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Some topological properties of e-space and description of τ-closed sets

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Abstract. In this paper we study preserving properties of the e-space under the normal functors Π^n, SP^n and $\exp_n$. We prove that when a topological space X is an e-space, the spaces X^n, SP^nX and $\exp_nX$ are also e-spaces. We also study the behavior of e-continuity of mappings, proving that the functors Π^n, SP^n and $\exp_n$ preserve the e-continuity. In addition, we introduce the notions of τ-boundary points, τ-cluster points and τ-boundary of a set and study many of their properties.

1. Introduction and preliminary notes

Throughout the paper all spaces are assumed to be T_1-spaces. The paper [1] introduced and investigated e-spaces and e-continuous mappings. In what follows, we enrich such related studies of e-spaces, investigating preserving properties of the e-space under some normal functors. We consider the following functors: Π^n, SP^n and $\exp_n$.

For a given finite number n, the operation of raising spaces to the n-th power extended to a covariant functor in the category of topological spaces and their continuous mappings. This n-th power denoted by Π^n, i.e. $\Pi^n(X) = X^n$ for a topological space X. For a mapping $f\colon X \to Y$ we obtained the mapping

$$\Pi^n f\colon X^n \to Y^n$$

defined by the formula

$$\Pi^n f(x_1, x_2, \dots, x_n) = (f(x_1), f(x_2), \dots, f(x_n)).$$

The functor Π^n is a normal functor in the category of compact spaces and their continuous mappings [9].

It is known that a permutation group is the group of all permutations, that is one-to-one mappings $X \to X$. A permutation group of a set X is usually denoted by $S(X)$. Especially, if $X = \{1, 2, \dots, n\}$, then $S(X)$ is denoted by S_n.

Let X^n be the n-th power of a compact space X. The permutation group S_n of all permutations acts on the n-th power X^n as permutation of coordinates. The set of all orbits of this action with the quotient topology is denoted by SP^nX. The orbit of $(x_1, x_2, \dots, x_n) \in X^n$ is denoted by $[(x_1, x_2, \dots, x_n)]$. Thus, points

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of the space SP^nX are finite subsets (equivalence classes) of the product X^n. The space SP^nX is called the *n-permutation degree* of the space X. The n-th permutation degree is a quotient of X^n. Therefore, the quotient mapping is denoted by $\pi_n^s\colon X^n \to SP^nX$ and defined by the rule

$$\pi_n^s(x_1, x_2, \dots, x_n) = [(x_1, x_2, \dots, x_n)],$$

where $(x_1, x_2, \dots, x_n) \in X^n$ [2, 3, 5, 6].

Let G be a subgroup of the permutation group S_n. The group G acts on the n-th power of the space X as permutation of coordinates. The set of all orbits of this action with the quotient topology is denoted by SP^n_GX. The space SP^n_GX is called *G-permutation degree* of the space X. Similar to the above discussion, the orbit of $(x_1, x_2, \dots, x_n) \in X^n$ is denoted by $[(x_1, x_2, \dots, x_n)]_G$ and the corresponding quotient mapping is denoted by $\pi^s_{n,G}\colon X^n \to SP^n_GX$ and defined by the rule

$$\pi^s_{n,G}(x_1, x_2, \dots, x_n) = [(x_1, x_2, \dots, x_n)]_G,$$

where $(x_1, x_2, \dots, x_n) \in X^n$. Consider any continuous mapping $f\colon X \to Y$. For an equivalence class $[(x_1, x_2, \dots, x_n)] \in SP^n_GX$ we put

$$SP^n_Gf[(x_1, x_2, \dots, x_n)] = [(f(x_1), f(x_2), \dots, f(x_n))].$$

Thereby, a mapping $SP^n_Gf\colon SP^n_GX \to SP^n_GY$ is defined. It is easy to check that the operation SP^n_G so constructed is a normal functor in the category of compact spaces and their continuous mappings. This functor is called the *functor of G-permutation degree* [9].

Denote by $\exp X$ the set of all nonempty closed subsets of a T_1-space X. The family $\mathcal{B}$ of all sets in the form of

$$O\langle U_1, U_2, \dots, U_n\rangle = \{F :\ F \in \exp X,\ F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \emptyset,\ i = 1, 2, \dots, n\},$$

where $U_1, U_2, \dots, U_n$ is a sequence of open subsets of X, generates the topology on $\exp X$. This topology is called the *Vietoris topology*. The set $\exp X$ with the Vietoris topology is called *the exponential space* or *the hyperspace of the space X* [10]. Denote by $\exp_n X$ the set of all non-empty closed subsets of X of cardinality not greater than the natural number n, i.e.

$$\exp_n X = \{F :\ F \in \exp X,\ |F| \leq n\}.$$

Put

$$\exp_\omega X = \cup\{\exp_n X :\ n = 1, 2, \dots\}$$

and

$$\exp_c X = \{F \in \exp X :\ F \text{ is compact in } X\}.$$

It is clear that $\exp_n X \subset \exp_\omega X \subset \exp_c X \subset \exp X$ for any topological T_1-space X.

In this paper, we study the properties of e-spaces and e-continuity mappings under the normal functors Π^n, SP^n and $\exp_n$. Also, based on the notion of τ-closure, we insert the new meanings of τ-boundary points, τ-cluster points and τ-boundary of a set and study many of their properties. Especially, in Section 2, we study the preserving properties of e-spaces under the normal functors Π^n, SP^n and $\exp_n$, that is we prove that if X is an e-space, then the spaces X^n, SP^nX and $\exp_nX$ are also e-spaces. In Section 3, we study the meaning of e-continuity under the view of the above functors, proving that they preserve such kind of continuity and finally, in Section 4, we introduce and study the notions of τ-boundary points, τ-cluster points and τ-boundary of a set.

2. Main results on *e*-spaces

Let X be a topological space and let A be a subset of X. We denote the closure of A in X by cl_XA or $[A]$. If there is no confusion with the considered space, we simply write $\mathrm{cl}A$. A set G in a topological space X is called *extremely open* (briefly *e-open*) if G and its closure $\mathrm{cl}_X G$ are open subsets of X. A subset of a topological space is called *e-closed* if its complement is an e-open [1]. Clearly every clopen set in a topological space is an e-open set, but not conversely. For example $\mathbb{R}\setminus\{a\}$ is an e-open subset of $\mathbb{R}$ (for each $a \in \mathbb{R}$) which is not a clopen set.

The following lemma showed that the intersection of every two e-open sets is an e-open set.

Lemma 2.1. [1] *Suppose that V is an e-open and U is an open subset of a topological space X. Then*

$$cl_X(V \cap U) = cl_X V \cap cl_X U.$$

Since the intersection of two e-open sets in a topological space (X, τ) is an e-open set by Lemma 2.1, the set of all e-open subsets of X forms a base for a topology τ_e on X. This means that τ_e is weaker topology with respect to the original topology τ. Whenever τ_e coincides with τ (i.e., $\tau = \tau_e$), we call the space X an *e-space*.

It is known that every homeomorphic image of an e-space is an e-space (see [1, Proposition 2.11]). We generalized this proposition as follows.

Proposition 2.2. *Let X and Y be topological spaces and let $f\colon X \to Y$ be a clopen and onto mapping. If X is an e-space, then Y is so.*

Proof. Take an arbitrary open subset U of Y. Since f is continuous, $f^{-1}(U)$ is an open subset of X. Since X is an e-space, $f^{-1}(U)$ can be represented as a union of e-open subsets G_a, $a \in M$, of X, i.e.

$$f^{-1}(U) = \bigcup_{\alpha\in M} G_\alpha.$$

(Each G_a and the closures $\mathrm{cl}_X G_a$, $a \in M$, are open in X.) Then

$$U = \bigcup_{\alpha\in M} f(G_\alpha).$$

By condition of the proposition f is an open mapping and so all $f(G_\alpha)$ and $f(\mathrm{cl}_X G_\alpha)$, $a \in M$, are open subsets of Y. It suffices to prove that $\mathrm{cl}_Y f(G_\alpha)$, $a \in M$, are open subsets of Y. Let $a \in M$. Since f is closed mapping we have

$$\mathrm{cl}_Y f(G_\alpha) = f(\mathrm{cl}_X G_\alpha).$$

This means that Y is an e-space. Proposition 2.2 is proved. □

Example 2.3. *In Proposition 2.2 the condition of clopenness of mapping is essential. We consider the mapping $f\colon \mathbb{R}_d \to \mathbb{R}_{st}$ defined by the formula $f(x) = x$, where $\mathbb{R}_d$ is the set of all real numbers with a discrete topology and $\mathbb{R}_{st}$ is the real line. Clearly, f is a continuous mapping and $\mathbb{R}_d$ is an e-space. But $\mathbb{R}_{st}$ is not e-space.*

Lemma 2.4. *Let X be a topological space and let $G_1, G_2, \ldots, G_n$ be e-open subsets in X. Then the Cartesian product $G_1 \times G_2 \times \ldots \times G_n$ is an e-open in X^n.*

Proof. Put $G = G_1 \times G_2 \times \ldots \times G_n$. We show that G and $[G]$ are open subsets in X^n. Since $G_1, G_2, \ldots, G_n$ (respectively, $[G_1], [G_2], \ldots, [G_n]$) are open, the set G (respectively $[G_1] \times [G_2] \times \ldots \times [G_n]$) is open in X^n too. It is known that the following equality holds:

$$[G] = [G_1] \times [G_2] \times \ldots \times [G_n].$$

It is easy to verify that $[G]$ is an open in X^n. Lemma 2.4 is proved. □

Theorem 2.5. *Let (X, τ) be a topological space and let n be a natural number. If X is an e-space, then X^n is an e-space.*

Proof. Let U be an arbitrary nonempty open subset of X^n. For every $x = (x_1, x_2, \dots, x_n) \in U$ there exist open neighborhoods $U_1(x), U_2(x), \dots, U_n(x)$ of $x_1, x_2, \dots, x_n$ respectively, such that

$$x \in U_1(x) \times U_2(x) \times \dots \times U_n(x) \subset U.$$

We have

$$\bigcup_{x \in U} (U_1(x) \times U_2(x) \times \dots \times U_n(x)) = U.$$

Let $B = \{G_\alpha : \alpha \in M\}$ be a base for the topology τ_e. Since X is an e-space there is an index subset M_i of M, such that

$$U_i(x) = \bigcup_{\alpha_i \in M_i} G_{\alpha_i}$$

for each $i = 1, 2, \dots, n$. By virtue of property of Cartesian product we have

$$U_1(x) \times U_2(x) \times \dots \times U_n(x) = \bigcup_{\alpha_i \in M_i, i=1,\dots,n} (G_{\alpha_1} \times G_{\alpha_2} \times \dots \times G_{\alpha_n}).$$

By Lemma 2.4 the set $G_{\alpha_1} \times G_{\alpha_2} \times \dots \times G_{\alpha_n}$ is an e-open. Thus U can be represented as a union of e-open subsets of X^n. Therefore X^n is an e-space. Theorem 2.5 is proved. □

Theorem 2.6. *Let X be an e-space and n be a natural number. Then the space of n-permutation degree SP^nX is an e-space.*

Proof. Let X be an e-space. Then by Theorem 2.5 the space X^n is also e-space. It is known that the mapping $\pi_n^s : X^n \to SP^nX$ is clopen, i.e. the image of every open (closed) subset of X^n under the mapping π_n^s is open (respectively closed) in SP^nX [8]. By virtue of Proposition 2.2, the space of n-permutation degree SP^nX is an e-space. Theorem 2.6 is proved. □

Corollary 2.7. *Let X be an e-space and n be a natural number. Then the space of G-permutation degree SP^n_GX is an e-space.*

Lemma 2.8. *Let $U_1, U_2, \dots, U_n$ be e-open subsets of a topological space X. Then the subset $O\langle U_1, U_2, \dots, U_n\rangle$ is e-open in* $\exp X$.

Proof. It is enough to show that the subset $[O\langle U_1, U_2, \dots, U_n\rangle]$ is open in $\exp X$, because by definition of Vietoris topology the subset $O\langle U_1, U_2, \dots, U_n\rangle$ is open in $\exp X$. For every subsets $U_1, U_2, \dots, U_n$ of X the following equality is true [7]:

$$O\langle [U_1], [U_2], \dots, [U_n]\rangle = [O\langle U_1, U_2, \dots, U_n\rangle].$$

Since the subsets $U_1, U_2, \dots, U_n$ of X are e-open, the subset $O\langle [U_1], [U_2], \dots, [U_n]\rangle$ is open in $\exp X$. Then the subset $[O\langle U_1, U_2, \dots, U_n\rangle]$ is also open in $\exp X$. Lemma 2.8 is proved. □

Theorem 2.9. *If a topological space X is an e-space, then the exponential space $\exp_\omega X$ is an e-space.*

Proof. Consider the family

$$\mathcal{B} = \{O\langle V_1, V_2, \dots, V_k\rangle : V_i \text{ is e-open subset in } X \text{ for each } i = 1, \dots, k \text{ and } k \in \mathbb{N}\}.$$

By virtue of Lemma 2.8 all elements of $\mathcal{B}$ are e-open subsets of $\exp_n X$. It is sufficient to show that the family $\mathcal{B}$ is a base of $\exp_\omega X$. Let $U_1, U_2, \dots, U_k$ be open subsets of X. Take an arbitrary point $F \in \exp_\omega X$ of $O\langle U_1, U_2, \dots, U_k\rangle$. Put

$$M_1 = F \cap U_1 = \{x_{11}, x_{12}, \dots, x_{1r_1}\}$$

$$M_2 = F \cap U_2 = \{x_{21}, x_{22}, \dots, x_{2r_2}\}$$

$$\dots$$

$$M_k = F \cap U_k = \{x_{k1}, x_{k2}, \dots, x_{kr_k}\}.$$

It is clear that $F = \bigcup\limits_{i=1}^{k} M_i$. Since X is an e-space there exist e-open subsets $V_{11}, V_{12}, \dots, V_{1r_1}$ of X such that $x_{1i} \in V_{1i} \subset U_1$ for all $i = 1, 2, \dots, r_1$. Put

$$V_1 = \bigcup_{i=1}^{r_1} V_{1i},\ V_2 = \bigcup_{i=1}^{r_2} V_{2i},\ \dots,\ V_k = \bigcup_{i=1}^{r_k} V_{ki}.$$

In this case $V_1, V_2, \dots, V_k$ are e-open subsets of X as a union of e-open subsets. It is easy to check that $F \in O\langle V_1, V_2, \dots, V_k\rangle \subset O\langle U_1, U_2, \dots, U_k\rangle$. Theorem 2.9 is proved. □

Corollary 2.10. *If a topological space X is an e-space, then for each $n \in \mathbb{N}$ the exponential subspace $\exp_n X$ is an e-space.*

3. *e*-continuity of mappings

Definition 3.1. [1] *Let X and Y be topological spaces and $f\colon X \to Y$ be a mapping. We say that f is e-continuous at a point $x \in X$ if for each open set V in Y containing $f(x)$ there exists an e-open set U in X containing x such that $f(U) \subset V$. A mapping $f\colon X \to Y$ is called e-continuous if it is e-continuous at each point of X.*

Clearly every e-continuous mapping is continuous, but the converse is not necessarily true in general. In fact if the mapping $id\colon \mathbb{R} \to \mathbb{R}$ is the identity, then it is continuous but not e-continuous. Whenever X is an e-space, then every continuous mapping on X is e-continuous.

Theorem 3.2. *Let X and Y be topological spaces and n be a natural number. If a mapping $f\colon X \to Y$ is an e-continuous, then the mapping $\Pi^n f\colon X^n \to Y^n$ is e-continuous.*

Proof. Let $f\colon X \to Y$ be an e-continuous mapping. For any point $x = (x_1, x_2, \dots, x_n) \in X^n$ we have

$$\Pi^n f(x) = (f(x_1), f(x_2), \dots, f(x_n)).$$

Choose an arbitrary open neighborhood V of $(f(x_1), f(x_2), \dots, f(x_n)) \in Y^n$. In this case there is an open subset V_i of Y containing of $f(x_i)$ such that

$$V_1 \times V_2 \times \dots \times V_n \subset V.$$

Since $f\colon X \to Y$ is e-continuous there is an e-open subset G_i of X such that $x_i \in G_i$ and $f(G_i) \subset V_i$ for all $i = 1, 2, \dots, n$. We have

$$\Pi^n f(G_1 \times G_2 \times \dots \times G_n) = f(G_1 \times G_2 \times \dots \times G_n) \subset V.$$

By Lemma 2.4 the set $G_1 \times G_2 \times \dots \times G_n$ is an e-open in X^n. On the other hand the point x belongs to $G_1 \times G_2 \times \dots \times G_n$. Therefore, $\Pi^n f$ is an e-continuous. Theorem 3.2 is proved. □

Corollary 3.3. *The functor Π^n preserves the e-continuity of mappings.*

Theorem 3.4. *Let X and Y be topological spaces and n be a natural number. If a mapping $f\colon X \to Y$ is an e-continuous, then the mapping $SP^n f\colon SP^n X \to SP^n Y$ is also e-continuous.*

Proof. Let $f\colon X \to Y$ be an e-continuous mapping. For any orbit $[x] = [(x_1, x_2, \dots, x_n)] \in SP^n X$ we have

$$SP^n f([x]) = [(f(x_1), f(x_2), \dots, f(x_n))].$$

Take an arbitrary open neighborhood V of $SP^n f([x])$. In this case for each $i = 1, \ldots, n$ there is an open subset V_i of Y containing of $f(x_i)$ such that

$$[V_1 \times V_2 \times \ldots \times V_n] \subset V,$$

where

$$[V_1 \times V_2 \times \ldots \times V_n] = \{[z] : \text{ there exists } \sigma \in S_n \text{ such that } z_i \in V_{\sigma(i)} \text{ for all } i = 1, \ldots, n\}.$$

By e-continuity of f there exists an e-open subset G_i of X such that $x_i \in G_i$ and $f(G_i) \subset V_i$ for all $i = 1, 2, \ldots, n$. Hence

$$SP^n f([G_1 \times G_2 \times \ldots \times G_n]) = [f(G_1) \times f(G_2) \times \ldots \times f(G_n)] \subset [V_1 \times V_2 \times \ldots \times V_n] \subset V.$$

Clearly, $[G_1 \times G_2 \times \ldots \times G_n]$ is an open neighborhood of $[x]$. We state that $\mathrm{cl}([G_1 \times G_2 \times \ldots \times G_n])$ is open in $SP^n X$. Since π_n^s is closed mapping we have

$$\mathrm{cl}([G_1 \times G_2 \times \ldots \times G_n]) = \mathrm{cl}(\pi_n^s(G_1 \times G_2 \times \ldots \times G_n)) = \pi_n^s(\mathrm{cl}(G_1 \times G_2 \times \ldots \times G_n)).$$

Therefore $\mathrm{cl}([G_1 \times G_2 \times \ldots \times G_n])$ is an open subset as the image of an open set under the open mapping π_n^s. Theorem 3.4 is proved. □

Corollary 3.5. *The functor SP^n preserves the e-continuity of mappings.*

Theorem 3.6. *Let X and Y be topological spaces and n be a natural number. If a mapping $f\colon X \to Y$ is an e-continuous, then $\exp_n f\colon \exp_n X \to \exp_n Y$ is an e-continuous mapping.*

Proof. Take an arbitrary point $F \in \exp_n X$ with $F = \{x_1, x_2, \ldots, x_n\}$. Let

$$\Phi = \exp_n f(F) = \{f(x_1), f(x_2), \ldots, f(x_m)\} \ (m \le n).$$

Choose an arbitrary neighborhood $O\langle V_1, V_2, \ldots, V_k\rangle$ of Φ, where $V_1, V_2, \ldots, V_k$ are open subsets of Y. Consider the following subsets of Y:

$$M_1 = \Phi \cap V_1 = \{f(x_{11}), f(x_{12}), \ldots, f(x_{1r_1})\};$$

$$M_2 = \Phi \cap V_2 = \{f(x_{21}), f(x_{22}), \ldots, f(x_{2r_2})\};$$

$$\ldots$$

$$M_k = \Phi \cap V_k = \{f(x_{k1}), f(x_{k2}), \ldots, f(x_{kr_k})\},$$

where $\bigcup\limits_{i=1}^{k} M_i = F$. Since f is e-continuous there exits an e-open set U_{1i} in X with $\{x \in F : \ f(x) = f(x_{1i})\} \subset U_{1i}$ such that $f(U_{1i}) \subset V_1$ for each $i = 1, 2, \ldots, r_1$. For the e-open set $U_1 = \bigcup\limits_{i=1}^{r_1} U_{1i}$ we have $f(U_1) \subset V_1$. Similarly, we will construct e-open sets $U_2, U_3, \ldots, U_k$ as above such that $f(U_j) \subset V_j$ for all $j = 2, 3, \ldots, k$. By Lemma 2.8 the set $O\langle U_1, U_2, \ldots, U_k\rangle$ is e-open in $\exp_n X$. It is easy to check that $F \in O\langle U_1, U_2, \ldots, U_k\rangle$. In addition the following inclusion holds:

$$\exp_n f(O\langle U_1, U_2, \ldots, U_k\rangle) \subset O\langle V_1, V_2, \ldots, V_k\rangle.$$

Thus the mapping $\exp_n f$ is e-continuous. Theorem 3.6 is proved. □

Corollary 3.7. *The functor $\exp_n$ preserves the e-continuity of mappings.*

4. On τ-boundary points of a set

τ-closed sets were introduced by I. Juhasz in 1980 in his book [11]. In 2016, O. Okunev [12] introduced the concept of τ-closure of a set and connected its properties with τ-continuous mappings introduced by A. Arkhangelsky in [13]. In 2023, the definitions of τ-open sets and τ-interiors of a set are introduced and new properties of τ-continuous mappings are studied [2]. In this section we introduce the definitions of τ-boundary points, τ-cluster points and τ-boundary of a set and study several of their properties.

Definition 4.1. [11] *Let X be a topological T_1-space. A set $F \subset X$ is called τ-closed in X if for each $B \subset F$ such that $|B| \leq \tau$, the closure of the set B in X lies in F.*

It is known that every closed subset of a topological space is τ-closed. But the opposite is not always true and the following Example 4.2 verifies this claim. Moreover, I. Juhasz in his work [11] proved that the tightness of a topological space X does not exceed τ if and only if every τ-closed subset is closed.

Example 4.2. [2] *On the real line $\mathbb{R}$ we will assume that all sets whose complement is countable are open, and we will also declare the empty set is open, i.e. the set of all real numbers $\mathbb{R}$ has the following topology:*

$$\theta = \{\emptyset\} \cup \{U : \ U \subset \mathbb{R}, \ |\mathbb{R} \backslash U| \leq \omega\}$$

Since every set whose cardinality of its complement does not exceed ω is open in this topological space, then an arbitrary countable set $B \subset X$ is closed. Let's choose an arbitrary subset $M \subset \mathbb{R}$. Then every subset $B \subset M$, whose cardinality does not exceed ω, coincides with its closure, which means that $\mathrm{cl}_X B \subset M$ for all $|B| \leq \omega$. From the arbitrariness of the set M it follows that each subset of this space is ω-closed. In particular, the set of all irrational numbers in this space is ω-closed, but not closed.

Definition 4.3. [2] *A set $F \subset X$ is called τ-open in X if its complement $X \backslash F$ is τ-closed. Any τ-open set containing a point $x \in X$ is called a τ-neighborhood of this point.*

Definition 4.4. [2] *Let X be a topological T_1-space. The τ-interior of a subset A of X is the union of all τ-open subsets of A, i.e.,*

$$Int_\tau A = \bigcup \{U : U \subset A \text{ and } U \text{ is } \tau\text{-open}\}.$$

It is known that the set $\mathrm{Int}_\tau A$ is a τ-open subset. In general, for a subset A of a topological space X τ-interior of A is not open in X, i.e. $\mathrm{Int}_\tau A \neq \mathrm{Int} A$. We consider the topological space in Example 4.2. As a subset A we take the set of all rational numbers $\mathbb{Q}$. Then $\mathrm{Int}_\omega \mathbb{Q} = \mathbb{Q}$, but $\mathrm{Int}\mathbb{Q} = \emptyset$.

Proposition 4.5. [2] *Let X be a topological T_1-space. Then for every subset $A \subset X$ we have the following equality*

$$Int_\tau A = X \backslash [X \backslash A]_\tau.$$

Definition 4.6. [2] *The τ-closure of a subset A of a T_1-space X is defined as follows:*

$$[A]_\tau = \bigcup \{\mathrm{cl}_X B : \ B \subset A, \ |B| \leq \tau\}.$$

Recall that a subset A is *τ-dense* in X if $[A]_\tau = X$ [12]. For any subsets A and B of the space X the following relation holds: if $A \subset B$, then $[A]_\tau \subset [B]_\tau$.

Example 4.7. *On the set of real numbers with the natural topology, we choose the set of all rational numbers. Let's find its ω-closure*

$$[\mathbb{Q}]_\omega = \bigcup \{\mathrm{cl}_{\mathbb{R}} B : B \subset \mathbb{Q}, \ |B| \leq \omega\}.$$

As a subset $B \subset \mathbb{Q}$, $|B| \leq \omega$ we take the set itself $\mathbb{Q}$, the closure of which coincides with the set of real numbers. This means $[\mathbb{Q}]_\omega = \mathbb{R}$, and we can conclude that the set of rational numbers on the Euclidean line is ω-dense.

Definition 4.8. [13] *A mapping $f : X \to Y$ is called τ-continuous if for every set $A \subset X$ such that $|A| \leq \tau$ the mapping $f|_A : A \to Y$ is continuous.*

Theorem 4.9. [4] *For a mapping $f : X \to Y$ of arbitrary topological T_1-spaces X and Y the following statements are equivalent:*

1) *$f : X \to Y$ is τ-continuous;*
2) *For every closed F of Y, the preimage $f^{-1}(F)$ is τ-closed in X;*
3) *For every τ-closed F of Y, the preimage $f^{-1}(F)$ is τ-closed in X;*
4) *$f([A]_\tau) \subset [f(A)]_\tau$, for every $A \subset X$;*
5) *$[f^{-1}(B)]_\tau \subset f^{-1}([B]_\tau)$, for every $B \subset Y$.*

Theorem 4.10. *Let X be a topological T_1-space. The operator of τ-closure has the following properties:*

1) *$[X]_\tau = X$;*
2) *$[\varnothing]_\tau = \varnothing$;*
3) *$A \subset [A]_\tau$;*
4) *$[A \cup B]_\tau = [A]_\tau \cup [B]_\tau$;*
5) *$[[A]_\tau]_\tau = [A]_\tau$.*

Proof. Any space X can be covered by its subsets, the cardinality of which does not exceed the cardinal number τ. This means that every space coincides with its τ-closure. The only subset of the empty set is the empty set whose closure is equal to itself. And this means that it is equal to its τ-closure. Properties 1) and 2) have been proven.

Property 3) is true, since any set A can be covered by subsets B whose cardinality does not exceed τ, i.e.

$$A = \bigcup\{B : B \subset A,\ |B| \leq \tau\} \subset \bigcup\{\mathrm{cl}_X B : B \subset A,\ |B| \leq \tau\} = [A]_\tau.$$

For property 4), since $A \subset A \cup B$ and $B \subset A \cup B$, we have $[A]_\tau \subset [A \cup B]_\tau$ and $[B]_\tau \subset [A \cup B]_\tau$. It follows that

$$[A]_\tau \cup [B]_\tau \subset [A \cup B]_\tau.$$

Due to the fact that every set is a subset of its τ-closure, $A \subset [A]_\tau$, $B \subset [B]_\tau$ (property 3)), we have $A \cup B \subset [A]_\tau \cup [B]_\tau$. Thus, $[A \cup B]_\tau = [A]_\tau \cup [B]_\tau$.

Let us prove property 5). Let us represent the τ-closure of the subset A as $[A]_\tau = \bigcup\{\mathrm{cl}_X B : B \subset A,\ |B| \leq \tau\}$ and denote it by M. Then

$$[[A]_\tau]_\tau = [M]_\tau = \bigcup\{\mathrm{cl}_X C : C \subset M,\ |C| \leq \tau\}.$$

Any subset $C \subset M$ whose cardinality does not exceed a cardinal number τ can be expressed through its elements, i.e.,

$$C = \{x_s : s \in S,\ |S| \leq \tau\}.$$

Since $C \subset \bigcup\{\mathrm{cl}_X B : B \subset A,\ |B| \leq \tau\}$, then from the family $\{\mathrm{cl}_X B : B \subset A,\ |B| \leq \tau\}$ we can select a subfamily

$$\{\mathrm{cl}_X B_s : x_s \in \mathrm{cl}_X B_s,\ s \in S\}.$$

We denote the union of all elements of the family $\{B_s : x_s \in \mathrm{cl}_X B_s,\ s \in S\}$ by B^*. Since the cardinality of the set B_s does not exceed the cardinal number τ for each $s \in S$, then the cardinality of B^* does not exceed the cardinal number τ. In this case, we have $C \subset cl_X B^*$. By properties of the closure of a set, we have $cl_X C \subset cl_X B^*$. It means that $cl_X C \subset [A]_\tau$ for all $C \subset M$ with $|C| \leq \tau$. Therefore

$$[[A]_\tau]_\tau \subset [A]_\tau.$$

The reverse inclusion $[A]_\tau \subset [[A]_\tau]_\tau$ follows from property 3). Theorem 4.10 is proved. □

Theorem 4.11. *Let X and Y be topological spaces and $A \subset X$, $B \subset Y$. Then the following equality is valid:*

$$[A \times B]_\tau = [A]_\tau \times [B]_\tau.$$

Proof. Let $z_0 = (x_0, y_0) \in [A \times B]_\tau$. Consider the projection mappings $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$. Then $pr_X(z_0) = x_0 \in pr_X([A \times B]_\tau)$. By virtue of property 4) of Theorem 4.9,

$$pr_X([A \times B]_\tau) \subset [pr_X(A \times B)]_\tau = [A]_\tau.$$

Hence, $x_0 \in [A]_\tau$. Similarly $pr_Y(z_0) = y_0 \in pr_Y([A \times B]_\tau) \subset [pr_Y(A \times B)]_\tau = [B]_\tau$. Based on this, $z_0 \in [A]_\tau \times [B]_\tau$ and

$$[A \times B]_\tau \subset [A]_\tau \times [B]_\tau.$$

On the other side, let $z_0 = (x_0, y_0) \in [A]_\tau \times [B]_\tau$. We prove that $z_0 \in [A \times B]_\tau$. By definition of τ-closure

$$(x_0, y_0) \in (\bigcup\{\mathrm{cl}_X C : C \subset A, |C| \le \tau\}) \times (\bigcup\{\mathrm{cl}_Y D : D \subset B, |D| \le \tau\}).$$

This means that there are sets $\mathrm{cl}_X C_0 \in \{\mathrm{cl}_X C : C \subset A, |C| \le \tau\}$, which contains the element x_0, and similarly $\mathrm{cl}_Y D_0 \in \{\mathrm{cl}_Y D : D \subset B, |D| \le \tau\}$, which in turn contains y_0. Hence, $(x_0, y_0) \in \mathrm{cl}_X C_0 \times \mathrm{cl}_Y D_0 = \mathrm{cl}_{X \times Y}(C_0 \times D_0)$. Then,

$$(x_0, y_0) \in \bigcup\{\mathrm{cl}_{X \times Y}(C \times D) : C \times D \subset A \times B, |C \times D| \le \tau\},$$

which coincides with $(x_0, y_0) \in [A \times B]_\tau$. It follows from this that

$$[A]_\tau \times [B]_\tau \subset [A \times B]_\tau$$

and therefore, $[A]_\tau \times [B]_\tau = [A \times B]_\tau$. Theorem 4.11 is proved. □

Corollary 4.12. *The product of two sets is τ-closed if and only if it's factors are τ-closed.*

Corollary 4.13. *The product of two sets is τ-dense if and only if it's factors are τ-dense.*

Definition 4.14. *Let X be a topological space. A point $x \in X$ is called a τ-cluster of a subset A of X if each τ-neighborhood of the point x has a non-empty intersection with A.*

Theorem 4.15. *The set of all τ-cluster points of a subset A of X coincides with the τ-closure of the set A.*

Proof. Put

$$A^* = \{x \in X : x \text{ is } \tau\text{-cluster point of } A\}.$$

Firstly, we prove the inclusion $A^* \subset [A]_\tau$. Assume that there is a τ-cluster point x of A such that $x \notin [A]_\tau$, i.e., $x \in X \setminus [A]_\tau$. The set $X \setminus [A]_\tau$ is τ-open, which means that it is a τ-neighborhood of the point x. But the set $X \setminus [A]_\tau$ don't meet with the set A. This is a contradiction that x is cluster point of A. Therefore, the set of all τ-cluster points of a set A is a subset of the τ-closure of the set A.

Now we will show $[A]_\tau \subset A^*$. Let us assume that there is a point $x \in [A]_\tau$ and there exists its τ-neighborhood $O_\tau(x)$ that does not intersect with A. Hence, $A \subset X \setminus O_\tau(x)$. Since $X \setminus O_\tau(x)$ is a τ-closed subset then $X \setminus O_\tau(x) = [X \setminus O_\tau(x)]_\tau$ and $x \notin [X \setminus O_\tau(x)]_\tau$. By property 3) of Theorem 4.10 we have $[A]_\tau \subset [X \setminus O_\tau(x)]_\tau$. This means that $x \notin [A]_\tau$, which contradicts the choice of point x. Theorem 4.15 is proved. □

Definition 4.16. *For a topological space X the τ-density defines the smallest cardinal number of the form $|A|$, where A is a τ-dense subset X, i.e.*

$$d_\tau(X) = \min\{|A| : A \textit{ is a } \tau\textit{-dense in } X\}.$$

The density of a topological space X does not always coincide with the τ-density of the space X. The following example verifies this claim.

Example 4.17. *Let X be a set of cardinality of a hypercontinuum on which a countably closed topology is given, i.e., let us declare closed all subsets of the set X whose cardinality does not exceed ω. Then the subset $A \subset X$ is everywhere dense in X if and only if $|A| > \omega$, that is $d(X) = \omega_1$, while the only ω-dense subset X is itself X, that is, $d_\omega(X) = \omega_2$.*

Definition 4.18. [14] *The largest cardinal number $\kappa \geq \omega$ such that any family of pairwise disjoint nonempty open subsets of a space X has cardinality $\leq \kappa$ is called the Suslin's number or the cellularity of the space X and is denoted by $c(X)$. If $c(X) = \omega$, then we say that the space X has the Suslins property.*

Definition 4.19. *The largest cardinal number $\kappa \geq \omega$ such that any family of pairwise disjoint non-empty τ-open subsets of a space X has cardinality $\leq \kappa$ is called the Suslins τ-number or the τ-cellularity of the space X and is denoted by $c_\tau(X)$.*

The Suslin's number does not coincide with the Suslin's τ-number, in general. The following example verifies this claim.

Example 4.20. *Let X be a set with $|X| = \omega_1$. Let us define a topology on this set by declaring closed all subsets of the set X whose cardinality does not exceed ω. Let us choose a system $\gamma = \{U_\alpha : \alpha \in A\}$ of pairwise disjoint non-empty open subsets of the space X. Let us also assume that the cardinality of the indexed set A is ω_1. Then there are elements such as $\alpha \in A$ and $\beta \in A$ with $U_\alpha \cap U_\beta = \varnothing$ and $X \backslash (U_\alpha \cap U_\beta) = X$. But due to De Morgan's laws $X \backslash (U_\alpha \cap U_\beta) = (X \backslash U_\alpha) \cup (X \backslash U_\beta)$. According to the condition of taking the topology, the cardinality of the sets $X \backslash U_\alpha$ and $X \backslash U_\beta$ is at most countable, and hence their union $(X \backslash U_\alpha) \cup (X \backslash U_\beta)$ does not exceed ω. Then $|X| = |X \backslash (U_\alpha \cap U_\beta)| = |(X \backslash U_\alpha) \cup (X \backslash U_\beta)| = \omega$, which contradicts the condition. The last means that $c(X) = \omega$.*

Let's check the Suslin's ω-number of this space. As a system of ω-open disjoint subsets, we choose all single-point sets of the space X. The last means that $c_\omega(X) = \omega_1$.

Definition 4.21. *We say that a point $x \in X$ is τ-boundary point of a subset A, if any τ-open neighborhood of x has a non-empty intersection with both the set A and its complement $X \backslash A$. The set of all τ-boundary points of a set $A \subset X$ is called the τ-boundary of a set A and denoted by $Fr_\tau A$.*

The following result derived easily by Definition 4.21.

Proposition 4.22. *The τ-boundary of a set $A \subset X$ is the intersection of the τ-closure of the set and τ-closure of its complement, i.e.,*

$$Fr_\tau A = [A]_\tau \cap [X \backslash A]_\tau.$$

Theorem 4.23. *The operator Fr_τ has the following properties:*

1) $Int_\tau A = A \backslash Fr_\tau A$;
2) $[A]_\tau = A \cup Fr_\tau A$;
3) $Fr_\tau(A \cup B) \subset Fr_\tau A \cup Fr_\tau B$;
4) $Fr_\tau(A \cap B) \subset Fr_\tau A \cap Fr_\tau B$;
5) $Fr_\tau(X \backslash A) = Fr_\tau A$;
6) $X = Int_\tau A \cup Fr_\tau A \cup Int_\tau(X \backslash A)$;
7) $Fr_\tau [A]_\tau \subset Fr_\tau A$;
8) $Fr_\tau Int_\tau A \subset Fr_\tau A$;
9) $Fr_\tau A = [A]_\tau \backslash Int_\tau A$;
10) *A is τ-open if and only if* $Fr_\tau A = [A]_\tau \backslash A$;
11) *A is τ-closed if and only if* $Fr_\tau A = A \backslash Int_\tau A$;
12) *A is τ-open-closed if and only if* $Fr_\tau A = \varnothing$.

Proof. 1) To prove property 1) we express the set $A \backslash Fr_\tau A$ as

$$A \backslash Fr_\tau A = A \backslash ([A]_\tau \cap [X \backslash A]_\tau)$$

by definition of the τ-boundary of a set. Then, by virtue of De Morgan's laws,

$$A\backslash Fr_\tau A = (A\backslash[A]_\tau) \cup (A\backslash[X\backslash A]_\tau).$$

Since $A \subset [A]_\tau$, then the difference $A\backslash[A]_\tau$ is empty, which means that $A\backslash Fr_\tau A = A\backslash[X\backslash A]_\tau$. By Proposition 4.5, $A\backslash Fr_\tau A = \mathrm{Int}_\tau A$. Property 1) has been proven.

2) Using Proposition 4.22, we write the equality

$$A \cup Fr_\tau A = A \cup ([A]_\tau \cap [X\backslash A]_\tau).$$

Let us apply the law of distributivity of a union of sets to the second part of the equality and obtain

$$A \cup Fr_\tau A = A \cup ([A]_\tau \cap [X\backslash A]_\tau) = ([A]_\tau \cup A) \cap (A \cup [X\backslash A]_\tau),$$

i.e.,

$$A \cup Fr_\tau A = ([A]_\tau \cup A) \cap (A \cup [X\backslash A]_\tau).$$

Due to property 3) of Theorem 4.10 $A \subset [A]_\tau$ and $X\backslash A \subset [X\backslash A]_\tau$. Therefore, $[A]_\tau \cup A = [A]_\tau$ and $A \cup [X\backslash A]_\tau = X$. This means, $A \cup Fr_\tau A = [A]_\tau \cap X$ which in turn is equal to $[A]_\tau$. Property 2) has been proven.

3) Let us express the τ-boundary of the set $A \cup B$ in the following way:

$$Fr_\tau(A \cup B) = [A \cup B]_\tau \cap [X\backslash(A \cup B)]_\tau.$$

By virtue of De Morgan's laws and Property 4) of Theorem 4.10, we obtain

$$[A \cup B]_\tau \cap [X\backslash(A \cup B)]_\tau = ([A]_\tau \cup [B]_\tau) \cap [(X\backslash A) \cap (X\backslash B)]_\tau$$

or

$$Fr_\tau(A \cup B) = ([A]_\tau \cup [B]_\tau) \cap [(X\backslash A) \cap (X\backslash B)]_\tau.$$

According to property 3) of Theorem 4.10

$$([A]_\tau \cup [B]_\tau) \cap [(X\backslash A) \cap (X\backslash B)]_\tau \subset ([A]_\tau \cup [B]_\tau) \cap ([X\backslash A]_\tau \cap [X\backslash B]_\tau).$$

The set $([A]_\tau \cup [B]_\tau) \cap ([X\backslash A]_\tau \cap [X\backslash B]_\tau)$, in turn, is equal to the set $([A]_\tau \cap [X\backslash A]_\tau) \cup ([B]_\tau \cap [X\backslash B]_\tau)$, which by definition coincides with the set $Fr_\tau A \cup Fr_\tau B$, that is $Fr_\tau(A \cup B) \subset Fr_\tau A \cup Fr_\tau B$. Property 3) has been proven.

4) We have $Fr_\tau(A \cap B) = [A \cap B]_\tau \cap [X\backslash(A \cap B)]_\tau$. Using De Morgan's laws and property 4) of Theorem 4.10 we obtain

$$Fr_\tau(A \cap B) = [A \cap B]_\tau \cap [X\backslash(A \cap B)]_\tau = [A \cap B]_\tau \cap ([(X\backslash A)]_\tau \cup [(X\backslash B)]_\tau).$$

Also,

$$[A \cap B]_\tau \cap ([(X\backslash A)]_\tau \cup [(X\backslash B)]_\tau) \subset ([A]_\tau \cap [B]_\tau) \cap ([X\backslash A]_\tau \cup [X\backslash B]_\tau).$$

The set $([A]_\tau \cap [B]_\tau) \cap ([X\backslash A]_\tau \cup [X\backslash B]_\tau)$, in turn, is equal to the set $([A]_\tau \cap [X\backslash A]_\tau) \cap ([B]_\tau \cap [X\backslash B]_\tau)$, which by definition coincides with the set $Fr_\tau A \cap Fr_\tau B$, that is $Fr_\tau(A \cap B) \subset Fr_\tau A \cap Fr_\tau B$. Property 4) has been proven.

5) To prove property 5), we express $Fr_\tau(X\backslash A)$ as

$$Fr_\tau(X\backslash A) = [X\backslash A]_\tau \cap [X\backslash(X\backslash A)]_\tau.$$

Then

$$Fr_\tau(X\backslash A) = [X\backslash A]_\tau \cap [X\backslash(X\backslash A)]_\tau = [X\backslash A]_\tau \cap [A]_\tau = Fr_\tau A.$$

The property 5) has been proven.

6) By the above property 1) we have

$$\mathrm{Int}_\tau A \cup Fr_\tau A \cup \mathrm{Int}_\tau(X\backslash A) = (A\backslash Fr_\tau A) \cup Fr_\tau A \cup ((X\backslash A)\backslash Fr_\tau(X\backslash A)).$$

By the above property 5) we have

$$(A\backslash Fr_\tau A)\cup Fr_\tau A\cup((X\backslash A)\backslash Fr_\tau(X\backslash A)) = A\cup Fr_\tau A\cup(X\backslash A) = X.$$

Property 6) has been proven.

7) Let $x\in Fr_\tau[A]_\tau$. Then, as a consequence $x\in[[A]_\tau]_\tau\cap[[X\backslash A]_\tau]_\tau$. By virtue of Proposition 4.22 and the property 5) of Theorem 4.10

$$[[A]_\tau]_\tau\cap[[X\backslash A]_\tau]_\tau = [A]_\tau\cap[X\backslash A]_\tau = Fr_\tau A.$$

The last means that $x\in Fr_\tau A$ and thus, $Fr_\tau[A]_\tau\subset Fr_\tau A$. Property 7) has been proven.

8) We have

$$Fr_\tau \mathrm{Int}_\tau A = [\mathrm{Int}_\tau A]_\tau\cap[X\backslash \mathrm{Int}_\tau A]_\tau.$$

According to Proposition 4.5,

$$[\mathrm{Int}_\tau A]_\tau\cap[X\backslash \mathrm{Int}_\tau A]_\tau = [\mathrm{Int}_\tau A]_\tau\cap[[X\backslash A]_\tau]_\tau.$$

By property 5) of Theorem 4.10,

$$[\mathrm{Int}_\tau A]_\tau\cap[[X\backslash A]_\tau]_\tau = [\mathrm{Int}_\tau A]_\tau\cap[X\backslash A]_\tau.$$

Since $\mathrm{Int}_\tau A\subset A$, then

$$[\mathrm{Int}_\tau A]_\tau\cap[X\backslash A]_\tau\subset[A]_\tau\cap[X\backslash A]_\tau,$$

that in turns $[A]_\tau\cap[X\backslash A]_\tau = Fr_\tau A$. Thus, $Fr_\tau\mathrm{Int}_\tau A\subset Fr_\tau A$. Property 8) has been proven.

9) Let $x\in Fr_\tau A$. Then by Proposition 4.22 $x\in[A]_\tau$ and $x\in[X\backslash A]_\tau$or, in other words, $x\in[A]_\tau$ and $x\notin X\backslash[X\backslash A]_\tau$. The last means that $x\in[A]_\tau\backslash(X\backslash[X\backslash A]_\tau)$. According to Proposition 4.5 $\mathrm{Int}_\tau A = X\backslash[X\backslash A]_\tau$. Therefore, $x\in[A]_\tau\backslash\mathrm{Int}_\tau A$ and hence

$$Fr_\tau A\subset[A]_\tau\backslash\mathrm{Int}_\tau A.$$

Let's select an arbitrary element x from $[A]_\tau\backslash\mathrm{Int}_\tau A$. Then by Proposition 4.5 $x\in[A]_\tau\backslash(X\backslash[X\backslash A]_\tau)$. Therefore, $x\in[A]_\tau$ and $x\in[X\backslash A]_\tau$. Then $x\in Fr_\tau A$. Property 9) has been proven.

10) *Necessity:* Let A be a τ-open subset of X. Then $\mathrm{Int}_\tau A = A$. Since $Fr_\tau A = [A]_\tau\backslash\mathrm{Int}_\tau A$, then

$$Fr_\tau A = [A]_\tau\backslash A.$$

Sufficiency: Let $A\subset X$ and $Fr_\tau A = [A]_\tau\backslash A$. Then, by property 9), $[A]_\tau\backslash A = [A]_\tau\backslash\mathrm{Int}_\tau A$ and $A = \mathrm{Int}_\tau A$. Hence, A is τ-open set. Property 10) has been proven.

11) *Necessity:* Let A be a τ-closed subset of X. Then $[A]_\tau = A$. Since $Fr_\tau A = [A]_\tau\backslash\mathrm{Int}_\tau A$, then

$$Fr_\tau A = A\backslash\mathrm{Int}_\tau A.$$

Sufficiency: Let $A\subset X$ and $Fr_\tau A = A\backslash\mathrm{Int}_\tau A$. Then, by property 9), $[A]_\tau\backslash\mathrm{Int}_\tau A = A\backslash\mathrm{Int}_\tau A$ and $A = [A]_\tau$. Therefore, A is a τ-closed set. Property 11) has been proven.

12) *Necessity:* Let A be a τ-open-closed subset of X. Then $[A]_\tau = A$ and $\mathrm{Int}_\tau A = A$. Since $Fr_\tau A = [A]_\tau\backslash\mathrm{Int}_\tau A$, then $Fr_\tau A = \varnothing$.

Sufficiency: Let $A\subset X$ and $Fr_\tau A = \varnothing$. Then, by property 9), $[A]_\tau\backslash\mathrm{Int}_\tau A = \varnothing$ and $\mathrm{Int}_\tau A = [A]_\tau$. Therefore, A is τ-open-closed set. Property 12) has been proven. Finally, Theorem 4.23 is proved. □

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