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Some ideal convergence of double sequence spaces of neutrosophic real numbers

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Abstract. We reveal idea of some new double sequence spaces, namely, ideal convergence of double sequence spaces in neutrosophic normed spaces. With the help of neutrosophic norm,we wish to define $\hat{\imath}_2$ - convergence and $\hat{\imath}_2^*$ - convergence of double sequences in neutrosophic normed spaces. Several basic properties and characterization theorems of these concepts would be investigated in neutrosophic normed spaces. Our purpose is also to introduce $\hat{\imath}_2$ - Cauchy and $\hat{\imath}_2^*$ - Cauchy double sequences in neutrosophic normed spaces. We have investigated some of the characterization theorems in neutrosophic normed spaces.

1. Introduction

Classical methods often fail to deal with many real - life problems due to uncertainties. To overcome such situations, Zadeh [23] talked about fuzzy set theory associated with only membership (truth) function. Thereafter, Atanassov [1] invented the notion of intuitionistic fuzzy set theory associating with membership & non - membership functions. For the purpose of solving naturalistic problems on decision making under uncertainty, Smarandache [18] introduced the notion of neutrosophic set theory associating with three independent functions, i.e., membership, non - membership & indeterminacy functions. Further, Smarandache [19] investigated on the applications of the neutrosophic set theory.

The use of intuitionistic fuzzy set theory is found in all fields where fuzzy set theory was investigated. George and Veeramani [4] defined fuzzy metric space and Park [14] generalized it. The hypothesis of statistical convergence for real number sequences was first scrutinized by Fast [3] and Schoenberg [16] individually. As a generalization of ordinary convergence and statistical convergence, Kostyrko et al. [10] invented $\hat{\imath}$-convergence. Karakaya et al. [5, 6] defined and studied $\hat{\imath}$-convergence and lacunary statistical convergence of sequences of functions in intuitionistic fuzzy normed spaces. We can find more research works on $\hat{\imath}$-convergence in [9, 11].

Neutrosophic metric space and neutrosophic normed space (in short, NNS) were investigated by Kirişci and Şimşek [7, 8]. Tripathy and Hazarika [20] introduced paranorm $\hat{\imath}$ - convergent sequence spaces. Mursaleen and Edely [12] introduced statistical convergence in double sequences. Savaş and Mursaleen [15] and

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Mursaleen et al. [13] scrutinized statistical and ideal convergence of double sequences for fuzzy numbers and intuitionistic fuzzy numbers respectively. Tripathy and Das [21] investigated on class of fuzzy number sequence spaces. The norm of $\hat{\imath}$-convergent of double sequence was introduced by Tripathy and Tripathy [22]. Das et al. [2] examined $\hat{\imath}$-convergence and $\hat{\imath}^*$-convergence of double sequences in $\mathbb{R}$. After getting motivations of these works, we introduce the ideal convergence of double sequence spaces in NSS. We investigate some of their basic properties and relationships with other convergence of sequence spaces. The article is subdivided as below. The following section shortly indicates known definitions and results which are related for investigation. In Section 3, we investigate the notion of $\hat{\imath}_2^*$-convergence of double sequences in a NSS. In S ection 4, we study notion of $\hat{\imath}_2^*$-convergence of double sequences in a NSS. Section 5 focuses on the concept of $\hat{\imath}_2$ and $\hat{\imath}_2^*$-Cauchy double sequences in a NSS. Conclusion appears in last section.

2. Preliminaries and definitions:

In this section necessary concepts and results have been procured.

Definition 2.1. ([17]) A continuous binary operation $\tilde{o} : [\mathbb{0}, \mathbb{1}] \times [\mathbb{0}, \mathbb{1}] \to [\mathbb{0}, \mathbb{1}]$ satisfying associative and commutative laws is called continuous t-norm (TN) if $\dot{x}\tilde{o}\mathrm{x} = \dot{\mathrm{x}}$ and $\dot{\mathrm{x}}\tilde{z}z \leq \dot{\mathrm{y}}\tilde{o}\dot{w}$ whenever $\dot{\mathrm{x}} \leq \dot{\mathrm{y}}$ and $\dot{z} \leq \dot{\mathrm{w}}$ for each $\dot{\mathrm{x}}, \dot{\mathrm{y}}, \dot{z}, \dot{\mathrm{w}} \in [\mathbb{0}, \mathbb{1}]$.

Definition 2.2. ([17]) *Given a continuous binary operation $\square : [\mathbb{0}, \mathbb{1}] \times [\mathbb{0}, \mathbb{1}] \to [\mathbb{0}, \mathbb{1}]$ which satisfies associative and commutative laws. Then it is called continuous t−co-norm (TC) if*

$$\dot{x}\square\mathbb{0} = \dot{x} \quad \text{and} \quad \dot{x}\square\dot{z} \leq \dot{y} = \dot{w} \quad \text{whenever} \quad \dot{x} \leq \dot{y} \text{ and } \dot{z} \leq \dot{w}$$

for each $\dot{x}, \dot{y}, \dot{z}, \dot{w} \in [\mathbb{0}, \mathbb{1}]$

Definition 2.3. ([8]) Take $\dot{F}$ as a vector space, $N = \{< \dot{u}, G(\hat{u}), \dot{B}(\hat{u}), Y(\hat{u})\rangle : \dot{u} \in \dot{F}\}$ be a NS where $N : \dot{F} \times \overline{\mathrm{R}}^+ \to [\mathbb{Q}, \mathbb{1}]$. Consider binary operations $\tilde{o}$ and $\cdot$ as, defined in Definition 2.1 and Definition 2.2. The four tuple $V = (\dot{F}, N, \acute{o}, \cdot)$ is known as, neutrosophic normed space (NNS) if the below conditions hold. For all $\dot{u}, \hat{v}, \in \dot{F}$ and $\lambda, \mu > 0$ and for each $\sigma \neq 0$,

(1) $\mathbb{0} \leq G(\mu, \lambda) \leq \mathbb{1}, \mathbb{0} \leq \dot{B}(\mu, \lambda) \leq \mathbb{1}, \mathbb{0} \leq Y(\mu, \lambda) \leq \mathbb{1}, \forall \lambda \in \bar{R}^+$,

(2) $G(\mu, \lambda) + \dot{B}(\mu, \lambda) + Y(\mu, \lambda) \leq 3$, (for $\lambda \in \bar{R}^+$),

(3) $G(\mu, \lambda) = \mathbb{1}$ (for $\lambda > \mathbb{0}$) if and only if $\dot{u} = \mathbb{0}$,

(4) $G(\hat{u}, \hat{v}, \lambda) = G(\hat{v}, \dot{u}, \lambda)$ (for $\lambda > \mathbb{0}$),

(5) $G(\hat{u}, \dot{v}, \lambda)\tilde{o}G(\hat{v}, \dot{u}, \lambda) \leq G(\hat{u}, y, \lambda + u(\forall \lambda, \mu >)$,

(6) $G(u, \dot{v}, \rightharpoonup) : [\mathbb{0}, \infty) \to [\mathbb{0}, \mathbb{1}]$ is continuous,

(7) $\lim_{\lambda \to \infty} G(\hat{u}, \dot{v}, \lambda) = \mathbb{1}(\forall \lambda > \mathbb{0})$,

(8) $B(\hat{u}, \hat{v}, \lambda) = \mathbb{0}$ (for $\lambda > \mathbb{0}$) if and only if $\dot{u} = \hat{v}$,

(9) $\dot{B}(u, \dot{v}, \lambda) = \dot{B}(\hat{v}, \dot{u}, \lambda)$(for $\lambda > \mathbb{0}$),

(10) $\dot{B}(\hat{u}, \dot{v}, \lambda) \cdot \dot{B}(\hat{v}, y, \mu) \geq \dot{B}(\hat{u}, y, \lambda + \mu)(\forall \lambda, \mu > \mathbb{0})$,

(11) $\dot{B}\dot{u}, \dot{v}, \cdot) : [\mathbb{0}, \infty) \to [\mathbb{0}, \mathbb{1}]$ is continuous,

(12) $\log_{\lambda \to \infty} \dot{B}(\hat{u}, \dot{v}, \lambda) = \mathbb{0}(\forall \lambda > \mathbb{0})$,

(13) $Y(u, \dot{v}, \lambda) = \mathbb{0}$ (for $\lambda > \mathbb{0}$) if and only if $\dot{u} = \dot{v}$,

(14) $Y(u, \dot{v}, \lambda) = Y(\dot{v}, \dot{u}, \lambda)(\forall \lambda > \mathbb{0})$,

(15) $Y(\hat{u}, \dot{v}, \lambda) \cdot Y(\dot{v}, y, \mu) \geq Y(u, y, \lambda + \mu)(\forall \lambda, \mu > \mathbb{0})$,

(16) $\dot{B}(\dot{u}, \dot{v}, \cdot) : [\mathbb{0}, \infty) \to [\mathbb{0}, \mathbb{1}]$ is continuous,

(17) $\lim_{\lambda \to \infty} Y(u, \hat{u}, \lambda) = \mathbb{1}(\forall \lambda > \mathbb{0})$,

(18) if $\lambda \leq \mathbb{0}$, then $G(u, \dot{v}, \lambda) = \mathbb{0}, B(u, \dot{v}, \lambda) = \mathbb{1}$ and $Y(\dot{u}, \dot{v}, \lambda) = \mathbb{1}$.

Then $\eta = (G, \dot{B}, Y)$ is called Neutrosophic Norm (NN).

Definition 2.4. ([3]) The asymptotic compactness of a subset C of N, represented by $\mu(C)$, is given below:

$$\mu(C) = \lim_{\check{n}} \frac{1}{\check{n}} |\{k \leq \check{n} : k \in C\}| \text{ as, } \check{n} \to \infty.$$

Consider $\dot{x} = (m_k)$, as, a number sequence which is statistically convergent to the number l if $C(\varepsilon) = \{k \leq \check{n} : |\dot{x}_k - l| > \varepsilon\}\} = 0$, for every $\varepsilon > 0$.

Then it is denoted by $st - lim\dot{x} = l$.

Definition 2.5. ([3]) A number of sequence $\dot{x} = (m_k)$ which is statistically Cauchy if, for each $\varepsilon > 0, \exists$ a number $M = M(\varepsilon)$ in order that

$$\lim \frac{1}{\check{n}} \left|\left\{\dot{f} \leq \check{n} : \left|\dot{x}_j - \dot{x}_M\right| \geq \varepsilon\right\}\right| = \mathbb{0} \text{ as, } \check{n} \to \infty$$
$$k(\varepsilon) = \{k \leq \check{n} : |\dot{x}_k - l| > \varepsilon\}\} = \mathbb{0}$$

Then it is denoted by st-lim $\dot{x} = l$.

Definition 2.6. ([9]) Consider a nonempty set ψ. Then a class $\psi_{\mathbb{1}} \subset P(\psi)$ is referred an ideal in ψ if and only if

(1) for each $\dot{A}, \dot{B} \in \psi_{\mathbb{1}}$, we have $A \cup \dot{B} \in \psi_{\mathbb{1}}$;

(2) for each $A \in \psi_{\mathbb{1}}$ and $\dot{B} \subset A$, we have $\dot{B} \in \psi_{\mathbb{1}}$,

where $P(\psi)$ denotes power set of ψ. $\psi_{\mathbb{1}}$ is called non-trivial ideal if $\psi \notin \psi_{\mathbb{1}}$.

Definition 2.7. ([9]) A nonempty group $\psi_{\dot{F}} \subset P(\psi)$ is said to be a filter on a nonempty set ψ if and only if

(1) $\theta \notin \psi_F$; where θ denotes the empty set,

(2) for each $\dot{A}, \dot{B} \in \psi_{\dot{F}}$, we have $\dot{A} \cap \dot{B} \in \psi_{\dot{F}}$;

(3) every $A \in \psi_{\dot{F}}$ and $A \subset \dot{B}$, we have $\dot{B} \in \psi_{\dot{F}}$.

Definition 2.8. ([9]) A non-trivial ideal ψ_I in ψ is referred an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletons, i.e., $\{\dot{x}\} \in \psi_{\mathbb{1}}$ for each $\dot{x} \in \psi$.

Definition 2.9. ([9] An admissible ideal $\psi_I \subset P(\mathbb{N})$ satisfies the condition (AP) if for every sequence $(A_{\check{n}})_{\check{n} \in \mathrm{N}}$ of pairwise disjoint sets from ψ_1, there are sets $\dot{\mathrm{B}}_{\check{n}} \subset \mathrm{N}, \check{n} \in \mathrm{N}$ in order that the symmetric difference $\dot{A}_{\check{n}} \Delta \dot{\mathrm{B}}_{\check{n}}$ is a finite set $\forall \check{n}$ and $\bigcup_{\check{n} \in \mathbb{N}} \dot{\mathrm{B}}_{\check{n}} \in \psi_{\mathbb{1}}$.

Definition 2.10. ([9] Let $\psi_I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in N. Then a sequence $\dot{\mathbf{x}} = (m_k)$ is assumed to be $\hat{\imath}$ convergent to L if, $\forall \varepsilon > \mathbb{0}$, the set

$$\{k \in \mathrm{N} : |m_k - L| \geq \varepsilon\} \in \hat{\imath}.$$

In this case, we write $\hat{\imath} - \lim \dot{x} = L$.

Definition 2.11. ([9]) Let $\psi_{\mathbb{1}} \subset 2^{\mathbb{N}}$ be an admissible ideal in N. A sequence $\dot{x} = (m_k)$ is assumed to be $\hat{\imath}-$ Cauchy if, for every $\varepsilon > \mathbb{0}, \exists$ a number $M = M(\varepsilon)$ such that

$$\{k \in N : |\dot{x}_k - \dot{x}_M| \geq \varepsilon\} \in \psi_I$$

3. $\hat{\imath}_2$ convergence in a NNS:

In this section we introduce the notion of ideal convergence of double sequences in a NSS. In every part of the article, we consider $N_{\hat{i}_2}$ as, non-trivial ideal of $N \times N$.

Definition 3.1. Let $(F, N, \tau, \cdot)$ be a NNS and η be NN. A double sequence $x = (s_{jk})$ of elements of F is called N_{I_2}-convergent to $L \in F$ with regard to η if, for each $\varepsilon > 0$ and $t > 0$, $\{(j,k) \in \mathbb{N} \times \mathbb{N} : N((s_{jk} - L, t) \leq 1 - \varepsilon)$ or $\eta((s_{jk} - L, t) \geq \varepsilon)\} \in I_2$ Here $N^{\eta}_{I_2}$- $\lim x = L$

Theorem 3.2. *Consider ($\dot{\mathrm{F}}, N, \tilde{o}, \cdot)$ as, a NNS and η be NN. Then, $\forall \varepsilon > \mathbb{0}$ and $t' > \mathbb{0}$, following are analogous to:*

(1) $N^{\eta}_{\hat{i}_2} - \lim \dot{x} = L.$

(2) $\left\{(j,k) \in N \times N : G\left(s_{jk} - L, t', t'\right) \leq \mathbb{1} - \varepsilon \in N_{\hat{\imath}_2}, \dot{B}\left(s_{jk} - L, t'\right) \geq \varepsilon \in N_{\hat{\imath}_2} \text{ and } Y\left(s_{jk} - L, t'\right) \geq \varepsilon\right\} \in N_{\hat{\imath}_2}.$

(3) $\left\{(j,k) \in N \times N : G\left(s_{jk} - L, t'\right) > \mathbb{1} - \varepsilon, \dot{B}\left(s_{jk} - L, t'\right) < \varepsilon \text{ and } Y\left(s_{jk} - L, t'\right) < \varepsilon\right\} \in N_{\hat{1}_2}).$

(4) $\left\{(j,k) \in N \times N : G\left(s_{jk} - L, t'\right) > \mathbb{1} - \varepsilon\right\} \in \dot{F}\left(N_{\hat{i}_2}\right), \left\{(j,k) \in N \times N : \dot{B}\left(s_{jk} - L, t'\right) < \varepsilon \in \dot{F}\left(N_{\hat{i}_2}\right)\right\}$ *and* $\left\{(\dot{j},k) \in N \times N : Y\left(s_{jk} - L, t\right) < \varepsilon\right\} \in \dot{F}\left(N_{\hat{i}_2}\right)\}.$

(5) $N^{\eta}_{\hat{1}_2} - \lim G\left(s_{jk} - L, t'\right) = \mathbb{1}, N^{\eta}_{\hat{1}_2} - \lim \dot{B}\left(s_{jk} - L, t'\right) = \mathbb{0}$ *and* $N^{\eta}_{\hat{i}_2} - \lim Y\left(s_{jk} - L, t'\right) = \mathbb{0}.$

Proof. The proof is a standard verification, so left. □

Theorem 3.3. Let$(\dot{\mathrm{F}}, N, \tilde{o}, \cdot)$ *be a NNS and η be NN. Assuming that a double sequence* $\dot{\mathrm{x}} = \left(\mathrm{s_{jk}}\right)$ *is* $N_{\hat{1}_2}-$ *convergent with regard to η, then* $N^{\eta}_{\hat{i}_2} - \lim \dot{\mathrm{x}}$ *is unique.*

Proof. Let $N^{\eta}_{,\hat{1}_2} - l$ im $\dot{\mathrm{x}}$ be not unique. Suppose that $N^{\eta}_{,\hat{1}_2} - \lim \dot{\mathrm{x}} = L_1$ and $N^{\eta}_{\hat{i}_2} - \lim \dot{\mathrm{x}} = \mathrm{L}_2$. Given $\varepsilon > \mathbb{0}$, choose $\check{r} > \mathbb{0}$ such that $(\mathbb{1} - \check{r})\tilde{o}(1 - \check{r}) > \mathbb{1} - \varepsilon$ and $\check{r} = \check{r} < \varepsilon$. Then, for any $t' > 0$, define the following sets as,:

$$k_{G.1}(\check{r}, t) = \left\{(\dot{j},k) \in N \times N : G\left(s_{jk} - \mathrm{L}, t'/2\right) \leq \mathbb{1} - \check{r}\right\},$$
$$k_{G.2}(\check{r}, t') = \left\{(j,k) \in N \times N : G\left(s_{jk} - \mathrm{L}, t'/2\right) \leq \mathbb{1} - \check{r}\right\},$$
$$k_{B.1}(\check{r}, t') = \left\{(j,k) \in N \times N : G\left(s_{jk} - \mathrm{L}, t', 2\right) \geq \check{r}\right\}$$
$$k_{B.2}(\check{r}, t') = \left\{(\dot{y},k) \in N \times N : G\left(s_{jk} - \mathrm{L}, t'/2\right) \geq \check{r}\right\},$$
$$k_{Y.11}(\check{r}, t') = \left\{(j,k) \in N \times N : G\left(s_{jk} - \mathrm{L}', t'/2\right) \geq \check{r}\right\},$$
$$k_{Y.2}(\check{r}, t') = \left\{(j,k) \in N \times N : G\left(s_{jk} - \mathrm{L}, t'/2\right) \geq \check{r}\right\}.$$

Since, $N^{\eta}_{\hat{i}_2} - \lim \dot{\mathrm{x}} = L_1$, we have
$k_{G,1}(\check{r}, t'), k_{\dot{B}.1}(\check{r}, t')$ and $k_{Y.1}(\check{r}, t) \in \hat{\imath}_2$.
Furthermore, using $N^{\eta}_{\hat{i}_2} - \lim \dot{x} = L_2$, we get
$k_{G.2}(\check{r}, t), k_{\dot{B}.2}(\check{r}, t')$ and $k_{Y.2}(\check{r}, t') \in \hat{1}_2$.

Next let $k_{\eta}(\check{r}, t') = (k_{G.1}(\check{r}, t') \cup k_{G.2}(\check{r}, t')) \cap (k_{\dot{B}.1}(\check{r}, t') \cup k_{\dot{B}.2}(\check{r}, t')) \cap (k_{Y.1}(\check{r}, t') \cup k_{Y.2}(\check{r}, t')) \in \hat{1}_2$. Then we find that $k_{\eta}(\check{r}, t) \in \hat{\imath}_2$. This denotes that its complement $k^c_{\eta}(\check{r}, t)$ is a non - empty set in $\dot{F}\left(\hat{\imath}_2\right)$. If $(\dot{j},k) \in k^c_{\eta}(\check{r}, t')$, then we have three possible cases. That is, $(j,k) \in k^c_{G.1}(\check{r}, t') \cap k_{G.2}(\check{r}, t')$ or $(\dot{j},k) \in k^c_{\dot{B}.1}(\check{r}, t') \cap k_{\dot{B}.2}(\check{r}, t)$ or$(\dot{j},k) \in k^c_{Y.1}(\check{r}, t') \cap k_{Y.2}(\check{r}, t')$. Then we have

$$G\left(L_{\mathbb{1}} - L_2, t'\right) \geq G\left(s_{jk} - L_1, \frac{t}{2}\right)\tilde{o}G\left(s_{jk} - L_2, \frac{t'}{2}\right) > (\mathbb{1} - \check{r})\tilde{o}(\mathbb{1} - \tilde{r}) > \mathbb{1} - \varepsilon$$

Since, $\varepsilon > 0$ is arbitrary, we get $G(L_1 - L_2, t') = 1$ for all $t' > 0$, which yields $L_1 = L_2$. On the contrary, if $(j,k) \in k^c_{B.1}(\check{r}, t) \cap k_{B.2}(\check{r}, t)$ and $(j,k) \in k^c_{Y,1}(\check{r}, t') \cap k_{Y.2}(\check{r}, t)$, then we may able to write that

$$\dot{B}(L_{11} - L_2, t') \leq \dot{B}\left(S_{jk} - L_n, \frac{t^2}{2}\right) \cdot \dot{B}\left(S_{jk} - L_{\lambda 2}, \frac{t^2}{2}\right) < \check{r}_- = \check{r} < \varepsilon$$

and $Y(L_1 - L_x, t') \leq Y\left(S_{jk} - L_1, \frac{t'}{2}\right) \cdot Y\left(S_{jk} - L_x, \frac{t'}{2}\right) < \check{r} = \check{r} < \varepsilon$ respectively.

Therefore, we have $B(L_{\lambda_1} - L_2, t') = 1$ for all $t' > 0$, which yields $L_1 = L_2$ and $Y(L_1 - L_2, t') = 1$, for all $t' > 0$, which yields $L_1 = L_2$. Therefore, in all the cases we conclude that $N^{\eta}_{\hat{1}_2} - \lim \dot{x}$ is unique. This completes the proof of the theorem. □

Theorem 3.4. *Let $(\dot{F}, N, \tilde{o}, \cdot)$ be a NNS and η be NN. If $\hat{\imath}_2$ is an admissible ideal, then we have the following:*

(1) *If $\eta - \lim s_{jk} = L$, then $N^{\eta}_{\hat{i}_2} - \lim s_{jk} = L_L$.*

(2) *If $N^{\eta}_{\hat{i}_2} - \lim s_{jkk} = L_1$ and $\mathbb{N}^{\eta}_{\hat{i}_2} - \lim t_{jk} = L_2$, then $N^{\eta}_{\hat{i}_2} - \lim\left(s_{jk} + t'_{jjk}\right) = \left(L_{\hat{i}_1} + L_2\right)$.*

(3) *If $N^{\eta}_{\hat{i}_2} - \lim s_{jk} = L$, then $N^{\eta}_{\hat{1}_2} - \lim \alpha S_{jk} = \alpha L$.*

Proof. (1) Suppose that $N^{\eta}_{\hat{i}_2} - \lim s_{j\mathfrak{k}} = L_{L^2}$. Then for each $\varepsilon > 0$ and $t' > 0$, there exists a positive integer n_0 such that

$G\left(s_{jk} - L, t'\right) > 1 - \varepsilon, \dot{B}\left(s_{jk} - L, t'\right) < \varepsilon$ and $Y\left(s_{jk} - L, t'\right) < \varepsilon$, for each $k > n_0$. Since, the set $A(\varepsilon) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : G(s_k - L, t') \leq 1 - \varepsilon \text{ or } \dot{B}\left(s_{jk} - L^L, t'\right) \geq \varepsilon \text{ or } Y\left(s_{jk} - L, t'\right) \geq \varepsilon\right\}$ contained in $\{1, 2, \ldots\ldots.., N - 1\}$ and the ideal $\hat{\hat{i}}_2$ is admissible, $A(\varepsilon) \in \hat{\imath}_2$. Hence $\mathbb{N} - lims_{jk} = L_L$.

(2) Let $N^{\eta}_{\hat{i}_2} - \lim s_{jk} = L_2$ and $N^{\eta}_{\hat{i}_2} - \lim t'_{jk} = L_2$. For a given $\varepsilon > 0$, choose $\check{r} > 0$ such that $(1 - \check{r})\tilde{o}(1 - \check{r}) > 1 - \varepsilon$ and $\check{r} \cdot \check{r} < \varepsilon$. Then, for any $t' > 0$, we define the following sets

$$\begin{aligned}
k_{G.1}(\check{r}, t') &= \left\{(j,k) \in N \times N : G\left(s_{jk} - L_{\hat{1}}, t'/2\right) \leq 1 - \check{r}\right\} \\
k_{G.2}(\check{r}, t') &= \left\{(j,k) \in N \times N : G\left(s_{jk} - L_1, t'/2\right) \leq 1 - \check{r}\right\} \\
k_{B.1}(\check{r}, t') &= \left\{(j,k) \in N \times N : G\left(s_{jk} - L_1, t'/2\right) \geq \check{r}\right\} \\
k_{B.2}(\check{r}, t') &= \left\{(j,k) \in N \times N : G\left(s_{jk} - L_1, t'/2\right) \geq \check{r}\right\} \\
k_{Y.1}(\check{r}, t') &= \left\{(j,k) \in N \times N : G\left(s_{jjk} - L_1\mathfrak{k}, t'/2\right) \geq \check{r}\right\} \\
k_{Y.2}(\check{r}, t') &= \left\{(j,k) \in N \times \mathbb{N} : G\left(s_{jk} - L_1, t'/2\right) \geq \check{r}\right\}
\end{aligned}$$

Since, $N^{\eta}_{\hat{1}_2} - \lim s_{jk} = L_1 1$, we have
$k_{G.1}(\check{r}', t'), k_{\dot{B}.1}(\check{r}, t')$ and $k_{Y.1}(\check{r}, t') \in \hat{\imath}_2$.
Furthermore, using $N^{\eta}_{\hat{i}_2} - \lim t_{jk} = L_2$, we get
$k_{G.2}(\check{r}, t'), k_{\dot{B}.2}(\check{r}, t')$ and $k_{Y.2}(\check{r}, t') \in \hat{\imath}_2$.

Let $k_{\eta}(\check{r}, t') = (k_{G.1}(\check{r}, t') \cup k_{G.2}(\check{r}, t')) \cap (k_{\dot{B}.1}(\check{r}, t') \cup k_{B.2}(\check{r}, t')) \cap (k_{Y.1}(\check{r}, t') \cup k_{Y.2}(\check{r}, t')) \in \hat{\imath}_2$. Then $k_{\eta}(\check{r}, t') \in \hat{\imath}_2$, which denotes that $K^c_{\eta}(\check{r}, t')$ is a non - empty set in $\dot{F}\left(\hat{1}\right)$. Now we have to show that $K^c_{\eta}(\check{r}, t') \subset \{(j,k) \in N \times N : G\left(\left(s_{jk} + t'_{jkk}\right) - (L_{L_1} + L_2), t'\right) > 1 - \varepsilon, \dot{B}\left(\left(s_{jk} + t'_{jk}\right) - (L_1 1 + L_2), t'\right) < \varepsilon$ and $Y\left(\left(s_{jk} + t'_{jkk}\right) - (L_1 + L_2), t') < \varepsilon\right\}$. If $(j,k) \in k^c_{\eta}(\check{r}, t')$, then we have $G\left(s_{jk} - L_1, \frac{t'}{2}\right) > 1 - \check{r}, G\left(t'_{j\mathfrak{k}} - L_2, \frac{t'}{2}\right) > 1 - \check{r}, \dot{B}\left(s_{jk} - L_{\mathfrak{m}}, \frac{t'}{2}\right) <$

ř, $\dot{B}\left(t'_{jk} - L_{\hat{n}2}, \frac{t'}{2}\right) <$ ř, $Y\left(s_{jk} - L_{n}, \frac{t'}{2}\right) <$ ř, $Y\left(t'_{jk} - L_2, \frac{t'}{2}\right) <$ ř. Therefore, $G\left(\left(s_{jk} + t'_{jk}\right) - (L_1 + L_2 2), t'\right) \geq G\left(s_{jk} - L_1, \frac{t'}{2}\right) \tilde{o} G\left(t'_{jk} - L_2, \frac{t'}{2}\right) > (1-ř)\tilde{o}(1-ř) > 1-\varepsilon$, $\dot{B}\left(\left(s_{jk} + t'_{jk}\right) - (L_{11} + L_2), t'\right) \leq \dot{B}\left(s_{jk} - L_1, \frac{t'}{2}\right) \cdot G\left(t'_{jk} - L_2, \frac{t'}{2}\right) <$ ř · ř $< \varepsilon$ and $Y\left(\left(s_{jk} + t'_{jk}\right) - (L_{11} + L_2), t'\right) \leq Y\left(s_{jk} - L_1, \frac{t'}{2}\right) \cdot G\left(t'_{jk} - L_2, \frac{t'}{2}\right) <$ ř · ř $< \varepsilon$.

This shows that k^c_η (ř, t′) $\subset \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : G\left(\left(s_{jk} + t'_{jk}\right) - (L_{11} + L_2), t'\right) > 1 - \varepsilon, \dot{B}\left(\left(s_{jk} + t'_{jk}\right) - (L_1 + L_2), t'\right) < \varepsilon \text{ and } Y\left(\left(s_{jk} + t'_{jk}\right) - (L_1 + L_2), t'\right) < \varepsilon\right\}$. Since, k^c_η (ř, t′) $\in \dot{F}\left(\hat{i}_2\right)$, $N^\eta_{I_2} - \lim\left(s_{jk} + t'_{jkk}\right) = (L_1 + L_2)$.

(3) It is obvious for $\alpha = 0$. Now let $\alpha \neq 0$. Then for a given $\varepsilon > 0$ and $t' > 0$,
$\dot{B}(\varepsilon) = \left\{(j, k) \in \mathbb{N} \times N : G\left(s_{jk} - L_{\hat{N}}, t'\right) > 1 - \varepsilon, \quad \dot{B}\left(s_{jk} - L, t'\right) < \varepsilon \text{ and } Y\left(s_{jk} - L, t'\right) < \varepsilon\right\} \in \dot{F}(\hat{\imath}_2)$
It is sufficient to prove that for each $\varepsilon > 0$ and $t' > 0$,
$\dot{B}(\varepsilon) \subset \left\{(j, k) \in \mathbb{N} \times \mathbb{N} : G\left(\alpha S_{jk} - \alpha L, t'\right) > 1 - \varepsilon, \dot{B}\left(\alpha S_{jk} - \alpha L, t'\right) < \varepsilon \text{ and } Y\left(\alpha s_{jk} - \alpha L, t'\right) < \varepsilon\right\}$.
Let $(j, k) \in \dot{B}(\varepsilon)$. Then we have
$G\left(s_{jk} - L', t'\right) > 1 - \varepsilon, \dot{B}\left(s_{jk} - L, t'\right) < \varepsilon$ and $Y\left(s_{jk} - L', t'\right) < \varepsilon$.
So, we have

$$G\left(\alpha s_{jk} - \alpha L', t'\right) = G\left(s_{jk} - L, \frac{t}{|\alpha|}\right) \geq G\left(s_{jk} - L, t'\right) \tilde{o} G\left(0, \frac{t}{|\alpha|} - t'\right) = G\left(s_{jk} - L, t'\right) \tilde{o} 1$$
$$= G\left(s_{jk} - L, t'\right) > 1 - \varepsilon.$$

Furthermore,

$$\dot{B}\left(\alpha s_{jk} - \alpha L, t'\right) = \dot{B}\left(s_{jk} - L, \frac{t}{|\alpha|}\right) \leq \dot{B}\left(s_{jk} - L_{k,}, t'\right) \cdot \dot{B}\left(0, \frac{t}{|\alpha|} - t'\right) = \dot{B}\left(s_{jk} - L, t'\right) \cdot 0$$
$$= \dot{B}\left(s_{jk} - L, t^t\right) < \varepsilon \text{ and}$$
$$Y\left(\alpha s_{jk} - \alpha L_n, t^t\right) = Y\left(s_{kk} - L, \frac{t}{|\alpha|}\right) \leq Y\left(s_{kk} - L_{k,}, t'\right) \cdot Y\left(0, \frac{t}{|\alpha|} - t'\right) = Y\left(s_{kk} - L_{k,}, t'\right) \cdot 0$$
$$= Y\left(s_{jk} - \mathbf{L}, t'\right) < \varepsilon.$$

Hence, we have
$\dot{B}(\varepsilon) \subset \left\{(j, k) \in N \times N : G\left(\alpha s_{jk} - \alpha L, t'\right) > 1 - \varepsilon, \dot{B}\left(\alpha s_{jk} - \alpha L, t'\right) < \varepsilon \text{ and } Y\left(\alpha s_{jk} - \alpha L, t'\right) < \varepsilon\right\}$ and from (2), we conclude that $N^\eta_{i_2} - \lim \alpha s_{jk} = \alpha L$. This completes the proof of the theorem. □

4. I^*_2-convergence in a NNS

We have introduced the concept of $\hat{\imath}^*_2$ - convergence of double sequences in a NNS in the following section.

Definition 4.1. Let $(\dot{F}, N, \tilde{o}, \cdot)$ be a NNS and η be NN. We say that a double sequence $\dot{x} = \left(s_{jk}\right)$ of elements in $\dot{F}$ is said to be $\hat{\imath}^*_2$-convergent to $L \in F$ with respect to η if $\exists$ a subset $k = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\}$ of $N \times N$ such that $k \in \dot{F}\left(\hat{\imath}_2\right)$ (i.e. $N \times N \setminus k \in \hat{\imath}^*_2$) and $\eta - \lim \dot{x}_m k_m = L$ as, $m \to \infty$. In this case we write $N^\eta_{\hat{\imath}^{**}_2} \lim \dot{x} = L$ and L_{is} is called the $\hat{\imath}^*_2 -$ limit of the sequence $\dot{x} = \left(s_{jk}\right)$ with respect to η.

Theorem 4.2. *Let $(\dot{F}, N, \tilde{0}, \cdot)$ be a NNS and η be NN. Let $\hat{\imath}^*_2$ be an admissible ideal. If $N^\eta_{\hat{\imath}^*_2} - \lim \dot{x} = L$, then $N^\eta_{\hat{\imath}^*_2} - \lim \dot{x} = L'$.*

Proof. Suppose that $N^\eta_{1^2_i} - \lim \dot{x} = L_{\hat{1}}$. Then define $k = \left\{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\right\} \in \dot{F}\left(\hat{\imath}_2\right)$ (i.e. $N \times N \setminus k = H$ (say) $\in \hat{1}^*_2$) such that $\eta - \lim s_{j_{mk} m} = L$ as, $m \to \infty$.

But then for each $\varepsilon > 0$ and $t > 0$, there exists a positive integer n_0 such that:
$G(s_{j_m k_m} - L, t') > \mathbb{1} - \varepsilon$, $\dot{B}(s_{j_m k_m} - L, t') < \varepsilon$, $Y(s_{j_m k_m} - L, t') < \varepsilon \forall m > N$
We know that the set:

$$\{(j_m, k_m) \in \mathbb{k} : G\left(s_{j_m k_m} - \tfrac{L}{\lambda}, t'\right) \le \mathbb{1} - \varepsilon, \dot{B}\left(s_{j_m k_m} - \tfrac{L}{\lambda}, t'\right) \ge \varepsilon \text{ or } Y(s_{j_m k_m} - _{\hat{N}}, t') \ge \varepsilon\}$$

is contained in $\{j_1 < j_2 < \cdots < j_{N-1}; k_1 < k_2 < \cdots < k_{N-\mathbb{1}}\}$ and since, the ideal $\hat{\imath}_2$ is admissible, we have:

$$\left\{(j_m, k_m) \in k : G\left(s_{j_m k_m} - L', t'\right) \le 1 - \varepsilon, \dot{B}\left(s_{j_m k_m} - L, t'\right) \ge \varepsilon \text{ or } Y\left(s_{j_m k_m} - L, t'\right) \ge \varepsilon\right\} \in \hat{\imath}_2^*.$$

Hence $\{(j, k) \in N \times \mathbf{N} : G(s_{i_m k_m} - L_{\hat{n}}, t') \le 1 - \varepsilon, B(s_{f_m k_m} - L_{\hat{n}}, t') \ge \varepsilon$ or $Y\left(s_{j_m k_m} - L, t'\right) \ge \varepsilon\right\} \subseteq H \cup \{j_{\mathbb{1}} < j_2 < \cdots < j_{N-1}; k_1 < k_2 < \cdots < k_{N-\mathbb{1}}\}) \in \hat{\imath}_2$ for all $\varepsilon > 0$ and $t' > 0$. Therefore, we conclude that $N^{\eta}_{\hat{\imath}_2^*} - \lim \dot{x} = L$. □

Remark 4.3. The following example shows that the converse of Theorem 4.2. may not be true in general.

Example 4.4. Let $(\mathbb{R}, \|\cdot\|)$ be denote the space of all real numbers with the usual norm and let $\dot{a} o \dot{b} = ab$ and $\dot{a} \cdot \dot{b} = \min\{\dot{a}, \dot{b}\} + 1 \forall \dot{a}, \dot{b} \in [0, \mathbb{1}]$, $G(\dot{x}, t') = \frac{t'}{t' + |\dot{x}|}$, $\dot{B}(\dot{x}, t') = \frac{|\dot{x}|}{t' + |\dot{x}|}$ and $Y(\dot{x}, t') = \frac{|\dot{x}|}{t' + |\dot{x}|}$.

Then $(\overline{R}, N, \tilde{o}, \cdot)$) is a NNS and η is NN.

Let $N \times \mathbb{N} = U_{i,j} \Delta_{ij}$ be a decomposition of $N \times \mathbb{N}$ such that, for any $(m, \check{n}) \in \mathbb{N} \times \mathbb{N}$, each Δ_{ij} contains infinitely many (i,j)'s, where $\dot{i} \ge m, j \ge \check{n}$ and $\Delta_{ij} \cap \Delta_{mn} = \emptyset$ for $(i, j) \ne (m, \check{n})$. Now, we define a sequence $s_{m\check{n}} = \frac{1}{ij}$ if $(m, \check{n}) \in \Delta_{ij}$. Then $G(s_{mn}, t') = \frac{t'}{t + |s_{mn}|} \to 1$, as, $m, n \to \infty$. $B(s_{mn}, t') = \frac{|s_{mn}|}{t' + |s_{mn}|} \to 0$and $Y(s_{mn}, t') = \frac{|s_{mn}|}{t' + |s_{mn}|} \to 0$, as, $m, n \to \infty$. Hence $N^{\eta}_{\hat{\imath}_2^*} - \lim \dot{x} = \mathbb{0}$, as, $m, \check{n} \to \infty$. Now suppose that $N^{\eta}_{\mathbf{N}_2^+} - \lim s_{m\check{n}} = 0$, as, $m, \check{n} \to \infty$.

Then, $\exists$ a subset $k = \{m_{\mathbb{1}} < m_2 < \cdots; \check{n}_{\mathbb{1}} < \check{n}_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that $k \in \dot{F}\left(\hat{\imath}_2\right)$ and $\eta - \lim s_{m_j n_f} = 0$ as, $\dot{j} \to \infty$. Since, $k \in \dot{F}\left(\hat{\imath}_2\right)$, there is a set $H \in \dot{F}\left(\hat{\imath}_2\right)$ such that $k = N \times N \backslash H$. Now, from the definition of $\hat{\imath}_2, \exists$, say, $p \in N$ such that

$$H \subset \left(\bigcup_{m=1}^{p}\left(\bigcup_{n=1}^{\infty} \Delta_{mn}\right)\right) \cup \left(\bigcup_{m=1}^{p}\left(\bigcup_{n=1}^{\infty} \Delta_{mn}\right)\right)$$

Then we have $\Delta_{(p+\mathbb{1}).(q+\mathbb{1})} \subset k$ and therefore $s_{m_j n_f} = \frac{1}{(p+1)^2} > 0$,

for infinitely many $(m_f, \check{n}_j)$'s from k. This results to $\eta - \lim s_{m_j n_j} = 0$, as, $j \to \infty$. Therefore, the assumption $N^{\eta}_{\hat{1}_2^*} - \lim s_{mn} = 0$, as, $m, \check{n} \to \infty$ is incorrect. Hence the converse of the theorem may not be true. This completes the proof of the theorem.

Remark 4.5. From the above example it is clear that $\hat{\imath}_2^*$ - convergence implies $\hat{\imath}_2$ - convergence but not necessarily converse. Now the question arises under what condition the converse may hold. For this we define the condition (AP) and see that under this condition the converse holds.

Definition 4.6. An admissible ideal $\hat{\imath}_2 \subset P(\mathbb{N} \times \mathbb{N})$ is said to satisfy the condition (AP) if for every sequence $(A_{\check{n}})_{\check{n} \in N}$ of pairwise disjoint sets from $\hat{\imath}_2$, there are sets $\dot{B}_{\check{n}} \subset N$ such that the symmetric difference $A_{\check{n}} \Delta \dot{B}_{\check{n}}$ is a finite set for every ň and $U_{\check{n} \in \mathbb{N}} \dot{B}_{\check{n}} \in \hat{\imath}_2$.

Theorem 4.7. *Let ($\dot{\mathrm{F}}, N, \tilde{o}, \cdot$) be a NNS and η be NN. The ideal $\hat{\mathrm{i}}_2$ satisfies the condition (AP). If $\dot{\mathrm{x}} = \left(s_{\mathrm{jk}}\right)$ is a sequence in F, $\mathbb{N}^{\eta}_{\hat{\imath}_2} - \lim \dot{\mathrm{x}} = \mathrm{L}$, then $\mathbb{N}^{\eta}_{\hat{\mathrm{I}}^*_2} - \lim \dot{\mathrm{x}} = \mathrm{L}$.*

Proof. Suppose $\hat{\imath}_2$ satisfies the condition (AP) and $N^{\eta}_{\hat{\imath}_2} - \lim \dot{\mathrm{x}} = \mathrm{L}$. Then for each $\varepsilon > \mathbb{0}$ and $\mathrm{t}' > 0$,

$\left\{(\mathrm{j}, \mathrm{k}) \in \mathrm{N} \times \mathrm{N} : G\left(s_{\mathrm{jk}} - \mathrm{L}_{\hat{n}}, \mathrm{t}'\right) \leq \mathbb{1} - \varepsilon, B\left(s_{\mathrm{jk}} - \mathrm{L}_{\hat{n}}, \mathrm{t}'\right) \geq \varepsilon \text{ or } Y\left(s_{\mathrm{jk}} - \underline{\lambda}^{\mathrm{L}}, \mathrm{t}'\right) \geq \varepsilon\right\} \in \hat{\mathrm{i}}_2$......(1).

We define the set $'A_p$ for $p \in \mathbb{N}$, and $\mathrm{t}' > 0$ as,

$\mathrm{A}_p = \left\{(\mathrm{f}, \mathrm{k}) \in \mathbb{N} \times \mathbb{N} : \mathbb{1} - \frac{1}{p} \leq G\left(s_{\mathrm{jt}} - \mathrm{L}, \mathrm{t}'\right) < 1 - \frac{\mathbb{1}}{p+\mathbb{1}}, \mathbb{1} - \frac{1}{p+\mathbb{1}} \leq \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \mathrm{L}, \mathrm{t}'\right) \leq \frac{\mathbb{1}}{p} \text{ or } \mathbb{1} - \frac{1}{p+\mathbb{1}} \leq Y\left(s_{\mathrm{jk}} - \mathrm{L}, \mathrm{t}'\right) \leq \frac{\mathbb{1}}{p}\right\}$

It is obvious that $\{\mathrm{A}_1, \mathrm{A}_2, \ldots\ldots.\}$ is countable and belongs to $\hat{\mathrm{i}}_2$ and $\mathrm{A}_{\dot{\imath}} \cap \mathrm{A}_{\mathrm{j}} = \varnothing$ for $\dot{\mathrm{i}} \neq \mathrm{j}$. By the condition (AP), there is a countable family of sets $\{\mathrm{A}_1, \mathrm{A}_2, \ldots\ldots\} \in \hat{\mathrm{i}}_2$ such that $'\mathrm{A}_{\mathrm{i}} \Delta \dot{\mathrm{B}}_{\mathrm{i}}$ is a finite set for each $\hat{\mathrm{i}} \in \mathrm{N}$ and $\dot{\mathrm{B}} = \bigcup_{i=j}^{\infty} \dot{B}_i \in \hat{\imath}_2$. From the definition of the associate filter $\dot{F}\left(\hat{\mathrm{I}}_2\right)$, there is a set $k \in \dot{F}\left(\hat{\imath}_2\right)$ such that $k = N \times \mathrm{N} \setminus \dot{\mathrm{B}}$.. In order to prove the theorem, we have to show that the sequence $\left(s_{jk}\right)_{(\mathrm{j},\mathrm{k}) \in \mathrm{k}}$ is convergent to L with respect to η. Let $\delta > 0$ and $\mathrm{t}' > 0$. Choose $q \in \mathrm{N}$ such that $\frac{1}{q} < \delta$. Then $\left\{(\mathrm{j}, \mathrm{k}) \in \mathrm{N} \times \mathrm{N} : G\left(s_{\mathrm{jk}} - \mathrm{L}, \mathrm{t}'\right) \leq 1 - \eta, \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \mathrm{L}_{\hat{n}}, \mathrm{t}'\right) \geq \eta \text{ or } Y\left(s_{\mathrm{jk}} - \mathrm{L}_n, \mathrm{t}'\right) \geq \eta\right\} \subset \{(\mathrm{j}, \mathrm{k}) \in \mathbf{N} \times \mathrm{N} : G\left(s_{\mathrm{jk}} - \mathrm{L}_{\hat{n}}, \mathrm{t}'\right) \leq 1 - \frac{1}{q}, B\left(s_{\mathrm{jk}} - \mathrm{L}_{\hat{n}}, \mathrm{t}'\right) \geq \frac{1}{q} \text{ or } Y\left(s_{\mathrm{jk}} - \mathrm{L}_{\hat{\lambda}}, \mathrm{t}'\right) \geq \frac{1}{q}\right\} \subset \mathrm{U}_{\mathrm{i}=1}^{q+1} A_{\mathrm{i}}$.

Since, $'^{i}_{i} \Delta \dot{\mathrm{B}}_i, \dot{\mathrm{i}} = 1, 2, \ldots\ldots, q+1$ are finite, $\exists\, (\mathrm{f}_0, \mathrm{k}_0) \in \mathbb{N} \times N$ such that

$\left(\cup_{i=1}^{q+1} \dot{B}_{\mathrm{i}}\right) \cap \left\{(\mathrm{j}, \mathrm{k}) : \dot{j} \geq \dot{f}_{\oplus} \text{ and } \mathrm{k} \geq \mathrm{k}_{\oplus}\right\} = \left(\cup_{i=1}^{q+\mathbb{1}} \mathrm{A}_i\right) \cap \left\{(\mathrm{j}, \mathrm{k}) : \dot{\mathrm{j}} \geq \dot{f}_{\oplus} \text{ and } \mathrm{k} \geq \mathrm{k}_{\mathbb{0}}\right\}$.

If $\mathrm{j} \geq \mathrm{j}_0, \mathrm{k} \geq \mathrm{k}_0$ and $\{(\mathrm{j}, \mathrm{k}) \in \mathbb{k}$, then $(\mathrm{j}, \mathrm{k}) \notin \cup_{i=1}^{q+\mathbb{1}} \dot{B}_{\mathrm{i}}$. Therefore, from (1), we have $(\mathrm{j}, \mathrm{k}) \notin \bigcup_{i=1}^{q+\mathbb{1}} A_{\mathrm{i}}$. Hence, $\forall \mathrm{j} \geq \dot{f}_0, \mathrm{k} \geq \mathrm{k}_{\mathbb{0}}$ and $(\mathrm{j}, \mathrm{k}) \in \mathrm{k}$, we have

$$G\left(s_{\mathrm{jk}} - \mathrm{L}, \mathrm{t}'\right) > 1 - \delta, B\left(s_{\mathrm{jk}} - \mathrm{L}', \mathrm{t}'\right) < \delta \text{ and } Y\left(s_{\mathrm{jk}} - \mathrm{L}_{\imath}, \mathrm{t}'\right) < \delta$$

Since, δ is arbitrary, we have $\mathbb{N}^{\eta}_{\hat{r}^*_2} - \lim \dot{\mathrm{x}} = \mathrm{L}$. This completes the proof of the theorem. □

Theorem 4.8. *Let ($\dot{\mathrm{F}}, N, \tilde{o}', \cdot$) be a NNS and η be NN. Then the following conditions are equivalent:*

(1) $N^{\eta}_{\hat{\imath}^*_2} - \lim \dot{x} = \mathrm{L}$.

(2) *There exist two sequences $\mathrm{y} = \left(\mathrm{u}_{\mathrm{jk}}\right)$ and $\mathrm{z} = \left(\hat{\mathrm{u}}_{\mathrm{jk}}\right)$ in F such that $\dot{\mathrm{x}} = \mathrm{y} + \check{\mathrm{z}}, \eta - \lim y = \mathrm{L}$ and the set $\{(\mathrm{j}, \mathrm{k}) : \dot{v}_{\mathrm{jt}} \neq \theta\} \in \hat{\mathrm{i}}_2$, where θ denotes the zero element of F.*

Proof. Let us consider that the condition (i) holds. Then $\exists$ a set $\mathrm{k} = \{(\mathrm{j}_m, \mathrm{k}_m) : \mathrm{j}_{\mathbb{1}} < \mathrm{j}_2 < \cdots ; \mathrm{t}_{\mathbb{1}} < \mathrm{t}_2 < \cdots\}$ of $\mathrm{N} \times \mathrm{N}$ such that

$\mathrm{k} \in \dot{\mathrm{F}}\left(\hat{\imath}^*_2\right)$ and $\eta - \lim \dot{x}_{\mathrm{j}_m \mathrm{k}_m} = \mathrm{L}$, as, $m \to \infty$.

We define the sequence $y = \left(\hat{u}_{\mathrm{jk}}\right)$ and $\check{z} = \left(\grave{\imath}_{\mathrm{jk}}\right)$ as, follows:

$$u_{jk} = \begin{cases} s_{jk}, \text{ if } (\mathrm{j}, k) \in k \\ \mathrm{L}; \text{ if } (\mathrm{j}, \mathrm{k}) \in \mathrm{k}^c \end{cases}$$

and $\grave{u}_{\mathrm{jk}} = s_{\mathrm{jk}} - \hat{u}_{\mathrm{jt}}$ for all $\{(\mathrm{j}, \mathrm{k}) \in \mathbb{N} \times N$. For given $\varepsilon > \mathbb{0}, \mathrm{t}' > \mathbb{0}$ and $(\mathrm{j}, \mathrm{k}) \in \mathrm{k}^c$, we have $\left\{(\mathrm{j}, \mathrm{k}) : \grave{u}_{\mathrm{jk}} \neq \theta\right\} \in \hat{\mathrm{I}}^*_2$.

Let us consider that the condition (ii) holds and $\mathrm{k} = \left\{(\mathrm{j}, \mathrm{k}) : \grave{\mathrm{j}}_{\mathrm{jk}} = \theta\right\}$. Clearly, $\mathrm{k} \in \dot{\mathrm{F}}\left(\hat{\imath}_2\right)$ is an infinite set. Let $\mathrm{k} = \left\{(\mathrm{j}_m, \mathrm{t}_m) : \dot{j}_1 < \dot{\mathrm{j}}_2 < \cdots ; \mathrm{t}_{\mathbb{1}} < \mathrm{t}_2 < \cdots\right\}$. Since, $\dot{\mathrm{x}}_{\mathrm{j}_m \mathrm{k}_m} = \mathrm{y}_{\mathrm{j}_m \mathrm{k}_m}$ and $\eta - \lim y_{\mathrm{j}_m \mathrm{k}_m} = \mathrm{L}, \eta - \lim \dot{x}_{\mathrm{j}_m \mathrm{k}_m} = \mathrm{L}$ as, $m \to \infty$. Hence $N^{\eta}_{1^*_i} - \lim \dot{x} = \mathrm{L}$. This completes the proof of the theorem. □

5. $\hat{\imath}_2$ and $\hat{\imath}_2^*$− Cauchy double sequences in NNS

We have defined $\hat{\imath}_2$ and $\hat{\imath}_2^*$-Cauchy double sequences on NNS in this section and have proved that $\hat{\mathrm{I}}_2$ convergence and $\hat{\imath}_2$-Cauchy are analogous on NNS.

Definition 5.1. Let $(F, N, \tilde{o}, \cdot)$ be a NNS and η be NN. Then a double sequence $\dot{\mathrm{x}} = \left(s_{\mathrm{jk}}\right)$ is said to be $\hat{\hat{\mathrm{I}}}_2-$ Cauchy with respect to. η if, for every $\varepsilon > 0$ and $\mathrm{t}' > 0, \exists N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that, for all $\dot{\mathrm{j}}, p \geq N, \mathrm{k}, q \geq M, \left\{(\mathrm{j}, \mathrm{k}) \in \mathbf{N} \times \mathrm{N} : G\left(s_{\mathrm{jk}} - \mathrm{L}, \mathrm{t}'\right) \leq 1 - \varepsilon \text{ or } B\left(s_{\mathrm{jk}} - \mathrm{L}^{\mathrm{L}}, \mathrm{t}'\right) \geq \varepsilon \text{ or } Y\left(s_{\mathrm{jk}} - \mathrm{L}_{\hat{n}}, \mathrm{t}'\right) \geq \varepsilon\right\}\right\} \in \hat{\mathrm{I}}_2$.

Definition 5.2. Let $(F, N, \tau, \cdot)$ be a NNS and η be NN. Then a double sequence $x = (s_{jk})$ is said to be I_2^*-Cauchy with respect to η if there exists a subset $K = (j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots$ of $N \times N$ such that $K \in F(I_2)$ and the subsequence $(s_{j_m k_m})$ is an ordinary Cauchy sequence with respect to η.

The following theorems are analogues to our Theorems 4.2 and Theorem 4.7, respectively and can be proved applying similar lines.

Theorem 5.3. *Let ($\dot{\mathrm{F}}, N, \tilde{\mathrm{o}}, \cdot$) be a NNS and η be NN. If a double sequence $\dot{\mathrm{x}} = \left(s_{jk}\right)$ is $\hat{\imath}_2^*$-Cauchy with respect to η, then it is $\hat{\imath}_2$-Cauchy with respect to η.*

Theorem 5.4. *Let ($\dot{F}, N, \tilde{o}, \cdot$)) be a NNS and η be NN. Let the ideal $\hat{\imath}_2$ satisfy the condition (AP). If a double sequence $\dot{\mathrm{x}} = \left(s_{jk}\right)$ is $\hat{\mathrm{i}}_2$-Cauchy with respect to η, then it is also $\hat{\mathrm{i}}_2^*$-Cauchy with respect to η.*

Now, we prove the following characterization.

Theorem 5.5. *Let ($\dot{\mathrm{F}}, N, \tilde{\mathrm{o}}, \cdot$) be a NNS and η be NN. Then a double sequence $\dot{\mathrm{x}} = \left(s_{\mathrm{jk}}\right)$ is $\hat{\mathrm{i}}_2$-convergent with respect to. η if and only if it is $\hat{\mathrm{I}}_2$-Cauchy with respect to η.*

Proof. Let $\dot{\mathrm{x}} = (s_{jk})$ be $\hat{\hat{\imath}}_2$-convergent to L with respect to η, i.e., $\mathrm{N}_{\hat{\mathrm{i}}_2}^{\eta}$-lim $\dot{\mathrm{x}} = L.$. Choose $\check{r} > 0$ such that $(\mathbb{1} - \check{r})\tilde{o}(\mathbb{1} - \check{r}) > 1 - \varepsilon$ and $\check{r} \cdot \check{r} < \varepsilon$. Then, for all $t' > 0.$, we have $'\mathrm{A} = \left\{(\dot{j}, \mathrm{k}) \in \mathrm{N} \times \mathrm{N} : G\left(s_{\mathrm{jk}} - \mathrm{L}', \mathrm{t}'\right) \leq 1 - \check{\mathrm{r}}\right.$ or $\left.\dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \mathrm{L}_{\lambda}, \mathrm{t}'\right) \geq \check{\mathrm{r}} \text{ or } Y\left(s_{\mathrm{jk}} - \mathrm{L}_1, \mathrm{t}'\right) \geq \check{\mathrm{r}}\right\} \in \hat{\mathrm{i}}_2$.

This denotes that

$\emptyset \neq \mathrm{A}^c = \left\{(\dot{j}, \mathrm{k}) \in \mathrm{N} \times \mathbb{N} : G\left(s_{\mathrm{jk}} - \mathrm{L}_n, \mathrm{t}'\right) > 1 - \check{\mathrm{r}}, \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \mathrm{L}_{\lambda}, \mathrm{t}'\right) < \check{\mathrm{r}} \text{ or } Y\left(s_{\mathrm{jk}} - \mathrm{L}_{\lambda}, \mathrm{t}'\right) < \check{\mathrm{r}}\right\} \in \dot{\mathrm{F}}\left(\hat{\imath}\right)$.Let (p, q) $\in \mathrm{A}^c$. Then we have $G\left(s_{pq} - \mathrm{L}, \mathrm{t}'\right) > 1 - \check{\mathrm{r}}$ or $\dot{\mathrm{B}}\left(s_{pq} - \mathrm{L}, \mathrm{t}'\right) < \check{\mathrm{r}}$ or $Y\left(s_{pq} - \mathrm{L}, \mathrm{t}'\right) <$ r.

Now let

$\dot{\mathrm{B}} = \left\{(\mathrm{j}, \mathrm{k}) \in \mathrm{N} \times \mathrm{N} : G\left(s_{\mathrm{jk}} - s_{pq}, \mathrm{t}'\right) \leq \mathbb{1} - \varepsilon \text{ or } B\left(s_{\mathrm{jk}} - s_{pq}, \mathrm{t}'\right) \geq \varepsilon \text{ or } Y\left(s_{\mathrm{jk}} - s_{pq}, \mathrm{t}'\right) \geq \varepsilon\right\} \in \hat{\mathrm{i}}_2$.

We have to show that $\dot{B} \subset \mathrm{A}$. Let $(\dot{j}, \mathrm{k}) \in \dot{\mathrm{B}}$. Then we have

$G\left(s_{\mathrm{jk}} - s_{pq}, \frac{\mathrm{t}'}{2}\right) \leq \mathbb{1} - \varepsilon, \dot{\mathrm{B}}\left(s_{\mathrm{k}} - s_{pq}, \frac{\mathrm{t}^{\mathrm{t}}}{2}\right) \geq \varepsilon$ or $Y\left(s_{\mathrm{jk}} - s_{pq}, \frac{\mathrm{t}'}{2}\right) \geq \varepsilon$.

We have two possible cases. We first consider that $G\left(s_{\mathrm{kk}} - s_{pq}, \mathrm{t}'\right) \leq 1 - \varepsilon$. Then we have $G\left(s_{\mathrm{kk}} - {}^{\mathrm{L}}, \frac{\mathrm{t}'}{2}\right) \leq$ 1- ř. Therefore $(\mathrm{j}, \mathrm{k}) \in {}'\mathrm{A}$. Otherwise, if $G\left(s_{\mathrm{jk}} - {}^{\mathrm{L}}, \frac{\mathrm{r}}{2}\right) >$ 1 - ř. Then

$\mathbb{1} - \varepsilon \geq G\left(s_{\mathrm{jk}} - s_{pq}, \mathrm{t}'\right) \geq G\left(s_{\mathrm{jk}} - \mathrm{L}', \frac{\mathrm{t}'}{2}\right) \tilde{o} G\left(s_{pq} - \mathrm{L}^{\mathrm{L}}, \frac{\mathrm{t}'}{2}\right) > (1 - \check{\mathrm{r}})\tilde{\mathrm{o}}(\mathbb{1} - \check{\mathrm{r}}) > 1 - \varepsilon,$

which is not possible. Hence $\dot{B} \subset {}'\mathrm{A}$.

Likewise, consider that $\dot{\mathrm{B}}\left(s_{\mathrm{jk}} - s_{pq}, \mathrm{t}'\right) \geq \varepsilon$. Then we have $\dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \mathrm{L}_n, \frac{\mathrm{t}'}{2}\right) \geq r$. Therefore $(\mathrm{j}, \mathrm{k}) \in {}'\mathrm{A}$. Otherwise, if $\dot{B}\left(s_{\mathrm{jk}} - \underline{L}, \frac{\frac{r}{r}}{2}\right) <$ r. Then

$$\varepsilon \leq \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - s_{pq}, \mathrm{t}'\right) \leq \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \mathrm{L}, \mathrm{t}'/2\right) \cdot \dot{\mathrm{B}}\left(s_{pq} - \mathrm{L}, \mathrm{t}'/2\right) < \check{\mathrm{r}} \cdot \check{\mathrm{r}} < \check{\mathrm{r}},$$

which is not possible. Hence $\dot{\mathrm{B}} \subset {}'\mathrm{A}$.

Similarly, considering $Y\left(s_{\mathrm{jk}} - s_{pq}, \mathrm{t}'\right) \geq \varepsilon$, we can show that $\dot{\mathrm{B}} \subset {}'\mathrm{A}$.

Sufficiency. Let $\dot{\mathrm{x}} = \left(s_{\mathrm{jk}}\right)$ be $\hat{\imath}_2$-Cauchy with respect to t but not $\hat{\mathrm{i}}_2$-convergence with respect to. η. Then $\exists$ M and N such that $A(\varepsilon, \mathrm{t}') = \left\{(\mathrm{j}, \mathrm{k}) \in \mathrm{N} \times \mathrm{N} : G\left(s_{\mathrm{jk}} - s_{MN}, \mathrm{t}'\right) \leq \mathbb{1} - \varepsilon \text{ or } \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - s_{MN}, \mathrm{t}'\right) \geq \varepsilon \text{ or } Y\left(s_{\mathrm{jk}} - s_{MN}, \mathrm{t}'\right) \geq \varepsilon\right\} \in \hat{\mathrm{i}}_2$ and

$$\begin{aligned} \dot{\mathrm{B}}(\varepsilon, \mathrm{t}') = \{(\mathrm{j}, \mathrm{k}) \in N \times N : \\ \{G\left(s_{\mathrm{jk}} - \mathrm{L}_n, \frac{\mathrm{t}'}{2}\right) > 1 - \varepsilon, \\ \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \mathrm{L}_n, \frac{\mathrm{t}'}{2}\right) < \varepsilon \text{ and} \\ Y\left(s_{\mathrm{jk}} - \frac{\mathrm{L}_n}{\mathrm{t}'} \frac{\mathrm{t}'}{2}\right) < \varepsilon\} \\ \in \hat{\mathrm{i}}_2. \end{aligned}$$

Equivalently, $\dot{\mathrm{B}}^c(\varepsilon, \mathrm{t}')\} \in \dot{\mathrm{F}}\left(\hat{\imath}_2\right)$. Since,

$$\begin{aligned} G\left(s_{\mathrm{jk}} - s_{MN}, \mathrm{t}'\right) &\geq 2G\left(s_{\mathrm{jk}} - \mathrm{L}_{\hat{\lambda}}, \frac{\mathrm{t}'}{2}\right) > 1 - \varepsilon, \\ \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - s_{MN}, \mathrm{t}'\right) &\leq 2\dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \mathrm{L}', \mathrm{t}'/2\right) < \varepsilon \\ Y\left(s_{\mathrm{jk}} - s_{MN}, \mathrm{t}'\right) &\leq 2Y\left(s_{\mathrm{jk}} - \mathrm{L}', \frac{\mathrm{t}'}{2}\right) < \varepsilon \end{aligned}$$

if $G\left(s_{\mathrm{jk}} - \mathrm{L}', \frac{\mathrm{t}^{\mathrm{t}}}{2}\right) > (1 - \varepsilon)/2, \dot{\mathrm{B}}\left(s_{\mathrm{jk}} - \frac{\mathrm{L}'}{}, \frac{\mathrm{t}'}{2}\right) < \varepsilon/2$ and $Y\left(s_{\mathrm{jk}} - \frac{\mathrm{L}'}{}, \frac{\mathrm{t}^{\mathrm{r}}}{2}\right) < \varepsilon/2$, respectively, we have $\mathrm{A}^c(\varepsilon, \mathrm{t}') \in \hat{\mathrm{i}}_2$ and so $\mathrm{A}(\varepsilon, \mathrm{t}') \in \dot{\mathrm{F}}\left(\hat{\imath}_2\right)$

which is a contradiction to $\dot{x}$ is $\hat{\imath}_2 - Cauchy$ with respect to. η. Hence $\dot{x}$ must be $\hat{\imath}_2$-convergent with respect to. NN η. This completes the proof of the theorem. □

Likewise, we can prove the following theorem.

Theorem 5.6. *Let $(\dot{\mathrm{F}}, N, \tilde{\mathrm{o}}, \cdot)$) be a NNS and η be NN. Then a double sequence $\dot{\mathrm{x}} = \left(s_{\mathrm{jk}}\right)$ is $\hat{\mathrm{i}}_2^*$-convergent with respect to. η if and only if it is $\hat{\mathrm{i}}_2^*$-Cauchy with respect to. η.*

6. Conclusion:

In this article, we have investigated a more general type of convergence for double sequences, that is, $\hat{\imath}_2$ - convergence as well as $\hat{\imath}_2$ - Cauchy in a more general setting, i.e. in a NNS. We have also studied $\hat{\imath}_2^*$ convergence as well as $\hat{\imath}_2^*$-Cauchy in a NNS. We have established that $\hat{\imath}_2$-convergence and $\hat{\imath}_2$-Cauchy are equivalent in a NNS. These definitions and results have provided new tools to deal with the convergence problems of double sequences occurring in many branches of science and engineering.

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