which satisfies (1.4), then f is quadratic.*

Proof. We put $x = y = 0$ in (1.4), to get

$$f(0) = 0. \tag{2.1}$$

Now we put $x = 0$ in (1.4) and we use the evenness of f,

$$36f(2y) = 16f(3y). \tag{2.2}$$

Thus (1.4) becomes,

$$f(x+3y) - f(x-3y) = 3(x+y) - 3(x-y), \tag{2.3}$$

for all $x, y \in X$.
Replacing x by y in (2.3), we get

$$f(4y) = 4f(2y).$$

Now we replace $2y = y$ in the above equation to get

$$f(2y) = 4f(y). \tag{2.4}$$

Hence (2.2) can be written as

$$f(3y) = 9f(y). \tag{2.5}$$

We substitute x by $x + y$ and y by $x - y$ in (2.3) and use (2.4) to get

$$f(2x - y) - f(x - 2y) = 3f(x) - 3f(y). \tag{2.6}$$

Now replacing y by $-y$ in (2.6)

$$f(2x + y) - f(x + 2y) = 3f(x) - 3f(y). \tag{2.7}$$

Replacing $y = 2y$ in (2.7) and using (2.4), we get

$$4f(x + y) - f(x + 4y) = 3f(x) - 12f(y). \tag{2.8}$$

We interchange x with y in (2.8) to get

$$4f(x + y) - f(4x + y) = 3f(y) - 12f(x). \tag{2.9}$$

Now we adding (2.8) and (2.9) yields

$$f(x + 4y) + f(4x + y) = 8f(x + y) + 9f(x) + 9f(y). \tag{2.10}$$

If we replace x by $x + y$ in (2.3), we have

$$f(x + 4y) - f(x - 2y) = 3f(x + 2y) - 3f(x). \tag{2.11}$$

Replacing $x = 2x$ in (2.11) and using (2.4), we get

$$f(x + 2y) = 3f(x + y) + f(x - y) - 3f(x). \tag{2.12}$$

Interchanging x and y in (2.12) to get

$$f(2x + y) = 3f(x + y) + f(x - y) - 3f(y). \tag{2.13}$$

Now adding (2.12) and (2.13) gives

$$f(x + 2y) + f(2x + y) = 6f(x + y) + 2f(x - y) - 3f(x) - 3f(y). \tag{2.14}$$

Replacing y by $-y$ we have

$$f(x - 2y) + f(2x - y) = 6f(x - y) + 2f(x + y) - 3f(x) - 3f(y). \tag{2.15}$$

Replacing $y = 2y$ in (2.14) and using (2.4), we arrive at

$$f(x + 4y) + 4f(x + y) = 6f(x + 2y) + 2f(x - 2y) - 3f(x) - 12f(y). \tag{2.16}$$

Interchanging x and y in (2.16) to get

$$f(4x + y) + 4f(x + y) = 6f(2x + y) + 2f(2x - y) - 3f(y) - 12f(x). \tag{2.17}$$

Now adding (2.16) with (2.17) and using (2.14) and (2.15), we have

$$f(4x + y) + f(x + 4y) = 32f(x + y) + 24f(x - y) - 39f(x) - 39f(y). \tag{2.18}$$

Comparing (2.10) and (2.18), we get

$$f(x + y) + f(x - y) = 2f(x) + 2f(y).$$

Thus, we have f is quadratic. □

Lemma 2.2. *If $f : X \to Y$ is an odd mapping which satisfies (1.4), then f is cubic.*

Proof. As f is odd we have $f(-x) = -f(x)$ which gives $f(0) = 0$.
Putting $x = 0$ in (1.4) we get

$$5f(3y) = 18f(2y) - 9f(y). \tag{2.19}$$

Replacing x by y in (1.4) yields

$$16f(3y) = 3f(4y) + 30f(2y). \tag{2.20}$$

Now replacing x by $3y$, we get

$$f(6y) = 4f(4y) - 5f(2y).$$

Replacing $2y$ by y, we have

$$f(3y) = 4f(2y) - 5f(y). \tag{2.21}$$

Now, multiplying (2.21) by 5 and subtracting it from (2.19), we get

$$f(2y) = 8f(y). \tag{2.22}$$

Thus, using the above equation in (2.19), we get

$$f(3y) = 27f(y). \tag{2.23}$$

Now, using (2.22) and (2.23) in (1.4), we have

$$f(x + 3y) - f(x - 3y) = 3f(x + y) - 3f(x - y) + 48f(x). \tag{2.24}$$

We replace x by $x + y$ to get

$$f(x + 4y) - f(x - 2y) = 3f(x + 2y) - 3f(x) + 48f(y). \tag{2.25}$$

Replacing x by $2x$ and using (2.22), we have

$$f(x + 2y) = 3f(x + y) + f(x - y) - 3f(x) + 6f(y). \tag{2.26}$$

Now we interchange x and y to get

$$f(2x + y) = 3f(x + y) - f(x - y) - 3f(y) + 6f(x). \tag{2.27}$$

We replace y by $-y$ which gives

$$f(2x - y) = 3f(x - y) - f(x + y) + 3f(y) + 6f(x). \tag{2.28}$$

Now, we add (2.27) and (2.28) to obtain

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$$

This completes the proof. □

3. Main resuls

3.1. The general solution

Theorem 3.1. *Let X and Y be two real vector spaces. A mapping $f : X \to Y$ satisfies (1.4) for all $x, y \in X$ if and only if there exist a symmetric and bi-additive mapping $B : X \times X \to Y$ and a mapping $F : X \times X \times X \to Y$ such that $f(x) = F(x, x, x) + B(x, x)$ for all $x \in X$, where F is symmetric for fixed one variable and is additive for fixed two variables.*

Proof. Let us decompose f into even and odd parts by putting

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)] \quad \text{and} \quad f_o(x) = \frac{1}{2}[f(x) - f(-x)],$$

for all $x \in X$. Then, we have

$$f(x) = f_e(x) + f_o(x),$$

for all $x \in X$. We can easily verify that $f_e(x)$ and $f_o(x)$ satisfy (1.4). Using Lemmas 2.1 and 2.2, we have $f_e(x)$ is quadratic and $f_o(x)$ is cubic. Thus, there exist a symmetric and bi-additive mapping $B : X \times X \to Y$ and a mapping $F : X \times X \times X \to Y$ such that $f_e(x) = B(x, x)$ and $f_o(x) = F(x, x, x)$, combining we get $f(x) = F(x, x, x) + B(x, x)$ for all $x \in X$, where F is symmetric for fixed one variable and is additive for fixed two variables.

Conversely, if there exist a symmetric and bi-additive mapping $B : X \times X \to Y$ and a mapping $F : X \times X \times X \to Y$ such that $f(x) = F(x, x, x) + B(x, x)$ for all $x \in X$, where F is symmetric for fixed one variable and is additive for fixed two variables, it is clear that f satisfies (1.4). □

3.2. The Hyers-Ulam stability

Throughout this section, let X and Y be a real vector space and a real Banach space, respectively.

In this section, the stability of the functional equation will be investigated. Before proceeding to the proof of the main result in this section, we shall need the following two results.

We define

$$Df(x, y) := 3f(x + 3y) - 3f(x - 3y) + 36f(2y) - 9f(x + y) + 9f(x - y) - 16f(3y),$$

for all $x, y \in X$.

Theorem 3.2. *Let $\phi : X \times X \to [0, \infty)$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(2^{i-1}x, 2^{i-1}x) + \phi(0, 2^{i-1}x)}{4^i},$$

converges and

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0,$$

for all $x \in X$ and $f(0) = 0$. Let us consider an even mapping $f : X \to Y$ which satisfies the inequality

$$||Df(x, y|| \le \phi(x, y), \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T : X \to Y$ which satisfies (1.4) and given by

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}. \tag{3.2}$$

Also, satisfies the inequality

$$\|f(x) - T(x)\| \leq \frac{1}{12} \sum_{i=0}^{\infty} \frac{\phi(2^{i-1}x, 2^{i-1}x) + \phi(0, 2^{i-1}x)}{4^i}, \tag{3.3}$$

for all $x \in X$.

Proof. Let us put $x = 0$ in (3.1). Then we get

$$\|36f(2y) - 16f(3y)\| \leq \phi(0, y),$$

for all $y \in X$. Replacing y by x, we get

$$\|36f(2x) - 16f(3x)\| \leq \phi(0, x), \tag{3.4}$$

for all $x \in X$. Now, let us put $y = x$ in (3.1)

$$\|3f(4x) + 24f(2x) - 16f(3x)\| \leq \phi(x, x), \tag{3.5}$$

for all $x \in X$. Using (3.4) and (3.5), we get

$$\|3f(4x) - 12f(2x)\| \leq \phi(x, x) + \phi(0, x),$$

for all $x \in X$. Dividing by 12 in the above inequality, we have

$$\left\| \frac{f(4x)}{4} - f(2x) \right\| \leq \frac{1}{12}[\phi(x, x) + \phi(0, x)],$$

for all $x \in X$. Replacing x by $\frac{1}{2}x$ in the above inequality, we have

$$\left\| \frac{f(2x)}{4} - f(x) \right\| \leq \frac{1}{12} \left[\phi\left(\frac{x}{2}, \frac{x}{2}\right) + \phi\left(0, \frac{x}{2}\right) \right], \tag{3.6}$$

for all $x \in X$. Now, replacing x by 2x in (3.6) and dividing by 4, we get

$$\left\| \frac{f(2^2x)}{4^2} - \frac{f(2x)}{4} \right\| \leq \frac{1}{12} \left[\frac{\phi(x, x) + \phi(0, x)}{4} \right].$$

Now, using (3.6) we have

$$\left\| \frac{f(2^2x)}{4^2} - f(x) \right\| \leq \frac{1}{12} \left[\frac{\phi(x, x) + \phi(0, x)}{4} + \phi\left(\frac{x}{2}, \frac{x}{2}\right) + \phi\left(0, \frac{x}{2}\right) \right].$$

Applying an induction argument to n, we obtain

$$\left\| \frac{f(2^n x)}{4^n} - f(x) \right\| \leq \frac{1}{12} \sum_{i=0}^{n-1} \frac{\phi(2^{i-1}x, 2^{i-1}x) + \phi(0, 2^{i-1}x)}{4^i}, \tag{3.7}$$

for all $x \in X$. Indeed

$$\left\| \frac{f(2^{n+1}x)}{4^{n+1}} - f(x) \right\| \leq \frac{1}{4} \left\| \frac{f(2^n 2x)}{4^n} - f(2x) \right\| + \left\| \frac{f(2x)}{4} - f(x) \right\|.$$

And, using (3.6) and (3.7), we get

$$\left\| \frac{f(2^{n+1}x)}{4^{n+1}} - f(x) \right\| \leq \frac{1}{4}\frac{1}{12}\sum_{i=0}^{n-1} \frac{\phi(2^i x, 2^i x) + \phi(0, 2^i x)}{4^i} + \frac{1}{12}\left[\phi(\frac{x}{2}), \frac{x}{2} + \phi(0, \frac{x}{2})\right]$$
$$= \frac{1}{12}\sum_{i=0}^{n} \frac{\phi(2^{i-1}x, 2^{i-1}x) + \phi(0, 2^{i-1}x)}{4^i},$$

for all $x \in X$. Now we claim $\frac{f(2^n x)}{4^n}$ is a Cauchy sequence. We replace x by $2^m x$ and divide (3.7) by 4^m to obtain that

$$\left\| 4^{-(n+m)} f(2^n 2^m x) - 4^{-m} f(2^m x) \right\| = 4^{-m} \left\| 4^{-n} f(2^n 2^m x) - f(2^m x) \right\|$$
$$\leq 4^{-m} \frac{1}{12} \sum_{i=0}^{n-1} \frac{\phi(2^{i-1}2^m x, 2^{i-1}2^m x) + \phi(0, 2^{i-1}2^m x)}{4^i}$$
$$\leq \frac{1}{12} \sum_{i=0}^{\infty} \frac{\phi(2^{i-1}2^m x, 2^{i-1}2^m x) + \phi(0, 2^{i-1}2^m x)}{4^{i+m}},$$

for all $x \in X$. Taking the limit as $m \to \infty$, we obtain $\frac{f(2^n x)}{4^n}$ is a Cauchy sequence in Y. As Y is complete, this sequence converges in Y.

Now, let us define $T : X \to Y$ by

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n},$$

for all $x \in X$. It is obvious that $T(-x) = T(x)$ for all $x \in X$. Now using (3.1), we have

$$\left\| DT(x, y) \right\| = \lim_{n\to\infty} 4^{-n} \left\| Df(2^n x, 2^n y) \right\| \leq 4^{-n} \phi(2^n x, 2^n y) = 0,$$

for all $x, y \in X$. Thus, according to Lemma 2.1, T is quadratic.

In (3.7), taking the limit as $n \to \infty$, we obtain

$$\|T(x) - f(x)\| \leq \frac{1}{12} \sum_{i=0}^{\infty} \frac{\phi(2^{i-1}x, 2^{i-1}x) + \phi(0, 2^{i-1}x)}{4^i},$$

for all $x \in X$. Now, we claim T is unique.

Suppose that there exists another quadratic mapping $\widetilde{T} : X \to Y$ which satisfies (1.4) and (3.3). As both T and $\widetilde{T}$ are quadratic, we have $\widetilde{T}(2^n x) = 4^n \widetilde{T}(x)$ and $T(2^n x) = 4^n T(x)$ for all $x \in X$. Thus, we obtain

$$\left\| \widetilde{T}(x) - T(x) \right\| = 4^{-n} \left\| \widetilde{T}(2^n x) - T(2^n x) \right\|$$
$$\leq 4^{-n} \left[\|\widetilde{T}(2^n x) - f(2^n x)\| + \left\| f(2^n x) - T(2^n x) \right\| \right]$$
$$\leq \frac{1}{6} \sum_{i=0}^{\infty} \frac{\phi(2^{i-1}2^n x, 2^{i-1}2^n x) + \phi(0, 2^{i-1}2^n x)}{4^{i+n}},$$

for all $x \in X$. By letting $n \to \infty$ in the above inequality, we have $\widetilde{T}(x) = T(x)$ for all $x \in X$ which gives the conclusion. □

Theorem 3.3. *Let $\phi : X \times X \to [0, \infty)$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{3\phi(0, 2^i x) + 10\phi(2^{i-1}x, 2^{i-1}x) + 10\phi(3 \cdot 2^{i-1}x, 2^{i-1}x)}{8^i}$$

converges and

$$\lim_{n\to\infty}\frac{\phi(2^n x, 2^n y)}{4^n} = 0,$$

for all $x, y \in X$.

Let us consider an odd mapping $f : X \to Y$ which satisfies the inequality

$$\left\|Df(x,y)\right\| \leq \phi(x,y), \tag{3.8}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $S : X \to Y$ which satisfies (1.4) and given by

$$S(x) = \lim_{n\to\infty}\frac{f(2^n x)}{8^n}. \tag{3.9}$$

Also satisfies the inequality

$$\left\|f(x) - S(x)\right\| \leq \frac{1}{96}\sum_{i=0}^{\infty}\left[\frac{3\phi(0,2^i x) + 10\phi(2^{i-1}x, 2^{i-1}x) + 10\phi(3\cdot 2^{i-1}x, 2^{i-1}x)}{8^i}\right]. \tag{3.10}$$

for all $x \in X$.

Proof. Since f is odd, we have $f(0) = 0$.

Putting $x = 0$ in (3.8), we get

$$\left\|-10f(3y) + 36f(2y) - 18f(y)\right\| \leq \phi(0,y).$$

Replacing y by x, we obtain

$$\left\|-10f(3x) + 36f(2x) - 18f(x)\right\| \leq \phi(0,x), \tag{3.11}$$

for all $x \in X$. Let us put $y = x$ in (3.8) to get

$$\left\|3f(4x) - 16f(3x) + 30f(2x)\right\| \leq \phi(x,x), \tag{3.12}$$

for all $x \in X$. Now putting $x = 3y$ in (3.8)

$$\left\|3f(6y) - 9f(4y) - 16f(3y) + 45f(2y)\right\| \leq \phi(3y,y).$$

We replace y by x in the above inequality, we obtain

$$\left\|3f(6x) - 9f(4x) - 16f(3x) + 45f(2x)\right\| \leq \phi(3x,x), \tag{3.13}$$

for all $x \in X$. Using (3.12) and (3.13), we get

$$\left\|3f(6x) - 12f(4x) + 15f(2x)\right\| \leq \phi(3x,x) + \phi(x,x),$$

for all $x \in X$. Replacing x by $\frac{1}{2}x$ in the above inequality, we obtain

$$\left\|3f(3x) - 12f(2x) + 15f(x)\right\| \leq \phi\left(\frac{3x}{2},\frac{x}{2}\right) + \phi\left(\frac{x}{2},\frac{x}{2}\right), \tag{3.14}$$

for all $x \in X$. It follows from (3.11) and (3.14) that

$$\left\|96f(x) - 12f(2x)\right\| \leq 10\phi\left(\frac{3x}{2},\frac{x}{2}\right) + 10\phi\left(\frac{x}{2},\frac{x}{2}\right) + 3\phi(0,x).$$

Now dividing by 96 in the above inequality, we get

$$\left\|\frac{f(2x)}{8}-f(x)\right\| \le \frac{1}{96}\left[3\phi(0,x)+10\phi\left(\frac{x}{2},\frac{x}{2}\right)+10\phi\left(\frac{3x}{2},\frac{x}{2}\right)\right], \tag{3.15}$$

for all $x \in X$. Now, we replace x by $2x$ and divide (3.15) by 8 to get

$$\left\|\frac{f(2^2x)}{8^2}-\frac{f(2x)}{8}\right\| \le \frac{1}{96}\left[\frac{3\phi(0,2x)+10\phi(x,x)+10\phi(3x,x)}{8}\right].$$

Using (3.15), we obtain

$$\left\|\frac{f(2^2x)}{8^2}-f(x)\right\| \le \frac{1}{96}\left[\frac{3\phi(0,2x)+10\phi(x,x)+10\phi(3x,x)}{8}+3\phi(0,x)+10\phi\left(\frac{x}{2},\frac{x}{2}\right)+10\phi\left(\frac{3x}{2},\frac{x}{2}\right)\right], \tag{3.16}$$

for all $x \in X$. Applying induction argument to n, we obtain

$$\left\|8^{-n}f(2^nx)-f(x)\right\| \le \frac{1}{96}\sum_{i=0}^{n-1}\frac{3\phi(0,2^ix)+10\phi(2^{i-1}x,2^{i-1}x)+10\phi(3\cdot 2^{i-1}x,2^{i-1}x)}{8^i}, \tag{3.17}$$

for all $x \in X$. Now, replacing x by 2^mx and dividing (3.17) by 8^m, we obtain

$$\begin{aligned}
&\left\|8^{-(n+m)}f(2^n2^mx)-8^{-m}f(2^mx)\right\| = 8^{-m}\left\|8^{-n}f(2^n2^mx)-f(2^mx)\right\| \\
&\le 8^{-m}\frac{1}{96}\sum_{i=0}^{n-1}\frac{3\phi(0,2^i2^mx)+10\phi(2^{i-1}2^mx,2^{i-1}2^mx)+10\phi(3\cdot 2^{i-1}2^mx,2^{i-1}2^mx)}{8^i} \\
&\le \frac{1}{96}\sum_{i=0}^{\infty}\frac{3\phi(0,2^i2^mx)+10\phi(2^{i-1}2^mx,2^{i-1}2^mx)+10\phi(3\cdot 2^{i-1}2^mx,2^{i-1}2^mx)}{8^{i+m}},
\end{aligned}$$

for all $x \in X$. The right-hand side of the above equation tends to zero as $m \to \infty$, thus $8^{-n}f(2^nx)$ is a Cauchy sequence in Y and hence converges. Let us define $S : X \to Y$ by $S(x) = \lim_{n\to\infty} 8^{-n}f(2^nx)$ for all $x \in X$. It is obvious that $S(-x) = -S(x)$ for all $x \in X$. As in the proof of Theorem 3.2, we have $\|DS(x,y)\| = 0$ for all $x, y \in X$. Thus, from Lemma 2.2, S is cubic.

In (3.17), taking the limit as $n \to \infty$, we obtain

$$\left\|S(x)-f(x)\right\| \le \frac{1}{96}\sum_{i=0}^{\infty}\left[\frac{3\phi(0,2^ix)+10\phi(2^{i-1}x,2^{i-1}x)+10\phi(3\cdot 2^{i-1}x,2^{i-1}x)}{8^i}\right].$$

Now, we claim S is unique. Let us consider another cubic mapping $\widetilde{S} : X \to Y$ satisfying (1.4) and (3.10). As we have, $\widetilde{S}(2^nx) = 8^n\widetilde{S}(x)$ and $S(2^nx) = 8^nS(x)$ for all $x \in X$, the rest of the proof is similar to the corresponding part of the proof Theorem 3.2. This completes the proof. □

Now, we prove the main theorem in this section.

Theorem 3.4. *Let $\phi : X \times X \to [0,\infty)$ be a function such that*

$$\sum_{i=0}^{\infty}\frac{\phi(2^{i-1}x,2^{i-1}x)+\phi(0,2^{i-1}x)}{4^i} \quad \textit{and} \quad \sum_{i=0}^{\infty}\frac{3\phi(0,2^ix)+10\phi(2^{i-1}x,2^{i-1}x)+10\phi(3\cdot 2^{i-1}x,2^{i-1}x)}{8^i}$$

converges and

$$\lim_{n\to\infty}\frac{\phi(2^nx,2^ny)}{4^n} = 0,$$

for all $x, y \in X$. Let us consider a mapping $f : X \to Y$ which satisfies the inequality

$$\left\|Df(x,y)\right\| \leq \phi(x,y), \tag{3.18}$$

for all $x, y \in X$ and $f(0) = 0$. Then there exist a unique cubic mapping $S : X \to Y$ and a unique quadratic mapping $T : X \to Y$ which satisfies (1.4) and

$$\begin{aligned}\left\|f(x) - S(x) - T(x)\right\| \leq \frac{1}{24}\sum_{i=0}^{\infty}\Big[&\frac{\phi(0,2^{i-1}x) + \phi(2^{i-1}x,2^{i-1}x) + \phi(0,-2^{i-1}x) + \phi(-2^{i-1}x,-2^{i-1}x)}{4^i}\\ &+ \frac{3\phi(0,2^ix) + 10\phi(2^{i-1}x,2^{i-1}x) + 10\phi(3\cdot 2^{i-1}x,2^{i-1}x)}{8^{i+1}}\\ &+ \frac{3\phi(0,-2^ix) + 10\phi(-2^{i-1}x,-2^{i-1}x) + 10\phi(-3\cdot 2^{i-1}x,-2^{i-1}x)}{8^{i+1}}\Big],\end{aligned} \tag{3.19}$$

for all $x \in X$.

Proof. Let us decompose f into even and odd parts by putting

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)] \quad \text{and} \quad f_o(x) = \frac{1}{2}[f(x) - f(-x)],$$

for all $x \in X$. Then we have $f_e(0) = 0$ and $f_o(0) = 0$ and $f(x) = f_e(x) + f_o(x)$.

Now, from (3.18), we obtain

$$\left\|Df_e(x,y)\right\| \leq \frac{1}{2}\left[\phi(x,y) + \phi(-x,-y)\right] \quad \text{and} \quad \left\|Df_o(x,y)\right\| \leq \frac{1}{2}\left[\phi(x,y) + \phi(-x,-y)\right],$$

for $x, y \in X$. Thus, in view of Theorems 3.2 and 3.3, there exist a unique quadratic function $T : X \to Y$ and a unique cubic function $S : X \to Y$, which are given by

$$T(x) = \lim_{n\to\infty}\frac{f_e(2^nx)}{4^n} \quad \text{and} \quad S(x) = \lim_{n\to\infty}\frac{f_o(2^nx)}{8^n},$$

for all $x \in X$. Also

$$\left\|T(x) - f_e(x)\right\| \leq \frac{1}{24}\sum_{i=0}^{\infty}\frac{\phi(0,2^{i-1}x) + \phi(2^{i-1}x,2^{i-1}x) + \phi(0,-2^{i-1}x) + \phi(-2^{i-1}x,-2^{i-1}x)}{4^i},$$

for all $x \in X$. And,

$$\begin{aligned}\left\|S(x) - f_o(x)\right\| \leq \frac{1}{192}\sum_{i=0}^{\infty}\Big[&\frac{3\phi(0,2^ix) + 10\phi(2^{i-1}x,2^{i-1}x) + 10\phi(3\cdot 2^{i-1}x,2^{i-1}x)}{8^{i+1}}\\ &+ \frac{3\phi(0,-2^ix) + 10\phi(-2^{i-1}x,-2^{i-1}x) + 10\phi(-3\cdot 2^{i-1}x,-2^{i-1}x)}{8^{i+1}}\Big].\end{aligned}$$

Thus, combining the above two inequality, we obtain (3.19). This completes the proof. □

Corollary 3.5. *Let X and Y be a real normed space and a Banach space, respectively and $\epsilon \geq 0$ be real number. If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\|Df(x,y)\right\| \leq \epsilon,$$

for all $x, y \in X$, then there exist a unique cubic mapping $S : X \to Y$ and a unique quadratic mapping $T : X \to Y$ satisfying (1.4) and

$$\left\|f(x) - S(x) - T(x)\right\| \leq \frac{125}{252}\epsilon,$$

for all $x \in X$.

Corollary 3.6. *Let X and Y be a real normed space and a Banach space, respectively and $\theta \geq 0$ and $p < 2$ be two real numbers. If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\|Df(x,y)\right\| \leq \theta\left(\|x\|^p + \|y\|^p\right),$$

for all $x, y \in X$ and $f(0) = 0$, then there exist a unique cubic mapping $S : X \to Y$ and a unique quadratic mapping $T : X \to Y$ satisfying (1.4) and

$$\left\|f(x) - S(x) - T(x)\right\| \leq \theta\|x\|^p \left[\frac{1}{2^p(2^2 - 2^p)} + \frac{1}{12}\frac{1}{2^3 - 2^p}\left\{3 + \frac{30}{2^p} + 10\left(\frac{3}{2}\right)^p\right\}\right],$$

for all $x \in X$.

Corollary 3.7. *Let X and Y be a real normed space and a Banach space with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively and $0 \leq a \leq \frac{1}{2}$ and $\delta \geq 0$ be two real numbers. If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\|Df(x,y)\right\|_2 \leq 2\delta\|x\|_1^p\|y\|_1^p,$$

for all $x, y \in X$ and $f(0) = 0$, then there exist a unique cubic mapping $S : X \to Y$ and a unique quadratic mapping $T : X \to Y$ satisfying (1.4) and

$$\left\|f(x) - S(x) - T(x)\right\| \leq \delta\|x\|_1^p \left[\frac{2}{3}\frac{1}{4 - 2^a} + \frac{1}{2}\frac{1}{8 - 2^a}\frac{\|x\|_1^p 2^{-2a}}{3}\left\{\frac{2}{4 - 2^{2a}} + \frac{5}{8 - 2^{2a}}\left(1 + 3^a\right)\right\}\right],$$

for all $x \in X$.

Acknowledgement

The authors sincerely thank the learned referees for their constructive comments. The authors acknowledge funding received from NBHM, DAE, India vide research grant no. 02011/19/2023/NBHM(R.P)/R&D-II/5949.

References

[1] J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, Cambridge, UK, 1989.
[2] T. Aoki, *On the stability of the linear transformation in Banach*, J. Math. Soc. Japan., **2** (1950), 64–65.
[3] J. Baker, *The stability od the cosine equation*, Proc. Amer. Math. Soc., **80** (1980), 411–416.
[4] I.-S. Chang, Y.-S. Jung, *Stability of a functional equation deriving from cubic and quadratic functions*, J. Math. Anal. Appl., **283** (2003), 491–500.
[5] P. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math., **27** (1984), 76–86.
[6] S. Czerwik, *On the stability of the quadratic mappings in normed spaces*, Abh. Math. Sem. Univ. Hamburg, **62** (1992), 59–64.
[7] D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser, Basel, 1986.
[8] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mapppings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[9] A. Grabiec, *The generalized Hyers-Ulam stability of a class of functional equations*, Publ. Math. Debrecen, **48** (1996), 217–235.
[10] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA., **27** (1941), 222–224.
[11] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[12] D.H. Hyers, G. Isac, Th. M. Rassias, *On the asymtoticity aspect of Hyers-Ulam stability of mapping*, Proc. Amer. Math. Soc., **126** (1998), 425–430.
[13] D.H. Hyers, Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math, **44** (1992), 125–153.
[14] K.-W. Jun, H.-M. Kim, *Remarks on the stability of additive functional equation*, Bull. Korean Math. Soc., **38** (2001), 678–687.
[15] K.-W. Jun, H.-M Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl., **274** (2002), 867–878.
[16] K.-W. Jun, Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality*, Math. Inequal. Appl., **4** (1) (2001), 93-118.
[17] S.-M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl., **222** (1998), 126–137.
[18] S.-M. Jung, *On the Hyers-Ulam-Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl., **232** (1999), 384–393.
[19] P.l. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math., **27** (1995), 368–372.
[20] T. Rassias, *On the stability of the linear mappings in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.
[21] T. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., **251** (2000), 264-284.
[22] S. M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Wiley-Interscience, New York, 1964.