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# Existence results for periodic fractional differential equation involving deformable derivative

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**Abstract.** In this paper, a differential equation problem with periodic boundary value conditions involving deformable derivative is considered. We demonstrate the existence of solution to this problem by applying the solution-tube method and Schauder fixed point theorem.

## 1. Introduction

Fractional calculus studies the eventuality of holding fractional order powers of differentiation and integration operators. Significant progress has been made in the field of fractional differential equations in the past few years since they have many important applications. In fact, fractional differential equations are widely employed to describe many phenomena in biology, physics, hydrology, chemistry, economy, nanotechnology and so on [3, 14, 17, 23–26].

The concept of deformable derivative, which is considered as fractional order derivative, was introduced and improved in [27] to surmount the lack of conformable derivative defined in [16] to include zero and negative numbers. As for the ordinary derivative, they define the deformable derivative by the help of a limit approach, in addition the ordinary derivative is linearly related to this derivative. Some studies about differential equations involving deformable derivative can be found in [1, 2, 9, 18–20].

In this work, we examine the following problem

$$\begin{cases} D^{\alpha} y(t) = \widetilde{f}(t, y(t)), & c \le t \le d, \\ y(c) = y(d), & \end{cases} \tag{1}$$

with $D^{\alpha}$ denotes the deformable fractional operator of order $\alpha$, $0 < \alpha \le 1$ and $\widetilde{f}$ is a continuous function. For the purpose of establishing existence results, we use the solution-tube method for the first time to

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deformable fractional calculus. In fact, we want to generalize the study done in [5] involving conformable derivative, where a local fractional nonlinear differential problem is treated, to the case of deformable derivative (see also [22] and [6] for the case of systems). Among the difficulties encountered in our case is that the deformable derivative has not the same properties as the conformable one mainly the product rule. We mention that the solution-tube is a successful method which has been applied to give existence results for ordinary differential problems [10, 11, 21], differential equations on times scales [4, 13, 15] and dynamic inclusions [12].

The remaining sections are structured in this way. In Section 2, the notion and properties of deformable fractional derivative and integral are presented, and we establish certain helpful outcomes. In the next section, we present the concept of solution-tube for problem (1) and show existence of solution to problem (1) with the help of this notion and Schauder fixed point theorem. We show also that, for our problem, there is an equivalence between the solution-tube method and that of upper and lower solutions. Finally, Section 4 is devoted to illustrative examples.

## 2. Preliminaries

Some necessary definitions and properties of the deformable fractional calculus (as given in [27]) which will be used in the remainder of the paper are presented in this part.

**Definition 2.1.** ([27]) *Let $y : [c, d] \to \mathbb{R}$ and $\alpha \in [0, 1]$. The deformable derivative of order $\alpha$ at $t \in (c, d)$ is defined by*

$$D^{\alpha}y(t) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta)y(t + \varepsilon\alpha) - y(t)}{\varepsilon},$$

*where $\alpha + \beta = 1$. The function $y$ is named $\alpha$-differentiable at $t$ if this limit exists.*

The following theorem presents the connection between the standard derivative and the deformable derivative.

**Theorem 2.2.** ([27]) *Let $y$ be defined on $(c, d)$. The function $y$ is differentiable at a point $t \in (c, d)$ if and only if it is $\alpha$-differentiable at that point for any $\alpha \in [0, 1]$. Furthermore, we have*

$$D^{\alpha}y(t) = \beta y(t) + \alpha Dy(t). \tag{2}$$

**Definition 2.3.** ([27]) *The $\alpha-$deformable integral of a continuous function $y$ defined on $[c, d]$ is given by*

$$I_c^{\alpha}y(t) = \frac{1}{\alpha} e^{-\frac{\beta}{\alpha}t} \int_c^t e^{\frac{\beta}{\alpha}s} y(s)ds, \quad \text{where } \alpha + \beta = 1,\ \alpha \in (0, 1].$$

**Remark 2.4.** *We get the usual derivative and the usual integral if $\alpha = 1$ and $\beta = 0$.*

The following theorem gathers the most important properties satisfied by the operators $D^{\alpha}$ and $I_c^{\alpha}$.

**Theorem 2.5.** ([18, 27]) *Let $\alpha, \alpha' \in (0, 1]$ such that $\alpha + \beta = 1$ and $\alpha' + \beta' = 1$. Then we have the followings:*

1. *If $y$ is $\alpha-$differentiable at a point $t$, thus $y$ is continuous at this point.*
2. *The operators $D^{\alpha}$ and $I_c^{\alpha}$ are linear.*
3. *Commutativity : $D^{\alpha}D^{\alpha'} = D^{\alpha'}D^{\alpha}$ and $I_c^{\alpha}I_c^{\alpha'} = I_c^{\alpha'}I_c^{\alpha}$.*
4. *$D^{\alpha}(k) = \beta k$, where $k$ is a constant.*
5. *$D^{\alpha}(yg) = D^{\alpha}(y)g + \alpha y Dg$.*
6. *If $y$ is continuous on $[c, d]$, then $I_c^{\alpha}y$ is $\alpha-$differentiable in $(c, d)$ and in addition*

$$D^{\alpha}(I_c^{\alpha}y)(t) = y(t), \tag{3}$$

$$I_c^{\alpha}(D^{\alpha}y)(t) = y(t) - y(c)e^{\frac{\beta}{\alpha}(c-t)}. \tag{4}$$

The following results are useful to demonstrate an existence result in the next section. Using the definition of the deformable and standard techniques, we can show the following result.

**Proposition 2.6.** *Let $\alpha \in (0,1]$ and $y : \mathbb{R} \to \mathbb{R}$, $\alpha$-differentiable at $t \in [c,d]$ and $y(t) \neq 0$. Then the absolute value function $\left|y\right|$ is $\alpha$-differentiable at $t$ and*

$$D^{\alpha}\left|y(t)\right| = \frac{y(t)}{\left|y(t)\right|}D^{\alpha}y(t).$$

In the following, the Banach space of all real valued continuous functions defined on $[c,d]$ is denoted by $C([c,d],\mathbb{R})$ and it is endowed with the norm

$$\left\|y\right\| = \sup_{t\in[c,d]}\left|y(t)\right|,$$

and we define the following space

$$\mathcal{F}^{\alpha} := \{y \text{ is } \alpha - \text{differentiable and } D^{\alpha}y \in C([c,d],\mathbb{R}),\ 0 < \alpha < 1\}.$$

**Lemma 2.7.** *Let $z \in \mathcal{F}^{\alpha}$. If $D^{\alpha}z(t) < 0$ on $\{t \in [c,d], z(t) > 0\}$ and if one of the two following inequalities is fulfilled*
*i) $z(c) \leq 0$,*
*ii) $z(c) \leq z(d)$,*
*then we have $z(t) \leq 0$ for all $t \in [c,d]$.*

*Proof.* Accept that there is $t' \in [c,d]$ so that $z(t') > 0$. In this case, there is $t_0 \in [c,d]$ with $z(t_0) = \max_{t\in[c,d]} z(t) > 0$. We have two cases.

*i)* If $t_0 > c$, then we can find an interval $[t_1,t_0] \subset [c,d]$ so that $z(t) > 0$, for all $t \in [t_1,t_0]$. It follows from the hypothesis $D^{\alpha}z(t) < 0$ and the definition of the deformable integral that $I^{\alpha}_{t_1}(D^{\alpha}z)(t_0) < 0$ but from Theorem 2.5, we have $I^{\alpha}_{t_1}(D^{\alpha}z)(t_0) = z(t_0) - e^{\frac{\beta}{\alpha}(t_1-t_0)}z(t_1)$. Contradiction with the fact that $z(t_0)$ is a maximum.

*ii)* The case when $t_0 = c$ is impossible since by the hypothesis $z(c) \leq 0$ or $z(c) \leq z(d)$. Then $z(t) \leq 0$ for every $t \in [c,d]$. □

**Proposition 2.8.** *Let $l \in C([c,d],\mathbb{R})$, $\alpha \in (0,1]$ and $q$ be a real constant. The function $y$ is a solution to problem*

$$\begin{cases} D^{\alpha}y(t) + qy(t) = l(t), & c \leq t \leq d, \\ c_0y(c) - d_0y(d) = \lambda_0, \end{cases} \tag{5}$$

*if and only if it is written in the next form*

$$y(t) := \Delta e^{-(\frac{\beta+q}{\alpha})(t-c)} + \frac{1}{\alpha}\int_c^t e^{-(\frac{\beta+q}{\alpha})(t-s)}l(s)ds, \tag{6}$$

*where $\Delta := \left(\frac{\lambda_0}{d_0\Omega} + \frac{1}{\alpha\Omega}\int_c^d e^{-(\frac{\beta+q}{\alpha})(d-s)}l(s)ds\right)$ and $\Omega = \frac{c_0}{d_0} - e^{-(\frac{\beta+q}{\alpha})(d-c)}$ (supposed different to zero).*

*Proof.* We suppose that $y$ is a solution of (5). Using the formula given in Theorem 2.2, we find

$$\begin{aligned} D^{\alpha}y(t) + qy(t) = \widetilde{f}(t,y(t)) &\Leftrightarrow \alpha Dy(t) + (\beta+q)y(t) = l(t) \\ &\Leftrightarrow Dy(t) + \frac{(\beta+q)}{\alpha}y(t) = \frac{1}{\alpha}l(t) \\ &\Leftrightarrow Dy(t)e^{\frac{(\beta+q)}{\alpha}t} + \frac{(\beta+q)}{\alpha}y(t)e^{\frac{(\beta+q)}{\alpha}t} = \frac{1}{\alpha}l(t)e^{\frac{(\beta+q)}{\alpha}t} \\ &\Leftrightarrow D(y(t)e^{\frac{(\beta+q)}{\alpha}t}) = \frac{1}{\alpha}l(t)e^{\frac{(\beta+q)}{\alpha}t}. \end{aligned}$$

Integrating from $c$ to $t$, we find

$$y(t)e^{\frac{(\beta+q)}{\alpha}t} - y(c)e^{\frac{(\beta+q)}{\alpha}c} = \frac{1}{\alpha}\int_c^t l(s)e^{\frac{(\beta+q)}{\alpha}s}\,ds.$$

The use of boundary conditions gives that

$$\begin{aligned} y(t) &= \left(\frac{\lambda_0}{d_0\Omega} + \frac{1}{\alpha\Omega}\int_c^d e^{-(\frac{\beta+q}{\alpha})(d-s)}l(s)ds\right)e^{-(\frac{\beta+q}{\alpha})(t-c)} + \frac{1}{\alpha}\int_c^t e^{-(\frac{\beta+q}{\alpha})(t-s)}l(s)ds. \\ &= \Delta e^{-(\frac{\beta+q}{\alpha})(t-c)} + \frac{1}{\alpha}\int_c^t e^{-(\frac{\beta+q}{\alpha})(t-s)}l(s)ds. \end{aligned}$$

Conversely, let $y : [c,d] \to \mathbb{R}$ be the function defined by (6). Applying $D^\alpha$ to either sides of the expression (6) and using the formula given in Theorem 2.2, we obtain

$$\begin{aligned} D^\alpha y(t) &= \alpha\left(-\Delta\frac{(\beta+q)}{\alpha}e^{-(\frac{\beta+q}{\alpha})(t-c)} - \frac{(\beta+q)}{\alpha}\int_c^t e^{-(\frac{\beta+q}{\alpha})(t-s)}l(s)ds + \frac{1}{\alpha}l(t)\right) \\ &\quad + \beta\left(\Delta e^{-(\frac{\beta+q}{\alpha})(t-c)} + \frac{1}{\alpha}\int_c^t e^{-(\frac{\beta+q}{\alpha})(t-s)}l(s)ds\right) \\ &= l(t) - q(\left(\Delta e^{-(\frac{\beta+q}{\alpha})(t-c)} + \frac{1}{\alpha}\int_c^t e^{-(\frac{\beta+q}{\alpha})(t-s)}l(s)ds\right) \\ &= l(t) - qy(t). \end{aligned}$$

So $D^\alpha y(t) + qy(t) = l(t)$. We verify now the limit conditions. We have

$$\begin{aligned} y(c) &= \Delta, \\ y(d) &= \Delta + \frac{1}{\alpha}\int_c^d e^{-(\frac{\beta+q}{\alpha})(d-s)}l(s)ds. \end{aligned}$$

Evaluating $c_0y(c) - d_0y(d)$ and using the fact that $\Omega = \frac{c_0}{d_0} - e^{-(\frac{\beta+q}{\alpha})(d-c)}$, we find

$$c_0y(c) - d_0y(d) = \lambda_0.$$

We deduce that $y$ given by (6) is a solution to (5). □

The expression of the solution $y$ can be written by using the Green function $\Pi$ defined by

$$\Pi(t,s) = \frac{e^{-\left(\frac{\beta+q}{\alpha}\right)(t-s)}}{\alpha(c_0 - d_0e^{-\left(\frac{\beta+q}{\alpha}\right)(d-c)})}\begin{cases} c_0, & c \le s \le t \le d, \\ d_0e^{-\left(\frac{\beta+q}{\alpha}\right)(d-c)}, & c \le t \le s \le d, \end{cases}$$

as follows

$$y(t) = \int_c^d \Pi(t,s)l(s)ds + \frac{\lambda_0}{d_0\Omega}e^{-(\frac{\beta+q}{\alpha})(t-c)}.$$

Proposition 2.8 is sufficient for our propose. However, it can be generalized to the case where $q$ is a function as we will see in the next proposition.

**Proposition 2.9.** *Let $l \in C([c,d],\mathbb{R})$, $q \in C([c,d],\mathbb{R})$ and $0 < \alpha \le 1$. The function $y$ is a solution to problem*

$$\begin{cases} D^\alpha y(t) + q(t)y(t) = l(t), & c \le t \le d, \\ c_0y(c) - d_0y(d) = \lambda_0, \end{cases} \tag{7}$$

*if and only if it is written in the following form*

$$y(t) := \frac{\psi(c)}{\psi(t)}\Delta' + \frac{1}{\alpha\psi(t)}\int_c^t \psi(s)l(s)ds,$$

*where*

$$\Delta' := \left(\frac{\lambda_0}{d_0\Omega'} + \frac{1}{\alpha\Omega'\psi(d)}\int_c^d \psi(s)l(s)ds\right), \quad \psi(t) := e^{\int \frac{(\beta+q(t))}{\alpha}dt},$$

*and* $\Omega' := \frac{c_0}{d_0} - \frac{\psi(c)}{\psi(d)}$ *(supposed different to zero).*

*Proof.* Suppose that $y$ is a solution to (7), then

$$\begin{aligned} D^\alpha y(t) + q(t)y(t) = \widetilde{f}(t, y(t)) &\Leftrightarrow \alpha Dy(t) + (\beta + q(t))y(t) = l(t) \\ &\Leftrightarrow Dy(t) + \frac{(\beta + q(t))}{\alpha}y(t) = \frac{1}{\alpha}l(t) \\ &\Leftrightarrow Dy(t)e^{\int \frac{(\beta+q(t))}{\alpha}dt} + \frac{(\beta + q(t))}{\alpha}y(t)e^{\int \frac{(\beta+q(t))}{\alpha}dt} = \frac{1}{\alpha}l(t)e^{\int \frac{(\beta+q(t))}{\alpha}dt} \\ &\Leftrightarrow D(y(t)e^{\int \frac{(\beta+q(t))}{\alpha}dt}) = \frac{1}{\alpha}l(t)e^{\int \frac{(\beta+q(t))}{\alpha}dt}. \end{aligned}$$

Integrating from $c$ to $t$ and setting $\psi(t) = e^{\int \frac{(\beta+q(t))}{\alpha}dt}$, we obtain

$$y(t)\psi(t) - y(c)\psi(c) = \frac{1}{\alpha}\int_c^t \psi(s)l(s)ds.$$

Employing the boundary conditions, we find that

$$y(t) = \frac{\psi(c)}{\psi(t)}\Delta' + \frac{1}{\alpha\psi(t)}\int_c^t \psi(s)l(s)ds.$$

The inverse implication can be obtained as in Proposition 2.8. □

## 3. Main Results

Our aim is to provide an existence result for problem (1). Thus we introduce the definition of solution-tube to (1).

**Definition 3.1.** *Let* $(w, \Gamma) \in \mathcal{F}^\alpha \times \mathcal{F}^\alpha$ *with* $\Gamma(t) \geq 0$ *for all* $t \in [c, d]$*. The couple* $(w, \Gamma)$ *is called a solution-tube to problem* (1) *if*

**(i)** $(x - w(t))(\widetilde{f}(t, x) - D^\alpha w(t)) \leq \Gamma(t)D^\alpha\Gamma(t)$ *for all* $t \in [c, d]$ *and for all* $x \in \mathbb{R}$ *when* $|x - w(t)| = \Gamma(t)$.

**(ii)** $D^\alpha w(t) = \widetilde{f}(t, w(t))$ *and* $D^\alpha\Gamma(t) = 0$ *for each* $t \in [c, d]$ *when* $\Gamma(t) = 0$.

**(iii)** $|w(d) - w(c)| \leq \Gamma(c) - \Gamma(d)$.

Let $\mathbf{Tub}(w, \Gamma)$ denote the set defined by

$$\mathbf{Tub}(w, \Gamma) := \left\{y \in \mathcal{F}^\alpha : \quad \left|y(t) - w(t)\right| \leq \Gamma(t), \ \forall t \in [c, d]\right\}.$$

The subsequent modified problem of problem (1) is considered

$$\begin{cases} D^\alpha y(t) + \alpha y(t) = \widetilde{f}(t, \widetilde{y}(t)) + \alpha\widetilde{y}(t), & c \leq t \leq d, \\ y(c) = y(d), \end{cases} \tag{8}$$

with

$$\widetilde{y}(t) = \begin{cases} \frac{\Gamma(t)}{\left|y(t)-w(t)\right|}(y(t)-w(t)) + w(t), & \text{when } \left|y(t)-w(t)\right| > \Gamma(t), \\ y(t), & \text{when } \left|y(t)-w(t)\right| \leq \Gamma(t). \end{cases}$$

**Remark 3.2.** *It is clear that a solution $y$ of (8) so that for every $t \in [c,d]$ : $\left|y(t)-w(t)\right| \leq \Gamma(t)$ is furthermore a solution for problem* (1).

**Lemma 3.3.** *The periodic problem*

$$\begin{cases} D^{\alpha}y(t) + \alpha y(t) &= l(t), \quad c \leq t \leq d, \\ y(c) &= y(d), \end{cases}$$

*with $l \in C([c,d],\mathbb{R})$, has exactly one solution $y \in \mathcal{F}^{\alpha}$ given by*

$$y(t) := \int_c^d \mathbf{\Pi}(t,s)l(s)ds,$$

*where*

$$\mathbf{\Pi}(t,s) = \frac{e^{-\frac{1}{\alpha}(t-s)}}{\alpha(1-e^{\frac{1}{\alpha}(d-c)})}\begin{cases} 1, & c \leq s \leq t \leq d, \\ e^{-\frac{1}{\alpha}(d-c)}, & c \leq t \leq s \leq d. \end{cases}$$

We define now the operator $A : C([c,d],\mathbb{R}) \to C([c,d],\mathbb{R})$ by

$$Ay(t) := \int_c^d \mathbf{\Pi}(t,s)(\widetilde{f}(t,\widetilde{y}(t)) + \alpha\widetilde{y}(t))ds.$$

We note that the fixed points of the operators $A$ coincide with the solutions of problem (8). Then, showing that problem (8) has solutions is equivalent to show that the operator $A$ has fixed points. To achieve this, we employ Schauder fixed point theorem. The following proposition insures the compactness of $A$.

**Proposition 3.4.** *If $(w,\Gamma) \in \mathcal{F}^{\alpha} \times \mathcal{F}^{\alpha}$ is a solution-tube of problem* (1)*, thus the operator $A : C([c,d],\mathbb{R}) \to C([c,d],\mathbb{R})$ is compact.*

*Proof.* First, we verify that $A$ is a continuous operator. Let $\{y_n\}$ be a sequence of $C([c,d],\mathbb{R})$ which converges to $y \in C([c,d],\mathbb{R})$. We have

$$\begin{aligned} \left|A(y_n(t)) - A(y(t))\right| &= \left|\int_c^d \mathbf{\Pi}(t,s)(\widetilde{f}(s,\widetilde{y_n}(s)) + \alpha\widetilde{y_n}(s))ds - \int_c^d \mathbf{\Pi}(t,s)(\widetilde{f}(s,\widetilde{y}(s)) + \alpha\widetilde{y}(s))ds\right| \\ &\leq M\left(\left|\int_c^d (\widetilde{f}(s,\widetilde{y_n}(s)) - \widetilde{f}(s,\widetilde{y}(s)))ds\right| + \alpha\int_c^d \left|\widetilde{y_n}(s) - \widetilde{y}(s)\right| ds\right), \end{aligned}$$

where $M := \max_{t,s\in[c,d]} |\mathbf{\Pi}(t,s)|$. From the definition of $\widetilde{y}$, we remark that $\widetilde{y}$ is bounded, i.e., it exists a constant $T > 0$ for which $\left\|\widetilde{y}\right\| \leq T$. Thus, we can find $N_0 \in \mathbb{N}$ to get $\left\|\widetilde{y_n}\right\| \leq T$ for all $n \geq N_0$. We state that $\widetilde{f}$ is uniformly continuous on $[c,d] \times B_T(0)$. Consequently, for $\varepsilon > 0$ fixed, we can find $\delta > 0$ so that if $|x' - x| < \delta < \frac{\varepsilon}{2\alpha M(d-c)}$, for every $x, x' \in \mathbb{R}$, then

$$\left|\widetilde{f}(s,x') - \widetilde{f}(s,x)\right| < \frac{\varepsilon}{2M(d-c)}, \quad \forall s \in [c,d].$$

By assumption, there is an index $N'$ such tat $\left\|\widetilde{y_n}(t) - \widetilde{y}(t)\right\| < \delta$ for $n > N'$. In this case, we have

$$\left|A(y_n(t)) - A(y(t))\right| < M\left(\frac{\varepsilon}{2M(d-c)}\int_c^d ds + \alpha\frac{\varepsilon}{2\alpha M(d-c)}\int_c^d ds\right) = \varepsilon.$$

Which ensures the continuity of $A$.

We verify now that the set $A(C([c,d],\mathbb{R}))$ is relatively compact. We start by showing that it is uniformly bounded. For $y \in C([c,d],\mathbb{R})$, we find

$$\begin{aligned}\left|A(y(t))\right| &= \left|\int_c^d \mathbf{\Pi}(t,s)(\widetilde{f}(s,\widetilde{y}(s)) + \alpha\widetilde{y}(s))ds\right| \\ &\le M\left|\int_c^d (\widetilde{f}(s,\widetilde{y}(s)) + \alpha\widetilde{y}(s))ds\right|.\end{aligned}$$

Using the fact that $\left\|\widetilde{y}\right\| \le T$ for all $s \in [c,d]$ and the continuity of $\widetilde{f}$ on $[c,d]\times B_T(0)$, we can find $L > 0$ so that $\left|\widetilde{f}(s,\widetilde{y}(s)\right| \le L$ for all $s \in [c,d]$. We obtain

$$\left|A(y(t))\right| \le M(L+\alpha T)(d-c).$$

Then the set $A(C([c,d],\mathbb{R}))$ is uniformly bounded.

Let us now show that $A(C([c,d],\mathbb{R}))$ is equicontinuous. Let $r_1, r_2 \in [c,d]$ with $r_2 > r_1$ and $y \in C([c,d],\mathbb{R})$, we have

$$\begin{aligned}\left|A(y(r_2)) - A(y(r_1))\right| &= \left|\int_c^d (\mathbf{\Pi}(r_2,s) - \mathbf{\Pi}(r_1,s))(\widetilde{f}(s,\widetilde{y}(s)) + \alpha\widetilde{y}(s))ds\right| \\ &\le (L+\alpha T)\left|\int_c^d \mathbf{\Pi}(r_2,s)ds - \int_c^d \mathbf{\Pi}(r_1,s)ds\right| \\ &= (L+\alpha T)\left|\int_c^{r_2} \mathbf{\Pi}(r_2,s)ds + \int_{r_2}^d \mathbf{\Pi}(r_2,s)ds - \int_c^{r_1} \mathbf{\Pi}(r_1,s)ds\right. \\ &\quad \left. - \int_{r_1}^d \mathbf{\Pi}(r_1,s)ds\right| \\ &= (L+\alpha T)\left|\int_c^{t_1} (\mathbf{\Pi}(r_2,s) - \mathbf{\Pi}(r_1,s))ds\right. \\ &\quad \left. + \int_{r_1}^{r_2} (\mathbf{\Pi}(r_2,s) - \mathbf{\Pi}(r_1,s))\, ds + \int_{r_2}^d (\mathbf{\Pi}(r_2,s) - \mathbf{\Pi}(r_1,s))\, ds\right| \\ &\le (L+\alpha T)\left(\frac{\left|e^{\frac{-1}{\alpha}r_2} - e^{\frac{-1}{\alpha}r_1}\right|}{\alpha(1-e^{\frac{1}{\alpha}(c-d)})}\left(\left|\int_c^{r_1} e^{\frac{s}{\alpha}}ds + \int_{r_2}^d e^{\frac{s+c-d}{\alpha}}ds\right|\right)\right) \\ &\quad + (L+\alpha T)\left|\int_{r_1}^{r_2} (\mathbf{\Pi}(r_2,s)ds - \mathbf{\Pi}(r_1,s))\, ds\right| \\ &\le 2(L+\alpha T)\left(K(d-c)\left|e^{\frac{-1}{\alpha}r_2} - e^{\frac{-1}{\alpha}r_1}\right| + M(r_2 - r_1)\right),\end{aligned}$$

where $K = \max\limits_{s\in[c,d]}\left\{\frac{\frac{1}{\alpha}s}{\alpha(1-e^{\frac{1}{\alpha}(c-d)})}, \frac{\frac{1}{\alpha}(s+c-d)}{\alpha(1-e^{\frac{1}{\alpha}(c-d)})}\right\}$. So $\left|A(y(r_2)) - A(y(r_2))\right| \to 0$ as $r_2 \to r_1$. Therefore, $A(C([c,d],\mathbb{R}))$ is equicontinuous. Applying Arzela-Ascoli Theorem, we conclude that $A(C([c,d],\mathbb{R}))$ is relatively compact subset of $C([c,d],\mathbb{R}))$. This implies that the operator $A$ is compact. □

**Theorem 3.5.** *If $(w,\Gamma) \in \mathcal{F}^\alpha \times \mathcal{F}^\alpha$ is a solution-tube of problem* (1)*, then there is at least one solution to problem* (1).

*Proof.* The operator $A$ is compact from Proposition 3.4. Using Schauder fixed point theorem, we conclude that $A$ possesses a fixed point $y \in \mathcal{F}^\alpha$. From Lemma 2.7, this fixed point is a solution to problem (8). Then it suffices to demonstrate that each solution $y$ to problem (8), verify $y \in \mathbf{Tub}(w,\Gamma)$. Let $\mathcal{S}$ be the set defined by

$$\mathcal{S} := \left\{t \in [c,d],\ \left|y(t) - w(t)\right| > \Gamma(t)\right\}.$$

We will show that $\mathcal{S} = \emptyset$. Let $t \in \mathcal{S}$. In the case when $\Gamma(t) > 0$, by using the definition of the solution-tube, we get

$$\begin{aligned}
D^\alpha\left(\left|y(t)-w(t)\right|-\Gamma(t)\right) &= \frac{(y(t)-w(t))(D^\alpha y(t)-D^\alpha w(t))}{\left|y(t)-v(t)\right|}-D^\alpha\Gamma(t)\\
&= \frac{(y(t)-w(t))}{\left|y(t)-w(t)\right|}\left(\widetilde{f}(t,\widetilde{y}(t))+\alpha\widetilde{y}(t)-\alpha y(t)-D^\alpha w(t)\right)-D^\alpha\Gamma(t)\\
&= \frac{(\widetilde{y}(t)-w(t))}{\Gamma(t)}\left(\widetilde{f}(t,\widetilde{y}(t))-D^\alpha w(t)\right)\\
&\quad+\alpha\frac{(y(t)-w(t))}{\left|y(t)-w(t)\right|}(\widetilde{y}(t)-y(t))-D^\alpha\Gamma(t)\\
&\le \frac{\Gamma(t)D^\alpha\Gamma(t)}{\Gamma(t)}+\alpha\frac{(y(t)-w(t))}{\left|y(t)-w(t)\right|}((\widetilde{y}(t)-w(t))\\
&\quad+(w(t)-y(t)))-D^\alpha\Gamma(t)\\
&\le \alpha(\Gamma(t)-\left|y(t)-w(t)\right|)\\
&< 0 \qquad\qquad (\text{since } t\in\mathcal{S}).
\end{aligned}$$

Consider now the case $t \in \mathcal{S}$ and $\Gamma(t) = 0$. From the definition of solution-tube, we get $D^\alpha\Gamma(t) = 0$. Therefore

$$\begin{aligned}
D^\alpha\left(\left|y(t)-v(t)\right|-\Gamma(t)\right) &= \frac{(y(t)-w(t))(D^\alpha y(t)-D^\alpha w(t))}{\left|y(t)-v(t)\right|}-D^\alpha\Gamma(t)\\
&= \frac{(y(t)-w(t))}{\left|y(t)-w(t)\right|}\left(\widetilde{f}(t,\widetilde{y}(t))+\alpha\widetilde{y}(t)-\alpha y(t)-D^\alpha w(t)\right)\\
&= \frac{(y(t)-w(t))}{\left|y(t)-w(t)\right|}\left(\widetilde{f}(t,w(t))+\alpha w(t)-\alpha y(t)-D^\alpha w(t)\right)\\
&= -\alpha\frac{(y(t)-w(t))}{\left|y(t)-w(t)\right|}(y(t)-w(t)) \quad \text{because } D^\alpha w(t)=\widetilde{f}(t,w(t))\\
&= -\alpha\left|y(t)-w(t)\right|\\
&< 0.
\end{aligned}$$

Let $z(t) := \left|y(t)-w(t)\right| - \Gamma(t)$. We have $D^\alpha z(t) < 0$ on $\mathcal{S} := \{t \in [c,d]\,,\ z(t) > 0\}$. In addition, since $(w,\Gamma)$ is a solution-tube to problem (1), we get

$$z(c) - z(d) \le |w(c) - w(d)| - (\Gamma(c) - \Gamma(d)) \le 0.$$

So $z(c) \le z(d)$. The conditions of Lemma 2.7 are satisfied, then we conclude that $z(t) \le 0$, for each $t \in [c,d]$ which means that $\mathcal{S} = \emptyset$. Then $y \in \mathbf{Tub}(w,\Gamma)$. Therefore $y$ is a solution to problem (1). □

In the rest of this section, we will show that the solution-tube method and the lower and upper solutions method are equivalent. To accomplish that, we start by presenting the next definition.

**Definition 3.6.** *A function $\overline{y} \in \mathcal{F}^\alpha$ is an upper solution of problem* (1) *if*

**i)** $D^\alpha\overline{y} \ge \widetilde{f}(t,\overline{y}), \quad \forall t \in [c,d]\,.$

**ii)** $\overline{y}(c) \ge \overline{y}(d)$.

*A function* $\underline{y} \in \mathcal{F}^{\alpha}$ *is a lower solution of problem* (1) *if*

**i)** $D^{\alpha}\underline{y} \leq \widetilde{f}(t, \underline{y}), \quad \forall t \in [c, d]$.

**ii)** $\underline{y}(c) \leq \underline{y}(d)$.

**Proposition 3.7.** *The following claims are equivalent:*

**(A)** *There exist* $\underline{y}, \overline{y}$ *a lower and an upper solutions to* (1) *such that* $\underline{y} \leq \overline{y}$.

**(D)** *There is* $(w, \Gamma)$ *a solution-tube of* (1).

*Proof.* Assume that (**D**) is satisfied. Let $\underline{y} = w - \Gamma$ and $\overline{y} = w + \Gamma$. Since $\left|\underline{y} - w\right| = \left|\overline{y} - w\right| = \Gamma$, from the definition of solution-tube, we have

$$\begin{cases} \left(\underline{y}(t) - \frac{\underline{y}+\overline{y}}{2}(t)\right)\left(\widetilde{f}(t, \underline{y}) - D^{\alpha}\underline{y}(t)\right) \leq \left(\frac{\overline{y}-\underline{y}}{2}(t)\right) D^{\alpha}\left(\frac{\overline{y}-\underline{y}}{2}\right)(t), \\ \left(\overline{y}(t) - \frac{\underline{y}+\overline{y}}{2}(t)\right)\left(\widetilde{f}(t, \overline{y}) - D^{\alpha}\overline{y}(t)\right) \leq \left(\frac{\overline{y}-\underline{y}}{2}(t)\right) D^{\alpha}\left(\frac{\overline{y}-\underline{y}}{2}\right)(t). \end{cases}$$

Using the fact that $\underline{y} \leq \overline{y}$, we can verify that

$$\begin{cases} D^{\alpha}\overline{y} \geq \widetilde{f}(t, \overline{y}) \\ D^{\alpha}\underline{y} \leq \widetilde{f}(t, \underline{y}). \end{cases}$$

.

In addition, from condition **iii)** of the definition of the solution-tube, we can verify that

$$\underline{y}(c) - \underline{y}(d) \leq 0 \leq \overline{y}(c) - \overline{y}(d).$$

We show now that if (**A**) is fulfilled, then (**D**) is also satisfied.
For this, let $w := \frac{\underline{y}+\overline{y}}{2}$ and $\Gamma(t) := \frac{\underline{y}-\overline{y}}{2}$. For $x \in \mathbb{R}$ so that $|x - w(t)| = \Gamma(t)$, i.e., $x = \underline{y}(t)$ or $x = \overline{y}(t)$, we have

$$\begin{aligned} (x - w(t))(\widetilde{f}(t, x) - D^{\alpha}w) &= \begin{cases} \left(\underline{y} - \frac{\underline{y}+\overline{y}}{2}\right)(\widetilde{f}(t, \underline{y}) - D^{\alpha}(\frac{\underline{y}+\overline{y}}{2})(t)) \\ \left(\overline{y} - \frac{\underline{y}+\overline{y}}{2}\right)(\widetilde{f}(t, \overline{y}) - D^{\alpha}(\frac{\underline{y}+\overline{y}}{2})(t)) \end{cases} \\ &\leq \begin{cases} \left(\frac{\underline{y}-\overline{y}}{2}\right)\left(D^{\alpha}\underline{y}(t) - D^{\alpha}(\frac{\underline{y}+\overline{y}}{2})(t)\right) \\ \left(\frac{\overline{y}-\underline{y}}{2}\right)\left(D^{\alpha}\overline{y}(t) - D^{\alpha}(\frac{\underline{y}+\overline{y}}{2})(t)\right) \end{cases} \\ &= \Gamma(t) D^{\alpha}\Gamma(t). \end{aligned}$$

Moreover, we can verify that $|w(d) - w(c)| \leq \Gamma(c) - \Gamma(d)$. So $(w, \Gamma)$ is a solution-tube for problem (1). □

**Corollary 3.8.** *If problem* (1) *has a lower and an upper solutions* $\underline{y}, \overline{y}$ *such that* $\underline{y} \leq \overline{y}$. *Then there is at least one solution* $y$ *to problem* (1) *verifying* $\underline{y} \leq y \leq \overline{y}$.

*Proof.* Let $w(t) = \frac{\underline{y}(t)+\overline{y}(t)}{2}$ and $\Gamma(t) = \frac{\underline{y}(t)-\overline{y}(t)}{2}$ for all $t \in [c, d]$. From Proposition 3.7, the couple $(w, \Gamma)$ is a solution-tube of (1). Theorem 3.5 implies that problem (1) possesses at least one solution $y$ satisfying $\left|y(t) - w(t)\right| \leq \Gamma(t)$. Consequently, we obtain $\underline{y}(t) \leq y(t) \leq \overline{y}(t)$, for all $t \in [c, d]$. □

## 4. Examples

**Example 4.1.** *For $0 < \alpha \leq 1$, study the problem below*

$$\begin{cases} D^{\alpha} y = \ln(t^2 + y^2)\sin(\pi y) + \alpha y & 0 \leq t \leq \pi, \\ y(0) = y(\pi). \end{cases} \tag{9}$$

*We have $\widetilde{f}(t, y) = \ln(t^2 + y^2)\sin(\pi y) + \alpha y$ is a continuous on $[0, \pi] \times \mathbb{R}$. We can verify that $(w, \Gamma) := (0, 2)$ is a solution-tube. In fact, $D^{\alpha}w(t) = 0$, $D^{\alpha}\Gamma(t) = 2\alpha$ and for $x \in \mathbb{R}$, so that $|x - w(t)| = \Gamma(t)$, i.e., $|x| = 2$ we find*

$$(x - w(t))\,(\widetilde{f}(t, x) - D^{\alpha}w(t)) = x\widetilde{f}(t, x) = 4\alpha \leq 4\alpha = \Gamma(t)D^{\alpha}\Gamma(t).$$

*Moreover $|w(0) - w(\pi)| \leq \Gamma(0) - \Gamma(\pi)$. Applying Theorem 3.5, we deduce that problem (9) has a solution $y$ so $\left|y(t)\right| \leq 2$.*

**Example 4.2.** *Examine the next problem*

$$\begin{cases} D^{\frac{2}{3}} y = \frac{2t^2 - y^3(t)}{3e^t + 7} + \frac{2}{3} & 0 \leq t \leq 2, \\ y(0) = y(2). \end{cases} \tag{10}$$

*Here $\widetilde{f}(t, y) = \frac{2t^2 - y^3(t)}{3e^t + 7} + \frac{2}{3}$ is continuous on $[0, 2] \times \mathbb{R}$. We can verify that $\underline{y}(t) = -2$ and $\overline{y}(t) = 2$ are upper and lower solutions, respectively. In fact, we have*

$$\begin{cases} D^{\frac{2}{3}}\overline{y}(t) = \frac{2}{3} & \geq & \widetilde{f}(t, \overline{y}) = \frac{2t^2 - 8}{3e^t + 7} + \frac{2}{3}, \quad \text{for } t \in [0, 2], \\ \overline{y}(0) = 2 & \geq & \overline{y}(2) = 2, \end{cases}$$

*and*

$$\begin{cases} D^{\frac{2}{3}}\underline{y}(t) = -\frac{2}{3} & \leq & \widetilde{f}(t, \underline{y}) = \frac{2t^2 + 8}{3e^t + 7} + \frac{2}{3}, \quad \text{for } t \in [0, 2], \\ \underline{y}(0) = -2 & \leq & \underline{y}(2) = -2. \end{cases}$$

*It follows from Corollary 3.8 that problem (10) possesses at least one solution $y$ with $-2 \leq y(t) \leq 2$ for $t \in [0, 2]$.*

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