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Willmore energy and total normalcy of knots

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Abstract. It is known that a knot is a closed, self-avoiding curve in 3-dimensional space. In the present article, we compute the total Normalcy and Willmore energy of knots under first order infinitesimal bending with modified orthogonal frame. Moreover, four illustrious examples with figures i.e. Trefoil knot, Figure eight knot, **P3Q2** knot and **P4Q3** knots, have been discussed and graphically presented to support the computations for bending parameter $\epsilon = 0$ and $\epsilon > 0$. Colors are used to illustrate normalcy values and Willmore energies at different points of these knots under infinitesimal bending.

1. Introduction

Infinitesimal deformation theory is one of the major fields of global differential geometry and is centre of attraction for many differential geometers nowadays. In particular, infinitesimal bending is being studied in plenty due to its applications in mechanics, physics, biology, medicine, architecture and many other interdisciplinary scientific areas (see for example, [5], [14], [15], [16], [17] and [18]).

Definition 1.1. *Let Two homeomorphic curves C and C' are called isotopic if there exists a continuous family of curves C_t depending on t, $(0 \leq t \leq 1)$, such that C_t is homeomorphic to C, $C_o = C$ and $C_1 = C'$. We say that C is a knot if it is homeomorphic to a circle, but is not isotopic to a circle.*

Precisely, a knot is a closed, self-avoiding curve in 3-dimensional space. Knot theory has become significant and been developed in different direction in the recent past. Majorly, there are four fields in applied knot theory: physical knot theory, knot theory in life sciences, computational knot theory and geometric knot

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theory. Out of these, physical knot theory ([2], [3]) incorporates a realistic model and is used to study the following question:
"How geometric and topological characteristics of filamentary structures, like magnetic flux tubes, polymers, vortex filaments, DNAs, affect their physical properties and fucntions ?"
Recently, in [4], L. H. Kauffman, L. S. Velimirović, M.S. Najdanović and S. R. Rančić studied infinitesimal bending of knots and observe variations in their energies (like Willmore energy and Mobius energy). On the other hand R. Rančić, M. Najdanović and L. Velimirović discussed the the total normalcy of the knot during the first order infintesimal bending and obtained expressions for the first variation of the same.
In [11], T. Sasai introduced an alternative frame, called modified orthogonal frame, in 1984 which is used in the methods which generally investigate the curves with singularities. General helices and Bertrand curves were studied with modified orthognal frame in [11]. It has been observed that for unit curvature i.e. for $\kappa = 1$, this frame coincides with Serret-Frenet frame. For more details of modified orthogonal frame, we refer to [6], [7] and [13]. The aim of this paper is to obtain the variations in Willmore energy and total normalcy under infinitesimal bending of knots with modified orthogonal frame.

2. Preliminaries

In this section, we give some basic definitions and results related to knots and infinitesimal bending of curves.

Definition 2.1. [10] *Assume that $C : \mu = \mu(u)$, $u \in \mathcal{I} \subseteq \mathcal{R}$ is a continuous regular curve included in the following family of curves*

$$C_\epsilon : \tilde{\mu}(u,\epsilon) = \mu_\epsilon(u) = \mu(u) + \epsilon\zeta(u),\ u \in \mathcal{I},\ (\epsilon \geq 0,\ \epsilon \longrightarrow 0)$$

Any family of curves C_ϵ is said to be an infinitesimal bending of a curve C if

$$ds_\epsilon^2 - ds^2 = o(\epsilon)$$

where $\zeta = \zeta(u)$, $\zeta \in C^1$, is called infinitesimal bending field of the curve C.

Following is the characterization for the bending field $\zeta(u)$ to be infinitesimal bending field for any curve C.

Theorem 2.2. [1] *A necessary and sufficient condition for bending field $\zeta(u)$ to be an infinitesimal bending field of a curve C is*

$$d\mu.d\zeta = 0$$

where . stands for the scalar product in $\mathcal{R}^3$.

Next theorem gives the infinitesimal bending field of a curve C.

Theorem 2.3. *The infinitesimal bending field for the continuous regular curve $C : \mu = \mu(u)$, $u \in \mathcal{I} \subseteq \mathcal{R}$ is given by*

$$\zeta(u) = \int [p(u)\boldsymbol{n}(u) + q(u)\boldsymbol{b}(u)]\, du$$

where $p(u)$ and $q(u)$ are arbitary integrable functions, and vectors $\boldsymbol{n}(u)$ and $\boldsymbol{b}(u)$ are unit principal normal and binormal vector fields of the curve C respectively.

Let us now consider a regular curve $C : \mu = \mu[u(s)]$ such that $\mu : [0, L] \longrightarrow \mathcal{R}^3$ is parametrized by arc length s which is of class C^α, $\alpha \geq 3$.Then there exists an orthonormal frame (Frenet-Serret frame) $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ satisfying the following classical Frenet equations.

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix} \tag{1}$$

where $'$ indicates the derivation with respect to arc length s, $\mathbf{t}$ is the unit tangent vector of μ i.e. $\mathbf{t} = \mu'$, $\mathbf{n}$ is the unit principal normal vector, $\mathbf{b}$ is the unit binormal vector, κ is the curvature and τ is the torsion of the curve μ respectively. We now give a brief idea about the modified orthogonal frame. Let $\mu(t)$ be a general analytic curve which is reparametrized by its arc length s. We further assume that the curvature is not zero at all points of the curve μ. This gives us an opportunity to define another orthogonal frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ given by

$$\left.\begin{aligned} \mathbf{T} &= \mu' \\ \mathbf{N} &= \mathbf{T}' \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} \end{aligned}\right\} \tag{2}$$

where $\mathbf{T} \times \mathbf{N}$ is the cross product of $\mathbf{T}$ and $\mathbf{N}$. It is easy to observe the following relations:

$$\mathbf{T} = \mathbf{t},\ \mathbf{N} = \kappa\mathbf{n},\ \mathbf{B} = \kappa\mathbf{b}. \tag{3}$$

From these equations, a simple calculation gives

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix} \tag{4}$$

where all derivatives are computed with respect to arclength s. If $\langle , \rangle$ is the standard inner product defined for $\mathcal{R}^3$, then we have

$$\langle \mathbf{T}, \mathbf{T} \rangle = 1,\ \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = \kappa^2,\ \langle \mathbf{T}, \mathbf{N} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = \langle \mathbf{B}, \mathbf{T} \rangle = 0. \tag{5}$$

The orthogonal frame given by (4) and satisfying (5) is known as modified orthogonal frame [11]. We observe that Frenet-Serret frame can be recovered from modified orthogonal frame by putting $\kappa = 1$ in it.

Let us assume that $C_\epsilon : \tilde{\mu}(s, \epsilon) = \mu_\epsilon(s) = \mu(s) + \epsilon\zeta(s)$ be an infinitesimal bending of any curve $C : \mu(s) : [0, L] \longrightarrow \mathcal{R}^3$. Then from [10], the vector field ζ can be written as

$$\zeta = \zeta_1\mathbf{t} + \zeta_2\mathbf{n} + \zeta_3\mathbf{b}$$

Using (3), the above equation can equivalently be transformed as

$$\zeta = \psi_1\mathbf{T} + \psi_2\mathbf{N} + \psi_3\mathbf{B} \tag{6}$$

where $\psi_1 = \zeta_1$, $\psi_2 = \left(\frac{\zeta_2}{\kappa}\right)$ and $\psi_3 = \left(\frac{\zeta_3}{\kappa}\right)$.

Definition 2.4. ([4, 8]) *The Willmore energy of a knot C is defined by*

$$\mathcal{W} = \frac{1}{2}\int_C \kappa^2 ds$$

where κ is the curvature of the knot.

Definition 2.5. *The total normalcy of a knot C is defined by the integral*

$$\omega = \int_C \sqrt{\kappa^2(s) + \tau^2(s)} ds$$

where κ and τ are the curvature and torsion of the knot C, respectively.

3. Basic results

We start this section with the following characterization theorem for any bending field to be infinitesimal bending field with modified orthogonal frame.

Theorem 3.1. *A necessary and sufficient condition for the bending field ζ of a curve C with modified orthogonal frame, to be an infinitesimal bending field, is*

$$\psi_1' - \psi_2\kappa^2 = 0. \tag{7}$$

Proof. It is known from [1], that the necessary and sufficient condition for any bending field ζ of the curve $C : \mu(s)$ to be infinitesimal, is

$$\mu'.\zeta' = 0 \ \ or \ \ \mathbf{t}.\zeta' = 0$$

which gives

$$\mathbf{T}.\zeta' = 0. \tag{8}$$

Now from (7), we have

$$\zeta' = \zeta_1'\mathbf{T} + \zeta_1\mathbf{T}' + \left(\frac{\zeta_2}{\kappa}\right)'\mathbf{N} + \left(\frac{\zeta_2}{\kappa}\right)\mathbf{N}' + \left(\frac{\zeta_3}{\kappa}\right)'\mathbf{B} + \left(\frac{\zeta_3}{\kappa}\right)\mathbf{B}'$$

Using (4) in the above equation, we derive

$$\zeta' = \zeta_1'\mathbf{T} + \zeta_1\mathbf{N} + \left(\frac{\zeta_2}{\kappa}\right)'\mathbf{N} + \left(\frac{\zeta_2}{\kappa}\right)\left(-\kappa^2\mathbf{T} + \frac{\kappa'}{\kappa}\mathbf{N} + \tau\mathbf{B}\right) + \left(\frac{\zeta_3}{\kappa}\right)'\mathbf{B} + \left(\frac{\zeta_3}{\kappa}\right)\left(-\tau\mathbf{N} + \frac{\kappa'}{\kappa}\mathbf{B}\right) \tag{9}$$

Combining (8) and (9), we get

$$\psi_1' - \psi_2\kappa^2 = 0$$

where $\psi_1 = \zeta_1$, $\psi_2 = \left(\frac{\zeta_2}{\kappa}\right)$. □

Remark 3.2. *It view of $\psi_1 = \zeta_1$, $\psi_2 = \left(\frac{\zeta_2}{\kappa}\right)$ in (7), equivalently we get back theorem-1.5 of [10] for the curve with Frenet-Serret frame i.e. a necessary and sufficient condition for the bending field ζ of a curve C with modified orthogonal frame, to be an infinitesimal bending field, is*

$$\zeta_1' - \kappa\zeta_2 = 0.$$

We now prove the following proposition which gives infinitesimal chnages in curvature and torsion i.e. $\delta\kappa$ and $\delta\tau$.

Proposition 3.3. *Under infinitesimal bending of a curve with modified orthogonal frame, following equations hold :*

$$\delta\kappa = \frac{1}{\kappa}\left[\frac{\kappa'\zeta_1}{\kappa} + \kappa\zeta_2 + \frac{\zeta_2''}{\kappa} - \frac{\tau'\zeta_3}{\kappa} - \frac{2\tau\zeta_3'}{\kappa} - \frac{\tau^2\zeta_2}{\kappa}\right] \tag{10}$$

$$\begin{aligned}\delta\tau = \kappa\left(\tau\zeta_2 + \zeta_3'\right) - \kappa'\left[\kappa\tau\zeta_1 + 2\tau\zeta_2' + \tau'\zeta_2 + \zeta_3'' - \tau^2\zeta_3 - \frac{\tau\zeta_1}{\kappa} - \frac{2\tau\zeta_2'}{\kappa^2} + \frac{\tau^2\zeta_3}{\kappa^2} - \frac{\tau'\zeta_2}{\kappa^2} - \frac{\zeta_3''}{\kappa^2}\right] \\ + \left[\kappa\left(\kappa\tau\zeta_1 + 2\tau\zeta_2' + \tau'\zeta_2 + \zeta_3'' - \tau^2\zeta_3\right)\right]'\end{aligned} \tag{11}$$

Proof. It is known from [10], that

$$\delta\mathbf{t} = \delta\mathbf{T} = \delta\mu' = (\delta\mu)' = \zeta'.$$

Now using equation (6) and theorem-3, we get

$$\delta\mathbf{T} = \zeta' = \left[\zeta_1 + \left(\frac{\zeta_2}{\kappa}\right)' + \frac{\kappa'\zeta_2}{\kappa^2} - \frac{\tau\zeta_3}{\kappa}\right]\mathbf{N} + \left[\frac{\zeta_2\tau}{\kappa} + \left(\frac{\zeta_3}{\kappa}\right)' + \frac{\kappa'\zeta_3}{\kappa^2}\right]\mathbf{B}.$$

Since $\delta\mathbf{T}' = (\delta\mathbf{T})'$, in view of the last equation and the relation (4), we get

$$\delta\mathbf{T}' = \left[\zeta_1 + \left(\frac{\zeta_2}{\kappa}\right)' + \frac{\kappa'\zeta_2}{\kappa^2} - \frac{\tau\zeta_3}{\kappa}\right]\left(-\kappa^2\mathbf{T} + \frac{\kappa'}{\kappa}\mathbf{N} + \tau\mathbf{B}\right) + \left[\zeta_1' + \left(\frac{\zeta_2}{\kappa}\right)'' + \left(\frac{\kappa'\zeta_2}{\kappa^2}\right)' - \left(\frac{\tau\zeta_3}{\kappa}\right)'\right]\mathbf{N}$$
$$+ \left[\frac{\zeta_2\tau}{\kappa} + \left(\frac{\zeta_3}{\kappa}\right)' + \frac{\kappa'\zeta_3}{\kappa^2}\right]\left(-\tau\mathbf{N} + \frac{\kappa'}{\kappa}\mathbf{B}\right) + \left[\left(\frac{\zeta_2\tau}{\kappa}\right)' + \left(\frac{\zeta_3}{\kappa}\right)'' + \left(\frac{\kappa'\zeta_3}{\kappa^2}\right)'\right]\mathbf{B}$$

which, after rearranging the terms, derives to

$$\delta\mathbf{T}' = \left[-\kappa^2\zeta_1 - \kappa^2\left(\frac{\zeta_2}{\kappa}\right)' - \kappa'\zeta_2 + \tau\kappa\zeta_3\right]\mathbf{T}$$
$$+ \left[\frac{\kappa'}{\kappa}\zeta_1 + \frac{\kappa'}{\kappa}\left(\frac{\zeta_2}{\kappa}\right)' + \frac{\kappa'}{\kappa}\left(\frac{\kappa'\zeta_2}{\kappa^2}\right) - 2\frac{\kappa'\tau\zeta_3}{\kappa^2} + \zeta_1' + \left(\frac{\zeta_2}{\kappa}\right)'' + \left(\frac{\kappa'\zeta_2}{\kappa^2}\right)' - \left(\frac{\tau\zeta_3}{\kappa}\right)' - \frac{\tau^2\zeta_2}{\kappa} - \tau\left(\frac{\zeta_3}{\kappa}\right)'\right]\mathbf{N}$$
$$\left[\tau\zeta_1 + \tau\left(\frac{\zeta_2}{\kappa}\right)' + 2\frac{\kappa'\tau\zeta_2}{\kappa^2} - \frac{\tau^2\zeta_3}{\kappa} + \frac{\kappa'}{\kappa}\left(\frac{\zeta_3}{\kappa}\right)' + \left(\frac{\tau\zeta_3}{\kappa}\right)' + \left(\frac{\zeta_3}{\kappa}\right)'' + \frac{\kappa'^2\zeta_3}{\kappa^3} + \left(\frac{\kappa'\zeta_3}{\kappa^2}\right)'\right].$$

On derivation of the terms, last equation simplifies to

$$\delta\mathbf{T}' = \left[-\kappa^2\zeta_1 - \kappa\zeta_2' - 2\kappa'\zeta_2 + \tau\kappa\zeta_3\right]\mathbf{T} + \left[\frac{\kappa'\zeta_1}{\kappa} + \left(\kappa - \frac{\tau^2}{\kappa}\right)\zeta_2 + \frac{\zeta_2''}{\kappa} - \frac{\tau'\zeta_3}{\kappa} - 2\frac{\tau\zeta_3'}{\kappa}\right]\mathbf{N}$$
$$+ \left[\tau\zeta_1 + \frac{\tau'\zeta_2}{\kappa} + 2\frac{\tau\zeta_2'}{\kappa} - \frac{\tau^2\zeta_3}{\kappa} + \frac{\zeta_3''}{\kappa}\right]\mathbf{B} \quad (12)$$

Also, we know that

$$\mathbf{T}' = \mathbf{t}' = \mathbf{N} = \kappa\mathbf{n}.$$

This implies that

$$\delta\kappa = \delta\mathbf{T}'.\mathbf{n} = \frac{1}{\kappa}\delta\mathbf{T}'.\mathbf{N}. \quad (13)$$

Therefore, combining equations (12) and (13), we get

$$\delta\kappa = \frac{1}{\kappa}\left[\frac{\kappa'\zeta_1}{\kappa} + \left(\kappa - \frac{\tau^2}{\kappa}\right)\zeta_2 + \frac{\zeta_2''}{\kappa} - \frac{\tau'\zeta_3}{\kappa} - 2\frac{\tau\zeta_3'}{\kappa}\right] \quad (14)$$

We now compute the variation in the torsion τ i.e. $\delta\tau$. From (4), we write

$$\mathbf{N}' = -\kappa^2\mathbf{T} + \frac{\kappa'}{\kappa}\mathbf{N} + \tau\mathbf{B}.$$

Taking small variation δ on both sides, we get

$$\delta\mathbf{N}' = -\kappa^2\delta\mathbf{T} - 2\kappa\delta\kappa\mathbf{T} + \frac{\kappa'}{\kappa}\delta\mathbf{N} + \delta\left(\frac{\kappa'}{\kappa}\right)\mathbf{N} + \tau\delta\mathbf{B} + \delta\tau\mathbf{B}$$

which gives

$$\delta\mathbf{N}'.\mathbf{B} = -\kappa^2\delta\mathbf{T}.\mathbf{B} + \frac{\kappa'}{\kappa}\delta\mathbf{N}.\mathbf{B} + \delta\tau$$

or

$$\delta\tau = \delta\mathbf{N}'.\mathbf{B} + \kappa^2\delta\mathbf{T}.\mathbf{B} - \frac{\kappa'}{\kappa}\delta\mathbf{N}.\mathbf{B}$$

Using the value of $\delta\mathbf{T}$ in the last equation, we arrive at

$$\delta\tau = \delta\mathbf{N}'.\mathbf{B} + \kappa^2\left[\frac{\zeta_2\tau}{\kappa} + \left(\frac{\zeta_3}{\kappa}\right)' + \frac{\kappa'\zeta_3}{\kappa^2}\right] - \frac{\kappa'}{\kappa}\delta\mathbf{N}.\mathbf{B}$$

On simplifying this equation, we get

$$\delta\tau = \delta\mathbf{N}'.\mathbf{B} + \kappa\tau\zeta_2 + \kappa^2\left(\frac{\zeta_3}{\kappa}\right)' + \kappa'\zeta_3 - \frac{\kappa'}{\kappa}\delta\mathbf{T}'.\mathbf{B}.$$

Now using equation (12) in the last equation, we find

$$\delta\tau = \delta\mathbf{N}'.\mathbf{B} + \kappa\tau\zeta_2 + \kappa^2\left(\frac{\zeta_3}{\kappa}\right)' + \kappa'\zeta_3 - \frac{\kappa'}{\kappa}\left(\tau\zeta_1 + \frac{\tau'\zeta_2}{\kappa} + 2\frac{\tau\zeta_2'}{\kappa} - \frac{\tau^2\zeta_3}{\kappa} + \frac{\zeta_3''}{\kappa}\right) \tag{15}$$

We shall compute $\delta\mathbf{N}'.\mathbf{B}$ in order to get the value of $\delta\tau$. Since $(\delta\mathbf{N})' = \delta\mathbf{N}'$, we have

$$\delta\mathbf{N}'.\mathbf{B} = (\mathbf{B}.\delta\mathbf{N})' - \mathbf{B}'.\delta\mathbf{N}$$

Putting the value of $\mathbf{B}'$ from (4) in the above equation and simplifying, we get

$$\delta\mathbf{N}'.\mathbf{B} = (\mathbf{B}.\delta\mathbf{N})' - \frac{\kappa'}{\kappa}\mathbf{B}.\delta\mathbf{N}. \tag{16}$$

It is easy to see that

$$\mathbf{B}.\delta\mathbf{N} = (\kappa\mathbf{b}).\delta(\kappa\mathbf{n}) = \kappa^2\mathbf{b}.\delta\mathbf{n} = \kappa^2\left(\frac{1}{\kappa}\mathbf{b}.\delta\mathbf{t}'\right).$$

In view of equation-17 of [10], the above equation transforms to

$$\mathbf{B}.\delta\mathbf{N} = \kappa\left(\kappa\tau\zeta_1 + 2\tau\zeta_2' + \tau'\zeta_2 + \zeta_3'' - \tau^2\zeta_3\right). \tag{17}$$

Combining equations (16) and (17) and then putting the value of $\delta\mathbf{N}'.\mathbf{B}$ in equation (15), we get the required value of $\delta\tau$ given by equation (11). This completes the proof. □

4. Willmore energy and Total Normalcy of a Knot under infinitesimal bending with modified orthogonal frame

In this section, we analyze the curvature based energy called Willmore energy of a deformed Knot and obtain its variation under infinitesimal bending with modified orthogonal frame. Moreover, we discuss total normalcy of a knot with modified orthogonal frame which is a measure of binormal indicatrix of a Knot.

First we prove the following:

Theorem 4.1. *The variation of the Willmore energy of a knot C with modified orthogonal frame under infinitesimal bending is*

$$\delta\mathcal{W} = \int_C \frac{1}{\kappa}\left[\kappa'\zeta_1 + \left(\kappa^2 - \tau^2 + \frac{1}{\kappa''}\right)\zeta_2 + \left(3\tau' + \frac{4\tau}{\kappa'}\zeta_3\right)\right] ds \tag{18}$$

Proof. From definition, the Willmore energy of a knot C is given by

$$\mathcal{W} = \frac{1}{2}\int_C \kappa^2 ds$$

Further, the Willmore energy of a deformed knot, denoted by $\mathcal{W}_\epsilon$ is given by [4]

$$\mathcal{W}_\epsilon = \mathcal{W} + \epsilon\delta\mathcal{W}$$

where $\delta\mathcal{W} = \int_C \kappa\delta\kappa ds$ Using equation (10) to find $\delta\mathcal{W}$, we get

$$\delta\mathcal{W} = \int_C \kappa\delta\kappa ds = \int_C \frac{1}{\kappa}\left[\zeta_1\kappa' + \left(\kappa^2 - \tau^2\right)\zeta_2 + \zeta_2'' - \zeta_3\tau' - 2\zeta_3'\tau\right]$$

After a bit of computation in the terms containing ζ_2'' and ζ_3', the last equation transforms to

$$\delta\mathcal{W} = \int_C \frac{1}{\kappa}\left[\kappa'\zeta_1 + \left(\kappa^2 - \tau^2 + \frac{1}{\kappa''}\right)\zeta_2 + \left(3\tau' + \frac{4\tau}{\kappa'}\zeta_3\right)\right]ds + \int_C\left[\frac{1}{\kappa}\left(\zeta_2' - 4\tau\zeta_3\right) - \frac{1}{\kappa'}\zeta_2\right]'$$

Now, since we assume that the knot and the infinitesimal bending field are smooth and $\zeta(0) = \zeta(L)$ for the infinitesimal bending of knots in order to get closed curves, we have

$$\delta\mathcal{W} = \int_C \frac{1}{\kappa}\left[\kappa'\zeta_1 + \left(\kappa^2 - \tau^2 + \frac{1}{\kappa''}\right)\zeta_2 + \left(3\tau' + \frac{4\tau}{\kappa'}\zeta_3\right)\right]ds$$

which completes the proof. □

Now we calculate the variation of the total normalcy of knots with modified orthogonal frame under infinitesimal bending.

Theorem 4.2. *Under infinitesimal bending of a knot C with modified orthogonal frame, variation of total normalcy $\delta\omega$ is given by*

$$\begin{aligned}\delta\omega = \int_C \Bigg\{&\left[\frac{\kappa'}{\kappa\sqrt{\kappa^2+\tau^2}} + \frac{\frac{\kappa'\tau^2}{\kappa} - \kappa\kappa'\tau^2}{\sqrt{\kappa^2+\tau^2}} - \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\kappa^2\tau\right]\zeta_1 \\ &+\left[\frac{\kappa^2-\tau^2}{\kappa\sqrt{\kappa^2+\tau^2}} + \left(\frac{1}{\kappa\sqrt{\kappa^2+\tau^2}}\right)'' + \frac{\kappa\tau^2 - \kappa'\tau\tau' + \frac{\kappa'\tau\tau'}{\kappa^2}}{\sqrt{\kappa^2+\tau^2}}\right. \\ &\left.-\left(\frac{\frac{2\kappa'\tau^2}{\kappa^2} - 2\kappa'\tau^2}{\sqrt{\kappa^2+\tau^2}}\right)' + \left(2\kappa\tau\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\right)' - \kappa\tau'\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\right]\zeta_2 \\ &+\left[\left(\frac{2\tau}{\kappa\sqrt{\kappa^2+\tau^2}}\right)' - \frac{\tau'}{\kappa\sqrt{\kappa^2+\tau^2}} + \frac{\kappa'\tau^3 - \frac{\kappa'\tau^3}{\kappa^2}}{\sqrt{\kappa^2+\tau^2}} - \left(\frac{\kappa\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\right. \\ &\left.+\left(\frac{\frac{\kappa'\tau}{\kappa^2} - \kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\right)'' + \kappa\tau^2\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)' - \left(\kappa\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\right)''\right]\zeta_3\Bigg\}ds \end{aligned} \tag{19}$$

Proof. From [10], the variation in the total normalcy is given by

$$\delta\omega = \int_C \frac{\kappa\delta\kappa + \tau\delta\tau}{\sqrt{\kappa^2+\tau^2}}ds$$

which may be written as the sum of the following two integrals

$$(\delta\omega)^1 = \int_C \frac{\kappa\delta\kappa}{\sqrt{\kappa^2+\tau^2}}ds, \quad (\delta\omega)^2 = \int_C \frac{\tau\delta\tau}{\sqrt{\kappa^2+\tau^2}}ds.$$

We shall compute the values of the above two integrals in order to get the variation of the total normalcy. Putting the value of $\delta\kappa$ from equation (14) in the last equation, we get

$$(\delta\omega)^1 = \int_C \left\{ \frac{\kappa'}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_1 + \frac{\kappa^2-\tau^2}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_2 - \frac{\tau'}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_3 \right\} ds + \int_C \left\{ \frac{1}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_2'' - \frac{2\tau}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_3' \right\} ds \quad (20)$$

Second integral in the above equation can be written as

$$\int_C \left\{ \frac{1}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_2'' - \frac{2\tau}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_3' \right\} ds = \int_C \left\{ \left(\frac{1}{\kappa\sqrt{\kappa^2+\tau^2}} \right)'' \zeta_2 + \left(\frac{1}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_2' - \left(\frac{1}{\kappa\sqrt{\kappa^2+\tau^2}} \right)' \zeta_2 \right)' \right\} ds + \int_C \left\{ \left(\frac{2\tau}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_3' \right)' + \left(\frac{2\tau}{\kappa\sqrt{\kappa^2+\tau^2}} \right)' \zeta_3 \right\} ds \quad (21)$$

Putting equation (4) in the equation (20), we get the first integral as

$$(\delta\omega)^1 = \int_C \left\{ \frac{\kappa'}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_1 + \left(\frac{\kappa^2-\tau^2}{\kappa\sqrt{\kappa^2+\tau^2}} + \left(\frac{1}{\kappa\sqrt{\kappa^2+\tau^2}} \right)'' \right)\zeta_2 + \left(\left(\frac{2\tau}{\kappa\sqrt{\kappa^2+\tau^2}} \right)' - \frac{\tau'}{\kappa\sqrt{\kappa^2+\tau^2}} \right)\zeta_3 \right\} ds + \int_C \left\{ \frac{1}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta\prime - \left(\frac{1}{\kappa\sqrt{\kappa^2+\tau^2}} \right)' \zeta_2 - \frac{2\tau}{\kappa\sqrt{\kappa^2+\tau^2}}\zeta_3 \right\}' ds \quad (22)$$

Similarly by putting the value of $\delta\tau$ from equation (11) we have

$$(\delta\omega)^2 = \int_C \frac{\tau\mathcal{A}}{\sqrt{\kappa^2+\tau^2}} ds + \int_C \frac{\tau\left[\kappa\left(\kappa\tau\zeta_1 + 2\tau\zeta_2' + \tau'\zeta_2 + \zeta_3'' - \tau^2\zeta_3\right)\right]'}{\sqrt{\kappa^2+\tau^2}} ds$$

where

$$\mathcal{A} = \kappa\tau\zeta_2 + \kappa\zeta_3' - \kappa\kappa'\tau\zeta_1 - 2\kappa'\tau\zeta_2' - \kappa'\tau'\zeta_2 - \kappa'\zeta_3'' + \kappa'\tau^2\zeta_3 + \frac{\kappa'\tau\zeta_1}{\kappa} + \frac{2\kappa'^{\tau\zeta_2'}}{\kappa^2} - \frac{\kappa'\tau^2\zeta_3}{\kappa^2} + \frac{\kappa'\tau'\zeta_2}{\kappa^2} + \frac{\kappa'\zeta_3''}{\kappa^2}.$$

Now, after a simple computation for the terms containing derivatives and grouping the terms of ζ_1, ζ_2 and $\zeta 3$, we get

$$(\delta\omega)^2 = \int_C \Bigg[\left(\frac{\frac{\kappa'\tau^2}{\kappa} - \kappa\kappa'\tau^2}{\sqrt{\kappa^2+\tau^2}} - \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \right)' \kappa^2\tau \right)\zeta_1 + \left(\frac{\kappa\tau^2 - \kappa'\tau\tau' + \frac{\kappa'\tau\tau'}{\kappa^2}}{\sqrt{\kappa^2+\tau^2}} - \left(\frac{\frac{2\kappa'\tau^2}{\kappa^2} - 2\kappa'\tau^2}{\sqrt{\kappa^2+\tau^2}} \right)' + \left(2\kappa\tau \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \right)' \right)' - \kappa\tau' \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \right)' \right)\zeta_2 + \left(\frac{\kappa'\tau^3 - \frac{\kappa'\tau^3}{\kappa^2}}{\sqrt{\kappa^2+\tau^2}} - \left(\frac{\kappa\tau}{\sqrt{\kappa^2+\tau^2}} \right)' + \left(\frac{\frac{\kappa'\tau}{\kappa^2} - \kappa'\tau}{\sqrt{\kappa^2+\tau^2}} \right)'' + \kappa\tau^2 \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \right)' - \left(\kappa \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \right)' \right)'' \right)\zeta_3 + \Bigg\{ \frac{\kappa\tau}{\sqrt{\kappa^2+\tau^2}}\zeta_3 + \frac{\left(\frac{\kappa'\tau}{\kappa^2-\kappa'\tau}\right)}{\sqrt{\kappa^2+\tau^2}}\zeta_3' - \left(\frac{\frac{\kappa'\tau}{\kappa^2-\kappa'\tau}}{\sqrt{\kappa^2+\tau^2}} \right)' \zeta_3 + \frac{\frac{2\kappa'\tau^2}{\kappa^2} - 2\kappa'\tau^2}{\sqrt{\kappa^2+\tau^2}}\zeta_2 + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\mathcal{B} - 2\kappa\tau \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \right)' \zeta_2 + \kappa \left(\frac{\tau}{\sqrt{\kappa^2+\tau 2}} \right)' \zeta_3' - \left(\kappa \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}} \right)' \right)' \zeta_3 \Bigg\}' \Bigg] ds \quad (23)$$

where $\mathcal{B} = \kappa^2\tau\zeta_1 + 2\tau\zeta_2' + \tau'\zeta_2 + \zeta_3'' - \tau^2\zeta_3$.

Now, since we assume that the knot and the infinitesimal bending field are smooth and $\zeta(0) = \zeta(L)$ for the infinitesimal bending of knots, addition of equations (22 and 23) gives the variation (19) of total normalcy of knots with modified orthogonal frame. □

5. Examples

In this section we produce some examples showing the comparison of Normalcy and Willmore energy of the initial knot ($\epsilon = 0$) and the deformed knot ($\epsilon > 0$) with modified orthogonal frame for Trefoil knot, Figure eight knot, two torus knots i.e. **P3Q2** and **P4Q3** knot. First we note that theorem-2 can be re-written with modified orthogonal frame as

Theorem 5.1. *The infinitesimal bending field for the continuous regular curve* $C : \mu = \mu(u)$, $u \in \mathcal{I} \subseteq \mathcal{R}$ *with modified orthogonal frame is given by*

$$\zeta(u) = \int [P(u)\boldsymbol{N}(u) + Q(u)\boldsymbol{B}(u)]\, du$$

where $P(u)$ *and* $Q(u)$ *are arbitary integrable functions.*

We now give examples as mentioned above by noting that $P(u) = p(u)/\kappa(u)$ and $Q(u) = q(u)/\kappa(u)$ in view of the above theorem-6 where $p(u)$ and $q(u)$ are written in the examples below which are explained as: curve definition, p and q functions for field, values for ϵ and calculated values for Normalcy and Willmore energy.

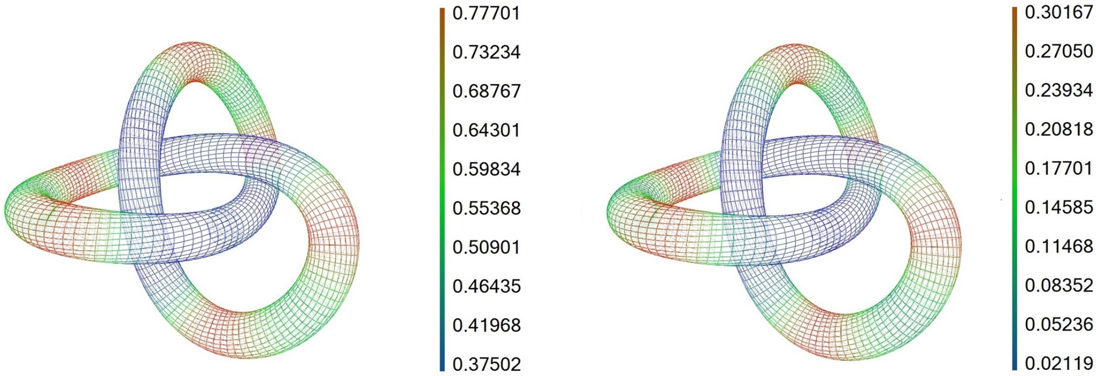

Figure 1: Trefoil total Normalcy (A) and Willmore energy (B) at $\epsilon = 0.00$

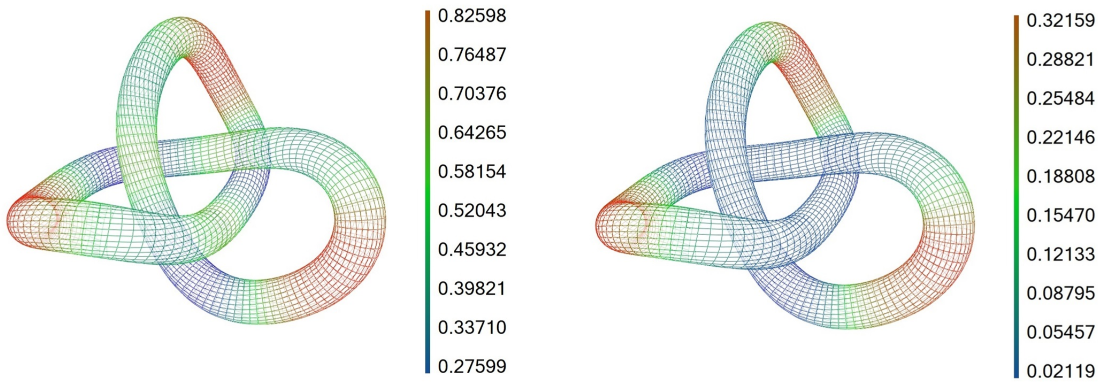

Figure 2: Trefoil total Normalcy (A) and Willmore energy (B) at $\epsilon = 0.80$

Example 5.2. *Trefoil Knot (Fig-1 and Fig-2):*

$\mu(u) = \{(sin(u) + 2 * sin(2u)), (cos(u) - 2 * cos(2u)), -1 * sin(3u)\};$
$p(u) = cos(3u),\ q(u) = sin(6u);$
$\epsilon = 0.00, \quad Total\ Normalcy = 15.62, \quad Willmore\ energy = 3.923;$
$\epsilon = 0.80, \quad Total\ Normalcy = 15.74. \quad Willmore\ energy = 3.025.$

Example 5.3. *Figure eight knot (Fig-3 and Fig-4):*

$\mu(u) = \{(cos(2u) + 2) * cos(3u)), (cos(2u) + 2) * sin(3u)), sin(4u)\};$
$p(u) = cos(6u),\ q(u) = sin(6u);$
$\epsilon = 0.00, \quad Total\ Normalcy = 23.72, \quad Willmore\ Energy = 5.98;$
$\epsilon = 2.00, \quad Total\ Normalcy = 36.85, \quad Willmore\ energy = 8.70.$

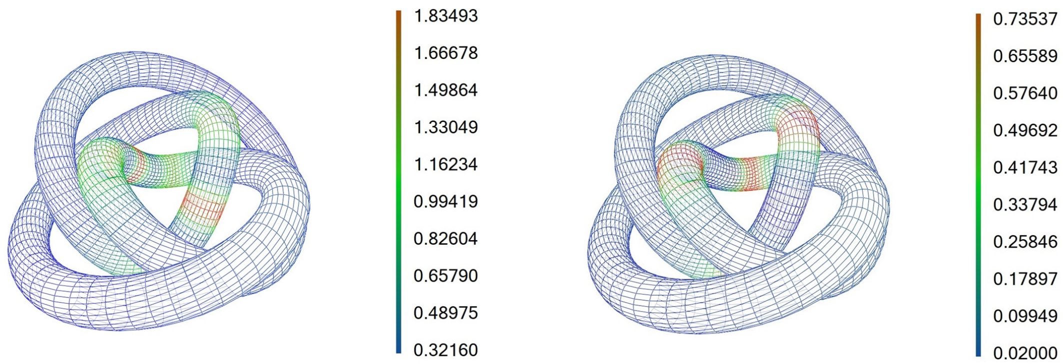

Figure 3: Figure eight knot total Normalcy (A) and Willmore energy (B) at $\epsilon = 0.00$

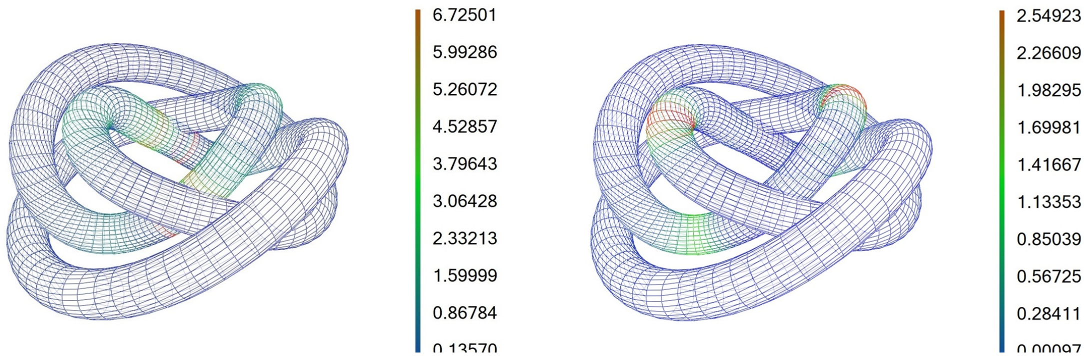

Figure 4: Figure eight knot total Normalcy (A) and Willmore energy (B) at $\epsilon = 2.00$

Example 5.4. *p3q2 torus knot (Fig-5 and Fig-6):*

$\mu(u) = \{(cos(2u) + 2) * cos(3u)), (cos(2u) + 2) * sin(3u)), -1 * sin(4u)\};$
$p(u) = cos(6u),\ q(u) = sin(6u);$
$\epsilon = 0.00, \quad Total\ Normalcy = 19.55, \quad Willmore\ Energy = 4.52;$
$\epsilon = 2.00, \quad Total\ Normalcy = 33.73, \quad Willmore\ energy = 7.90.$

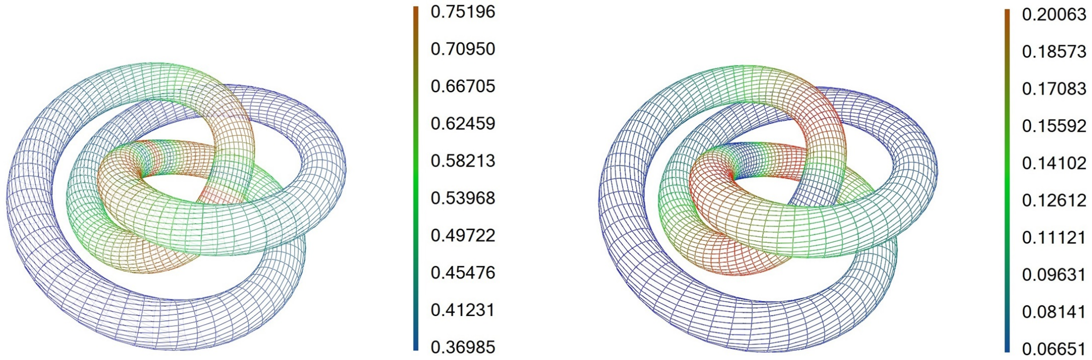

Figure 5: p3q2 total Normalcy (A) and Willmore energy (B) at $\epsilon = 0.00$

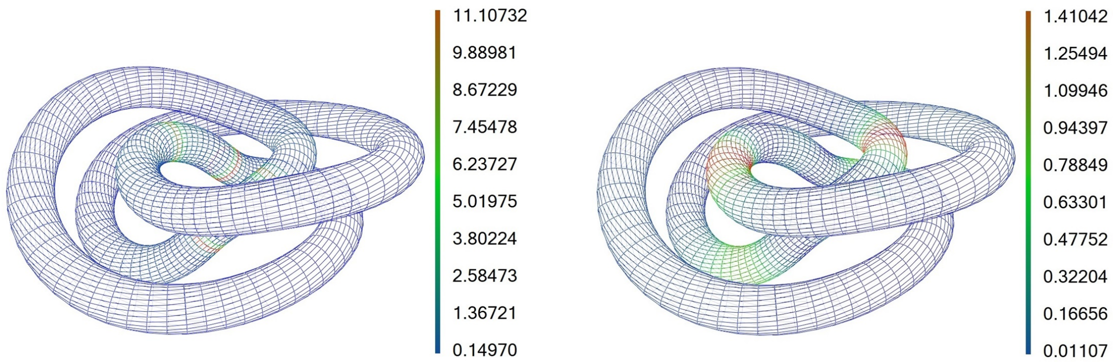

Figure 6: p3q2 total Normalcy (A) and Willmore energy (B) at $\epsilon = 2.00$

Example 5.5. *p4q3 torus knot (Fig-7 and Fig-8):*

$$\mu(u) = \{(cos(4u) + 2) * cos(3u)), (cos(4u) + 2) * sin(3u)), sin(4u)\};$$
$$p(u) = cos(2u),\ q(u) = sin(6u);$$
$$\epsilon = 0.00, \quad Total\ Normalcy = 39.80, \quad Willmore\ Energy = 6.58;$$
$$\epsilon = 1.20, \quad Total\ Normalcy = 42.92, \quad Willmore\ energy = 8.96.$$

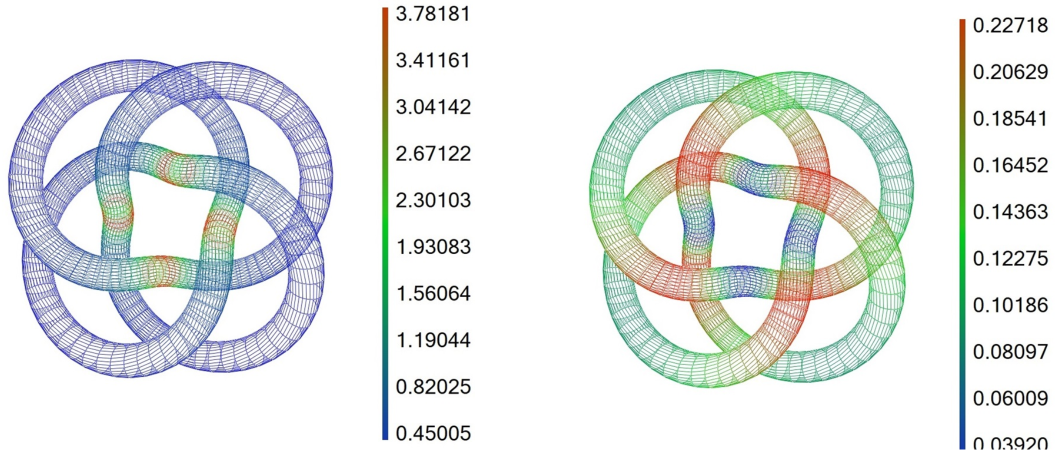

Figure 7: p4q3 total Normalcy (A) and Willmore energy (B) at $\epsilon = 0.00$

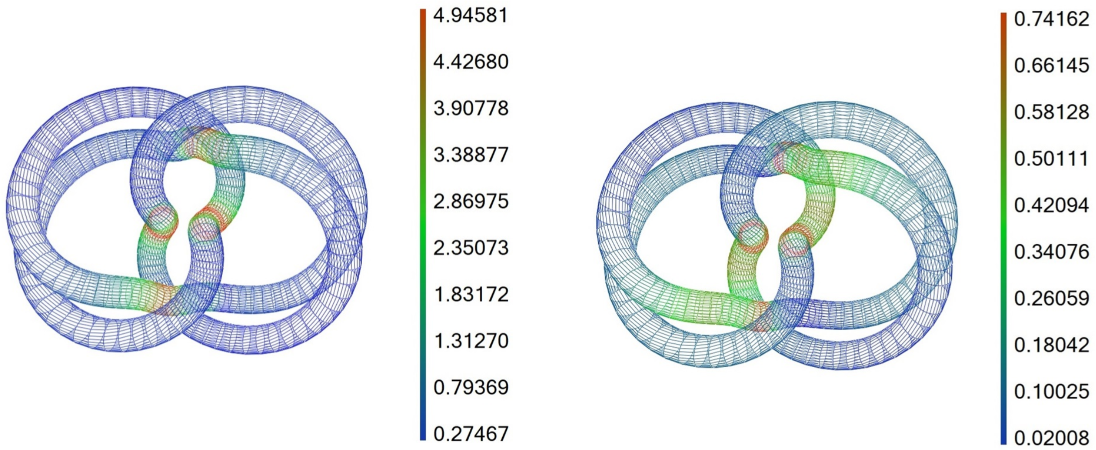

Figure 8: p4q3 total Normalcy (A) and Willmore energy (B) at $\epsilon = 1.20$

Remark 5.6. *It has been observed that all curves have given thickness and some semitransparency for better visualization effects with modified orthogonal frame.*

6. Significance of modified orthogonal frame

In differential geometry, there may exists curves whose curvature functions have discrete zero points due to which their principal normal and binormal vector fields become discontinuous. For example, in 3-dimensional Euclidean space E^3, the curve $y = x^3; z = 0$ has the following principal normal $\mathbf{n}$ and binormal $\mathbf{b}$([12])

$$\mathbf{n} = \left(-\frac{3x^3}{\left(|x|\,(1+9x^4)^{\frac{1}{2}}\right)}, \frac{x}{\left(|x|\,(1+9x^4)^{\frac{1}{2}}\right)}, 0 \right)$$

$$\begin{aligned} \mathbf{b} &= (0,0,1), \quad x > 0 \\ &= (0,0,-1), \quad x < 0 \end{aligned}$$

From the above example it is clear that the curvature is not always differentiable even if the related curve is analytic. This implies that sometimes it is hard to see if the curve determined by the given curvature and torsion, is analytic or not. Several analyses have been done to overcome this problem ([9, 19, 20]. In 1959, Nomizu proved that an analytic curve is always a Frenet curve ([9]). In [11], Sasai overcame this problem by defining a new orthogonal frame, called modified orthogonal frame, and asserted that the essential invariants for analytic curves are κ^2 and τ. This approach proved to be useful in dealing analytic curves with singularities also (see [12]).

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