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# Analytical approach for a coupled system of hybrid fractional integro-differential equations with Atangana–Baleanu–Caputo derivative

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**Abstract.** This paper aims to investigate the existence results for a coupled system of hybrid fractional integro-differential equations using the Atangana-Baleanu-Caputo operator. Solutions are derived from the defined hypotheses and the standard fixed point theorem. An example is provided to illustrate the theoretical results.

## 1. Introduction

The concept of fractional derivatives dates back to the 17th century when mathematicians like Leibniz and L'Hôpital discussed the possibility of generalizing derivatives to non-integer orders. This early curiosity laid the groundwork for what we now call fractional calculus. As a result, integer-order derivatives cannot describe processes with memory, which is the primary advantage of fractional derivatives over classical derivatives. Fractional differential equations provide a powerful tool for modeling numerous real-life dynamic processes, as they can describe their behavior more accurately. They have applications in signal and image processing, atmospheric diffusion of pollution, transmission of ultrasound waves, cellular diffusion processes, feedback amplifiers, and the effect of speculation on the profitability of stocks in financial markets, among many others. For more details on this topic, we refer the reader to [4, 7, 21–23, 26–30].

The $\mathcal{A}$tangana–$\mathcal{B}$aleanu–$\mathcal{C}$aputo ($\mathcal{ABC}$) fractional derivative is a new concept in fractional calculus, addressing the limitations of traditional derivatives in modeling real-world phenomena with memory and hereditary properties. Significant progress has been made in applying this derivative to various fields, including biological models, leading to insights into complex systems and the development of numerical methods for solving fractional differential equations. [2, 3, 6, 11, 15–17].

Studying hybrid fractional integro-differential equations is crucial for modeling complex real-life phenomena like viscoelastic materials, population dynamics, and finance. These equations offer a deeper

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understanding of systems with memory effects and can capture the long-term behavior of processes. This enables more accurate predictions and solutions in various scientific and engineering fields. They combine integral and differential calculus elements with fractional calculus, allowing for more precise and flexible representations of dynamic systems. Here are a few references: [1, 5, 8, 9, 18, 24, 25, 31, 35].

In their publication referenced as [32], Shah al. explored a specific class of fractional-order evolution control systems by applying controllability criteria, where the variable $t$ is confined to the interval $J = [0, T]$.

$$\begin{cases} {}_0^{mABC}\mathcal{D}^{\theta}[u(t) - \phi(t, u(t))] = Au(t) + Bx(t) + \psi(t, u(t)),\ 0 < \theta \leq 1, \\ u(0) = u_0, \end{cases}$$

where $A$ is the infinite small generator of an analytical semigroup of bounded linear operators on the Hilbert space say $H$, $x$ is the control variable function on $L^2[J, H]$, while $B$ is also a linear bounded operator from $H$ to $H$ and $\psi, \phi : J \times R \to R$.

Bashiri et al. [9] studied the existence of solutions for the system of fractional hybrid differential equations

$$\begin{cases} D^p\left[\dfrac{\theta(t) - w(t, \theta(t))}{u(t, \theta(t))}\right] = v(t, \vartheta(t)),\ t \in J, \\ \\ D^p\left[\dfrac{\vartheta(t) - w(t, \vartheta(t))}{u(t, \vartheta(t))}\right] = v(t, \theta(t)),\ t \in J,\ 0 < p < 1, \\ \theta(0) = 0, \quad \vartheta(0) = 0, \end{cases}$$

where $D^p$ denotes the Riemann-Liouville fractional derivative of order $p$, $J = [0, 1]$, and the functions $u : J \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $w : J \times \mathbb{R} \to \mathbb{R}$, $w(0, 0) = 0$ and $v : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain conditions.

Inspired by the aforementioned works, we build on their ideas in this paper to examine the existence results for problems of this form

$$\begin{cases} {}_0^{\mathcal{ABC}}\mathcal{D}_\iota^q\left[\dfrac{\chi(\iota) - \varpi(\iota, \chi(\iota))}{\omega(\iota, \chi(\iota))}\right] = h(\iota, \zeta(\iota), \mathfrak{G}\zeta(\iota)), \\ \\ {}_0^{\mathcal{ABC}}\mathcal{D}_\iota^q\left[\dfrac{\zeta(\iota) - \varpi(\iota, \zeta(\iota))}{\omega(\iota, \zeta(\iota))}\right] = h(\iota, \chi(\iota), \mathfrak{G}'\chi(\iota)), \\ \chi(0) = \chi_0, \quad \zeta(0) = \zeta_0, \end{cases} \tag{1}$$

where $\iota \in \Lambda : \Lambda = [0, T]$, $T > 0$ and $0 < q < 1$, ${}_0^{\mathcal{ABC}}\mathcal{D}_\iota^q$ is an $\mathcal{ABC}$ operator, the functions $\omega : \Lambda \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $\varpi : \Lambda \times \mathbb{R} \to \mathbb{R}$ satisfy certain conditions and $h : \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a specific function.

The terms $\mathfrak{G}\chi(\iota)$, $\mathfrak{G}'\chi(\iota)$ provided by:

$$\mathfrak{G}\chi(\iota) = \int_0^\iota I(\iota, \tau)\chi(\tau)d\tau,\ I \in C(Q, \mathbb{R}^+)\ \ and\ \ \mathfrak{G}'\chi(\iota) = \int_0^\iota J(\iota, \tau)\chi(\tau)d\tau,\ J \in C(Q_0, \mathbb{R}^+),$$

with $Q = \{(\iota, \varsigma) \in \mathbb{R}^2 : 0 \leq \varsigma \leq \iota \leq T\}$, $\quad Q_0 = \{(\iota, \varsigma) \in \mathbb{R}^2 : 0 \leq \varsigma \leq \iota \leq T\}$. And also we investigate the existence result for it special case when $\omega(\iota, .(\iota)) = 1$, as a second problem.

This research article is organized as follows: The basic properties of $\mathcal{ABC}$ derivative are presented in Section 2. Section 3 deals with the existence of the solutions for the hybrid problem and also for its special case when $\omega(\iota, .(\iota)) = 1$,. An example is provided in Section 4 to support the analytical results.

## 2. Preliminaries

This study will revisit some essential definitions of fractional calculus, which form the basis for our main results. Let's consider $\mathcal{X} = C(\Lambda, \mathbb{R})$ as a Banach space, with the norm denoted as $\|\chi\| = \max_{\iota \in \Lambda} |\chi(\iota)|$.

**Definition 2.1.** *[33] Let $\chi \in AC(\Lambda, \mathbb{R})$, be a solution of (1) if the mapping $\chi \to \frac{\chi}{\omega(\iota, \chi)}$ is completely continuous for every $\chi \in \mathbb{R}$ and $\chi$ satisfies equation (1), where $AC(\Lambda, \mathbb{R}) = \{z | \, z : \Lambda \to \mathbb{R}\}$ is completely continuous.*

**Definition 2.2.** *[34] The Mittag–Leffler function (one parameter) $E_q$ is defined as:*

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}, \; z \in \mathbb{C}, \; R(q) > 0, \tag{2}$$

**Definition 2.3.** *[33] Let $\chi \in H^1(0, T)$ and $q \in [0, 1]$. The $\mathcal{ABC}$ fractional derivative for function $\chi$ of order $q$ is defined as*

$${}_{0}^{\mathcal{ABC}}\mathcal{D}_{\iota}^{q}\chi(\iota) = \frac{\mathcal{B}(q)}{1-q} \int_0^{\iota} E_q\left[-\frac{q}{1-q}(\iota - \varsigma)^q\right]\chi'(\varsigma)d\varsigma. \tag{3}$$

*Here the normalization function is $\mathcal{B}(q)$ with $\mathcal{B}(0) = \mathcal{B}(1) = 1$.*

**Definition 2.4.** *[33] Let $\chi$ be a function, then the $\mathcal{AB}$ fractional integral of order $q \in (0, 1)$ is defined by*

$${}_{0}^{\mathcal{AB}}\mathcal{I}_{\iota}^{q}\chi(\iota) = \frac{1-q}{\mathcal{B}(q)}\chi(\iota) + \frac{q}{\mathcal{B}(q)}\, {}_{0}\mathcal{I}_{\iota}^{q}\chi(\iota), \tag{4}$$

*where*

$${}_{0}\mathcal{I}_{\iota}^{q}\chi(\iota) = \frac{1}{\Gamma(q)} \int_0^{\iota} (\iota - \varsigma)^{q-1}\chi(\varsigma)d\varsigma.$$

**Lemma 2.5.** *[33] If $0 < q < 1$, then ${}_{0}^{\mathcal{AB}}\mathcal{I}_{\iota}^{q}({}_{0}^{\mathcal{ABC}}\mathcal{D}_{\iota}^{q}\chi(\iota)) = \chi(\iota) - \chi(0)$.*

**Lemma 2.6.** *[13, 14] Let S be a non-empty, closed, convex, and bounded subset of a Banach algebra $\mathcal{X}$ and let $\mathfrak{M}, \mathfrak{N} : \mathcal{X} \to \mathcal{X}$ and $\mathfrak{L} : S \to \mathcal{X}$ be two operators fulfilling the following properties:*

*(1) $\mathfrak{M}$ and $\mathfrak{N}$ are Lipschitzians with Lipschitz constants $\alpha$ and $\beta$, respectively,*
*(2) $\mathfrak{L}$ is completely continuous,*
*(3) $\zeta = \mathfrak{M}\zeta\mathfrak{L}\chi + \mathfrak{N}\zeta, \Rightarrow \zeta \in S$ for all $\chi \in S$, and*
*(4) $\|\rho\|\|\varrho\|\mathfrak{T} < 1$, where $\mathfrak{T} = \|\mathfrak{L}S\| = \sup\{\|\mathfrak{L}(\zeta)\| : \zeta \in S\}$. Then the operator $\mathfrak{M}\zeta\mathfrak{L}\chi + \mathfrak{N}\zeta = \zeta$ has a solution in S.*

The following is a fixed-point theorem in Banach spaces, attributed to Burton [12].

**Lemma 2.7.** *[12] Let S be a nonempty, closed, convex, and bounded subset of a Banach space $\mathcal{X}$ and let $B : \mathcal{X} \to \mathcal{X}$ and $D : S \to \mathcal{X}$ be two operators such that*

*(i) B is a contraction with constant $\alpha < 1$,*
*(ii) D is completely continuous,*
*(iii) $\chi = B\chi + D\zeta \Rightarrow \chi \in S$ for all $\zeta \in S$.*

*Then the operator equation $B\chi + D\zeta = \chi$ has a solution in S.*

## 3. Main results

*3.1. Solution of fractional hybrid differential equation*

**Lemma 3.1.** *Assume that $\chi$, where $0 < q < 1$, and $\varpi, \omega \in C(\Lambda \times \mathbb{R}, \mathbb{R})$ such that $\omega(0, \chi(0)) \neq 0$ and $\varpi(0, \chi(0)) = 0$. Then the unique solution of the following initial value problem:*

$$\begin{cases} {}_{0}^{\mathcal{ABC}}\mathcal{D}_{\iota}^{q}\left[\dfrac{\chi(\iota) - \varpi(\iota, \chi(\iota))}{\omega(\iota, \chi(\iota))}\right] = \mathbb{H}(\iota), \\ \chi(0) = \chi_0, \end{cases} \tag{5}$$

*is*

$$\chi(\iota) = \omega(\iota,\chi(\iota))\left(\frac{\chi_0}{\omega(0,\chi_0)} + \frac{1-q}{\mathcal{B}(q)}\mathbb{H}(\iota) + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}\mathbb{H}(\varsigma)d\varsigma\right) + \varpi(\iota,\chi(\iota)) \tag{6}$$

*Proof.* Let $\chi(\iota)$ denote a solution of the problem (5) . By applying the operator ${}_0^{\mathcal{AB}}\mathcal{I}_\iota^q$ to both sides of (5), we obtain

$${}_0^{\mathcal{AB}}\mathcal{I}_\iota^q\, {}_0^{\mathcal{ABC}}\mathcal{D}_\iota^q\left[\frac{\chi(\iota)-\varpi(\iota,\chi(\iota))}{\omega(\iota,\chi(\iota))}\right] = {}_0^{\mathcal{AB}}\mathcal{I}_\iota^q\mathbb{H}(\iota),$$

So, based on the lemma 2.5, we can conclude that

$$\frac{\chi(\iota)-\varpi(\iota,\chi(\iota))}{\omega(\iota,\chi(\iota))} - \frac{\chi(0)-\varpi(0,\chi(0))}{\omega(0,\chi(0))} = {}_0^{\mathcal{AB}}\mathcal{I}_\iota^q\mathbb{H}(\iota).$$

Due to the fact that $\omega(0,\chi(0)) \neq 0$ and $\varpi(0,\chi(0)) = 0$, we have

$$\frac{\chi(\iota)-\varpi(\iota,\chi(\iota))}{\omega(\iota,\chi(\iota))} = \frac{\chi_0}{\omega(0,\chi_0)} + \frac{1-q}{\mathcal{B}(q)}\mathbb{H}(\iota) + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}\mathbb{H}(\varsigma)d\varsigma.$$

i.e

$$\chi(\iota) = \omega(\iota,\chi(\iota))\left(\frac{\chi_0}{\omega(0,\chi_0)} + \frac{1-q}{\mathcal{B}(q)}\mathbb{H}(\iota) + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}\mathbb{H}(\varsigma)d\varsigma\right) + \varpi(\iota,\chi(\iota))$$

We direct the readers to [19] for the equivalence reciprocal. □

**Theorem 3.2.** *Based on Lemma 3.1, our problem* (1) *can be equated to the following coupled integral equations:*

$$\begin{cases} \chi(\iota) = \omega(\iota,\chi(\iota))\left(\dfrac{\chi_0}{\omega(0,\chi_0)} + \dfrac{1-q}{\mathcal{B}(q)}h(\iota,\zeta(\iota),\mathfrak{G}'\zeta(\iota))\right. \\ \qquad \left. + \dfrac{q}{\mathcal{B}(q)\Gamma(q)}\displaystyle\int_0^\iota (\iota-\varsigma)^{q-1}h(\varsigma,\zeta(\varsigma),\mathfrak{G}'\zeta(\varsigma))d\varsigma\right) + \varpi(\iota,\chi(\iota)), \\ \zeta(\iota) = \omega(\iota,\zeta(\iota))\left(\dfrac{\zeta_0}{\omega(0,\zeta_0)} + \dfrac{1-q}{\mathcal{B}(q)}h(\iota,\chi(\iota),\mathfrak{G}\chi(\iota))\right. \\ \qquad \left. + \dfrac{q}{\mathcal{B}(q)\Gamma(q)}\displaystyle\int_0^\iota (\iota-\varsigma)^{q-1}h(\varsigma,\chi(\varsigma),\mathfrak{G}\chi(\varsigma))d\varsigma\right) + \varpi(\iota,\zeta(\iota)). \end{cases} \tag{7}$$

For the proof of our main results, we employ the following hypotheses:

($\mathcal{H}_1$) The function $\hbar : \Lambda \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfies the following properties:
- (a) $\forall(\chi,\varsigma) \in \mathcal{X} \times \mathcal{X}$, the function $\hbar(.,\chi,\varsigma)$ is strongly measurable.
- (b) $\forall\iota \in \Lambda$, the function $\hbar(\iota,.) : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is continuous.
- (c) For any $r > 0$, there exists a function $g \in L^\infty(\Lambda,\mathbb{R}^+)$ such that

$$\sup_{\|\chi\|\leq r,\ \|\zeta\|\leq j^*Tr} \|h(\iota,\chi,\zeta)\| \leq g(\iota), \quad \iota \in \Lambda,$$

and

$$i^* = \sup_{\iota\in\Lambda}\int_0^\iota I(\iota,\varsigma)d\varsigma < \infty, \quad j^* = \sup_{\iota\in\Lambda}\int_0^\iota J(\iota,\varsigma)d\varsigma < \infty.$$

($\mathcal{H}_2$) The mapping $\chi \longrightarrow \dfrac{\chi-\varpi(\iota,\chi)}{\omega(\iota,\chi)}$ is increasing in $\mathbb{R}$ almost everywhere.

($\mathcal{H}_3$) The functions $\omega : \Lambda \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $\varpi : \Lambda \times \mathbb{R} \to \mathbb{R}$, $\varpi(0,0) = 0$ are continuous and there exist two functions $\rho$ and $\varrho$ with bounds $\|\rho\|$ and $\|\varrho\|$, respectively, such that for all $\chi, \zeta \in \mathcal{X}$ and $\iota \in \Lambda$ we have

$$|\omega(\iota, \chi(\iota)) - \omega(\iota, \zeta(\iota))| \leq \rho(\iota)|\chi(\iota) - \zeta(\iota)|,$$

and

$$|\varpi(\iota, \chi(\iota)) - \varpi(\iota, \zeta(\iota))| \leq \varrho(\iota)|\chi(\iota) - \zeta(\iota)|.$$

**Definition 3.3.** *[10] A function $\hbar$ satisfies the local Lipshitz condition in $\chi(\iota)$, uniformly in $\iota$ on bounded intervals if for every $\iota' \geq 0$ and $\kappa \geq 0$ there is a constant $\ell(c, \iota')$ such that*

$$\|\hbar(\iota, \chi_1) - \hbar(\iota, \chi_2)\|_{\mathcal{X}} \leq \ell(c, \iota')\, \|\chi_1 - \chi_2\|\,,$$

*for all $\chi_1, \chi_2 \in \mathcal{X}$, and $\iota \in \Lambda$.*

**Theorem 3.4.** *Assuming that ($\mathcal{H}_1$)-($\mathcal{H}_3$) hold, there exists a solution for the system* (1) *if*

$$\|\rho\|\|\varrho\|\left(\left|\frac{\chi_0}{\omega(0,\chi_0)}\right| + \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}\right) < 1 \tag{8}$$

*Proof.* Set $\mathcal{X} = C(\Lambda, \mathbb{R})$ and a subset $S$ of $\mathcal{X}$ defined by $S = \{\chi \in \mathcal{X} |\ \|\chi\| \leq N\}$,

$$N = \frac{\left(\left|\frac{\chi_0}{\omega(0,\chi_0)}\right| + \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}\right)W_0 + \overline{W}_0}{1 - \left(\left|\frac{\chi_0}{\omega(0,\chi_0)}\right| + \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}\right)\|\rho\| - \|\varrho\|},$$

where $W_0 = \max_{\iota\in\Lambda} |\omega(\iota, 0)|$, $\overline{W}_0 = \max_{\iota\in\Lambda} |\varpi(\iota, 0)|$.

$S$ is a closed, convex, and bounded subset of the Banach algebra $\mathcal{X}$.

Let us define two operators $\mathfrak{M}, \mathfrak{R} : \mathcal{X} \to \mathcal{X}$ and $\mathfrak{L} : S \to \mathcal{X}$ by

$$\begin{aligned}
\mathfrak{M}\chi(\iota) &= \omega(\iota, \chi(\iota)) \\
\mathfrak{L}\chi(\iota) &= \frac{\zeta_0}{\omega(0,\zeta_0)} + \frac{1-q}{\mathcal{B}(q)}h(\iota, \chi(\iota), \mathfrak{G}'\chi(\iota)) \\
&\quad + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota - \varsigma)^{q-1}h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma))d\varsigma \\
\mathfrak{R}\chi(\iota) &= \varpi(\iota, \chi(\iota)).
\end{aligned}$$

so from the defined operators the system (1) can be written as

$$\begin{aligned}
\chi(\iota) &= \mathfrak{M}\chi(\iota)\mathfrak{L}\zeta(\iota) + \mathfrak{R}\chi(\iota), \\
\zeta(\iota) &= \mathfrak{M}\zeta(\iota)\mathfrak{L}\chi(\iota) + \mathfrak{R}\zeta(\iota),
\end{aligned}$$

Now, we demonstrate that the operators $\mathfrak{M}$, $\mathfrak{L}$, and $\mathfrak{R}$ satisfy the conditions of theorem 3.4. The proof consists of steps.

**Step I.** $\mathfrak{M}$, and $\mathfrak{R}$ are Lipschitz.
Let $\chi, \zeta \in \mathcal{X}$, by the hypothesis ($\mathcal{H}_3$) we have

$$\begin{aligned}
|\mathfrak{M}\chi(\iota) - \mathfrak{M}\zeta(\iota)| &= |\omega(\iota, \chi(\iota)) - \omega(\iota, \zeta(\iota))| \\
&\leq \rho(\iota)|\chi(\iota) - \zeta(\iota)| \\
&\leq \|\rho\|\|\chi - \zeta\|.
\end{aligned}$$

Hence, $\mathfrak{M}$ is a Lipshitzien function defined on $\mathcal{X}$ with a Lipschitz constant of $\|\rho\|$. Similarly, $\mathfrak{N}$ is also a Lipschitzian function on $\mathcal{X}$ with a Lipschitz constant of $\|\varrho\|$.

**Step II.** $\mathfrak{L}$ is completely continuous.
We will show that the operator $\mathfrak{L} : S \to \mathcal{X}$ is both compact and continuous from $S$ into $\mathcal{X}$. First, we establish the continuity of $\mathfrak{L}$ on $S$. Let $\chi_n$ be a sequence in $S$ that converges to a point $\chi \in S$ as $n \to \infty$. Since we know that $h$ is continuous mappings from $(\mathcal{H}_1)$, we can conclude that

$$h(\iota, \chi_n(\iota), \mathfrak{G}\chi_n(\iota)) \to h(\iota, \chi_(\iota), \mathfrak{G}\chi_(\iota)).$$

as $n \to \infty$. Since $\|h(\iota, \chi_n(\iota), \mathfrak{G}\chi_n(\iota)) - h(\iota, \chi_(\iota), \mathfrak{G}\chi_(\iota))\| \leq 2g(\varsigma)$, by the Lebesgue dominated convergence theorem, for each $\iota \in \Lambda$, we have

$$\begin{aligned}
\lim_{n\to\infty}(\mathfrak{L}\chi_n)(\iota) &= \lim_{n\to\infty}\Bigg[\frac{\zeta_0}{\omega(0,\zeta_0)} + \frac{1-q}{\mathcal{B}(q)}h(\iota, \chi_n(\iota), \mathfrak{G}'\chi_n(\iota)) \\
&\quad + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}h(\varsigma, \chi_n(\varsigma), \mathfrak{G}'\chi_n(\varsigma))d\varsigma\Bigg] \\
&= \frac{\zeta_0}{\omega(0,\zeta_0)} + \frac{1-q}{\mathcal{B}(q)}\lim_{n\to\infty}h(\iota, \chi_n(\iota), \mathfrak{G}'\chi_n(\iota)) \\
&\quad + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}\lim_{n\to\infty}h(\varsigma, \chi_n(\varsigma), \mathfrak{G}'\chi_n(\varsigma))d\varsigma \\
&= \frac{\zeta_0}{\omega(0,\zeta_0)} + \frac{1-q}{\mathcal{B}(q)}[h(\iota, \chi(\iota), \mathfrak{G}'\chi(\iota))] \\
&\quad + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}[h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma))]d\varsigma \\
&= (\mathfrak{L}\chi)(\iota)
\end{aligned}$$

It proves that $\mathfrak{L}$ is a continuous operator on $S$.
From the hypothesis $(\mathcal{H}_1)$ for any $\chi \in S$ and $\iota \in \Lambda$, we have

$$\begin{aligned}
|(\mathfrak{L}\chi)(\iota)| &= \left|\frac{\zeta_0}{\omega(0,\zeta_0)}\right| + \frac{1-q}{\mathcal{B}(q)}|h(\iota, \chi(\iota), \mathfrak{G}'\chi(\iota))| \\
&\quad + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}|h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma))|d\varsigma \\
&\leq \left|\frac{\zeta_0}{\omega(0,\zeta_0)}\right| + \frac{1-q}{\mathcal{B}(q)}|g(\varsigma)| + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}|g(\varsigma)|d\varsigma \\
&\leq \left|\frac{\zeta_0}{\omega(0,\zeta_0)}\right| + \frac{1-q}{\mathcal{B}(q)}\|g\| + \frac{1}{\mathcal{B}(q)\Gamma(q)}\iota^q\|g\|d\varsigma.
\end{aligned}$$

It leads to

$$|(\mathfrak{L}\chi)(\iota)| \leq \left|\frac{\zeta_0}{\omega(0,\zeta_0)}\right| + \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}, \quad \chi \in S,\ \iota \in \Lambda. \tag{9}$$

Then we can say that $\mathfrak{L}$ is uniformly bounded on $S$.

Now, we show that $\mathfrak{L}(S)$ is an equicontinuous set in $\mathcal{X}$. Let any $\chi \in S$ and $0 \leq \iota_1 < \iota_2 \leq T$. Then we have

$$\begin{aligned}
&|(\mathfrak{L}\chi)(\iota_2) - (\mathfrak{L}\chi)(\iota_1)| \\
&\leq \frac{1-q}{\mathcal{B}(q)} \|h(\iota_2, \chi(\iota_2), \mathfrak{G}'\chi(\iota_2)) - h(\iota_1, \chi(\iota_1), \mathfrak{G}'\chi(\iota_1))\| + \frac{q}{\mathcal{B}(q)\Gamma(q)} \\
&\times \left| \int_0^{\iota_2} (\iota_2 - \varsigma)^{q-1} h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma)) d\varsigma - \int_0^{\iota_1} (\iota_1 - \varsigma)^{q-1} h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma)) d\varsigma \right|
\end{aligned} \tag{10}$$

Since $h(\iota, \chi)$ is continuous on compact set $\Lambda \times [-R, R]$, it is uniformly continuous there and hence we have

$$|h(\iota_2, \chi(\iota_2), \mathfrak{G}'\chi(\iota_2)) - h(\iota_1, \chi(\iota_1), \mathfrak{G}'\chi(\iota_1))| \to 0, \ as \ |\iota_2 - \iota_1| \to 0, \ for \ every \ \chi \in S. \tag{11}$$

From the hypothesis $(\mathcal{H}_1)$,

$$\begin{aligned}
&\left| \int_0^{\iota_2} (\iota_2 - \varsigma)^{q-1} h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma)) d\varsigma - \int_0^{\iota_1} (\iota_1 - \varsigma)^{q-1} h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma)) d\varsigma \right| \\
&\leq \int_0^{\iota_1} [(\iota_2 - \varsigma)^{q-1} - (\iota_1 - \varsigma)^{q-1}] |h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma))| d\varsigma + \int_{\iota_1}^{\iota_2} (\iota_2 - \varsigma)^{q-1} |h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma))| d\varsigma \\
&\leq \int_0^{\iota_1} [(\iota_2 - \varsigma)^{q-1} - (\iota_1 - \varsigma)^{q-1}] |g(\varsigma)| d\varsigma + \int_{\iota_1}^{\iota_2} (\iota_2 - \varsigma)^{q-1} |g(\varsigma)| d\varsigma \\
&\leq \|g\| (\iota_1^q + (\iota_2 - \iota_1)^q - \iota_2^q + (\iota_2 - \iota_1)^q) \\
&\leq 2\|g\| (\iota_2 - \iota_1)^q.
\end{aligned}$$

Hence

$$\left| \int_0^{\iota_2} (\iota_2 - \varsigma)^{q-1} h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma)) d\varsigma - \int_0^{\iota_1} (\iota_1 - \varsigma)^{q-1} h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma)) d\varsigma \right| \to 0, \tag{12}$$

$$as \ |\iota_2 - \iota_1| \to 0, \ \chi \in S.$$

From equations (10), (11) and (12), we obtain that

$$|(\mathfrak{L}\chi)(\iota_2) - (\mathfrak{L}\chi)(\iota_1)| \to 0, as \ |\iota_2 - \iota_1| \to 0, \ for \ each \ \chi \in S.$$

Hence $\mathfrak{L}(S)$ is an equicontinuous set in $\mathcal{X}$. Since $\mathfrak{L}$ is a uniformly bounded and equicontinuous set in $\mathcal{X}$, from Ascoli-Arzela fixed point theorem, $\mathfrak{L}$ is completely continuous.

**Step III.** Let $\chi \in \mathcal{X}$. For any $\zeta \in S$, consider the equation in the form of operators $\chi = \mathfrak{M}\chi\mathfrak{L}\zeta + \mathfrak{N}\chi$. Let us prove that $\chi \in S$. From the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ and from the inequality (9), we have

$$\begin{aligned}
|\chi(\iota)| &\leq |\mathfrak{M}\chi(\iota)||\mathfrak{L}\zeta(\iota)| + |\mathfrak{N}\chi(\iota)| \\
&\leq (|\omega(\iota, \chi(\iota)) - \omega(\iota, 0) + \omega(\iota, 0)|) \times \left( \left| \frac{\chi_0}{\omega(0, \chi_0)} \right| + \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right) \frac{\|g\|}{\mathcal{B}(q)} \right) \\
&+ (|\varpi(\iota, \chi(\iota)) - \varpi(\iota, 0) + \varpi(\iota, 0)|) \\
&\leq (\|\rho\||\chi(\iota)| + W_0) \times \left( \left| \frac{\chi_0}{\omega(0, \chi_0)} \right| + \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right) \frac{\|g\|}{\mathcal{B}(q)} \right) \\
&+ (\|\varrho\||\chi(\iota)| + \overline{W}_0),
\end{aligned}$$

after taking the supremum over $\iota$ on $\Lambda$ and considering $(\mathcal{H}_1)$, it becomes evident that

$$\|\chi\| \leq \frac{\left(\left|\frac{\chi_0}{\omega(0,\chi_0)}\right| + \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}\right)W_0 + \overline{W}_0}{1-\left(\left|\frac{\chi_0}{\omega(0,\chi_0)}\right| + \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}\right)\|\rho\| - \|\varrho\|},$$

Therefore, $\|\chi\| \leq N$.

**Step IV.** The constants $\chi$, $\zeta$ and $\mathfrak{T}$ of Lemma 2.6 corresponding to the operators $\mathfrak{M}$, $\mathfrak{L}$ and $\mathfrak{R}$ defined before, respectively, are

$$\chi = \|\rho\|,\ \zeta = \|\varrho\| \ and\ \mathfrak{T} = \left|\frac{\chi_0}{\omega(0,\chi_0)}\right| + \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}$$

and from inequality (8), it follows that

$$\chi\zeta\mathfrak{T} = \|\rho\|\|\varrho\|\left(\left|\frac{\chi_0}{\omega(0,\chi_0)}\right| + \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}\right) < 1.$$

It is clear from **Step I** to **Step IV** that the conditions specified in Lemma 2.6 are satisfied. Therefore, the equation $\chi = \mathfrak{M}\chi\mathfrak{L}\zeta + \mathfrak{R}\chi$ has a fixed point in $S$, which serves as a solution to the coupled system (1). This completes the proof. □

### 3.2. *Special case: Solution for a fractional evolution system*

In this section, we suppose that $\omega(\iota,.(\iota)) = 1$, and we add $\mathfrak{A}$ which is the infinite small generator of an analytical semigroup of bounded linear operators in $\mathcal{X}$. Our system (1) becomes a fractional-order evolution system:

$$\begin{cases} {}_0^{\mathcal{ABC}}\mathcal{D}_\iota^q[\chi(\iota) - \varpi(\iota,\chi(\iota))] - \mathfrak{A}\chi(\iota) = h(\iota,\zeta(\iota),\mathfrak{G}\zeta(\iota)), \\ {}_0^{\mathcal{ABC}}\mathcal{D}_\iota^q[\zeta(\iota) - \varpi(\iota,\zeta(\iota))] - \mathfrak{A}\zeta(\iota) = h(\iota,\chi(\iota),\mathfrak{G}'\chi(\iota)), \\ \chi(0) = 0, \quad \zeta(0) = 0, \end{cases} \tag{13}$$

Based on Theorem 3.2 our solution to the system (13) is as follow

$$\begin{cases} \chi(\iota) = \varpi(\iota,\chi(\iota)) + \dfrac{1-q}{\mathcal{B}(q)}[\mathfrak{A}\chi(\iota) + h(\iota,\zeta(\iota),\mathfrak{G}'\zeta(\iota))] \\ \qquad + \dfrac{q}{\mathcal{B}(q)\Gamma(q)}\displaystyle\int_0^\iota (\iota-\varsigma)^{q-1}[\mathfrak{A}\chi(\varsigma) + h(\varsigma,\zeta(\varsigma),\mathfrak{G}'\zeta(\varsigma))]d\varsigma, \\ \zeta(\iota) = \varpi(\iota,\zeta(\iota)) + \dfrac{1-q}{\mathcal{B}(q)}[\mathfrak{A}\zeta(\iota) + h(\iota,\chi(\iota),\mathfrak{G}\chi(\iota))] \\ \qquad + \dfrac{q}{\mathcal{B}(q)\Gamma(q)}\displaystyle\int_0^\iota (\iota-\varsigma)^{q-1}[\mathfrak{A}\zeta(\varsigma) + h(\varsigma,\chi(\varsigma),\mathfrak{G}\chi(\varsigma))]d\varsigma. \end{cases} \tag{14}$$

Using the same hypothesis in $(\mathcal{H}_3)$ for $\varpi$. We also change $(\mathcal{H}_2)$ by $(\mathcal{H'}_2)$ and we add $(\mathcal{H}_4)$ as follow

$(\mathcal{H'}_2)$ The mapping $\chi \longrightarrow \chi - \varpi(\iota,\chi)$ is increasing in $\mathbb{R}$ almost everywhere.

$(\mathcal{H}_4)$ For constants $\gamma > 0$, we have

$$|\mathfrak{A}\chi(\iota)| \leq \gamma|\chi(\iota)|.$$

We will now demonstrate the existence theorem for the system. (13).

**Theorem 3.5.** *Assume that hypotheses* $(\mathcal{H}_1)$, $(\mathcal{H'}_2)$, $(\mathcal{H}_3)$ *and* $(\mathcal{H}_4)$ *hold. Then the fractional-order evolution system* (13) *has asolution defined on* $\Lambda$.

*Proof.* Set $\mathcal{X} = C(\Lambda, \mathbb{R})$ and a subset $S'$ of $\mathcal{X}$ defined by $S' = \{\chi \in \mathcal{X} | \, \|\chi\| \leq N'\}$,

Clearly, $S'$ is a closed, convex, and bounded subset of the Banach space $\mathcal{X}$. Let us define two operators $B : \mathcal{X} \to \mathcal{X}$ and $D : S \to \mathcal{X}$ by

$$B\chi(\iota) = \varpi(\iota, \chi(\iota)) + \frac{1-q}{\mathcal{B}(q)}\mathfrak{A}\chi(\iota) + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota - \varsigma)^{q-1}\mathfrak{A}\chi(\varsigma)d\varsigma$$
$$D\chi(\iota) = \frac{1-q}{\mathcal{B}(q)}h(\iota, \chi(\iota), \mathfrak{G}'\chi(\iota)) + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota - \varsigma)^{q-1}h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma))d\varsigma.$$

so from the defined operators the system (13) can be written as

$$\chi(\iota) = B\chi(\iota) + D\zeta(\iota),$$
$$\zeta(\iota) = B\zeta(\iota) + D\chi(\iota),$$

Now, we demonstrate that the operators $B$ and $D$ satisfy the conditions of Lemma 2.7. The proof consists of steps.

**Step I.** $B$ are Lipschitz.
Let $\chi, \zeta \in \mathcal{X}$, by the hypothesis $(\mathcal{H'}_2)$ and $(\mathcal{H}_3)$, we have

$$\begin{aligned}
|B\chi(\iota) - B\zeta(\iota)| &\leq |\varpi(\iota, \chi(\iota)) - \varpi(\iota, \zeta(\iota))| + \frac{1-q}{\mathcal{B}(q)}|\mathfrak{A}\chi(\iota) - \mathfrak{A}\zeta(\iota)| \\
&\quad + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota - \varsigma)^{q-1}|\mathfrak{A}\chi(\varsigma) - \mathfrak{A}\zeta(\varsigma)|d\varsigma \\
&\leq \|\varrho\||\chi(\iota) - \zeta(\iota)| + \left(\left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\gamma}{\mathcal{B}(q)}\right)|\chi(\iota) - \zeta(\iota)| \\
&\leq \phi(\iota)|\chi(\iota) - \zeta(\iota)| \\
&\leq \|\phi\||\chi(\iota) - \zeta(\iota)|,
\end{aligned}$$

with $\phi(\iota) = \left(\|\varrho\| + \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\gamma}{\mathcal{B}(q)}\right)$.
Thus, $B$ is a Lipschitz function defined on $\mathcal{X}$ with a Lipschitz constant of $\|\phi\|$.

**Step II.** $D$ is completely continuous.
We will demonstrate that the operator $D : S \to \mathcal{X}$ is both compact and continuous from $S$ to $\mathcal{X}$.

To prove that $D$ is a continuous operator on $S$, we use the same steps in **Step II.** in the proof of Theorem 3.4.

From the hypothesis $(\mathcal{H}_1)$ for any $\chi \in S$ and $\iota \in \Lambda$, we have

$$\begin{aligned}
|(D\chi)(\iota)| &= \frac{1-q}{\mathcal{B}(q)}|h(\iota, \chi(\iota), \mathfrak{G}'\chi(\iota))| + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota - \varsigma)^{q-1}|h(\varsigma, \chi(\varsigma), \mathfrak{G}'\chi(\varsigma))|d\varsigma \\
&\leq \frac{1-q}{\mathcal{B}(q)}|g(\varsigma)| + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota - \varsigma)^{q-1}|g(\varsigma)|d\varsigma \\
&\leq \frac{1-q}{\mathcal{B}(q)}\|g\| + \frac{1}{\mathcal{B}(q)\Gamma(q)}\iota^q\|g\|d\varsigma.
\end{aligned}$$

It leads to

$$|(D\chi)(\iota)| \leq \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}, \quad \chi \in S, \ \iota \in \Lambda. \tag{15}$$

Then we can say that $D$ is uniformly bounded on $S$.
We show that $D$ is an equicontinuous set in $\mathcal{X}$. (it's the same as in **Step II.** in the proof of Theorem 3.4).

Thus, $D(S)$ is an equicontinuous set in $\mathcal{X}$. Since $\mathfrak{L}$ is both uniformly bounded and equicontinuous in $\mathcal{X}$, the Ascoli-Arzelà fixed point theorem implies that $D$ is completely continuous.

**Step III.** Let $\chi \in \mathcal{X}$. For any $\zeta \in S$, consider the equation in the form of operators $\chi = B\chi + D\zeta$. Let us prove that $\chi \in S$. From the hypotheses $(\mathcal{H}_1)$, $(\mathcal{H}'_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and from (15), we have

$$\begin{aligned}
|\chi(\iota)| &\leq |B\chi(\iota)| + |D\zeta(\iota)| \\
&\leq (|\varpi(\iota,\chi(\iota)) - \varpi(\iota,0) + \varpi(\iota,0)|) + \frac{1-q}{\mathcal{B}(q)}|\mathfrak{A}\chi(\iota)| + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}|\mathfrak{A}\chi(\varsigma)|d\varsigma \\
&+ \frac{1-q}{\mathcal{B}(q)}|h(\iota,\chi(\iota),\mathfrak{G}'\chi(\iota))| + \frac{q}{\mathcal{B}(q)\Gamma(q)}\int_0^\iota (\iota-\varsigma)^{q-1}|h(\varsigma,\chi(\varsigma),\mathfrak{G}'\chi(\varsigma))|d\varsigma \\
&\leq \|\varrho\||\chi(\iota)| + \overline{W}_0 + \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\gamma|\chi(\iota)|}{\mathcal{B}(q)} + \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)},
\end{aligned}$$

after taking the supremum over $\iota$ on $\Lambda$ and considering $(\mathcal{H}_1)$, it becomes evident that

$$\|\chi\| \leq \frac{\overline{W}_0 + \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}}{1-|\varrho\| - \left(1-q+\frac{\iota^q}{\Gamma(q)}\right)\frac{\gamma}{\mathcal{B}(q)}} \leq N'.$$

which implies that $\chi \in S$. So, the last assumption of Lemma 2.7 has been proved. Therefore, all the conditions of Lemma 2.7 are satisfied, hence the operator $\chi = B\chi + D\zeta$ has a coupled fixed point on $S$. As a result, the system (13) has a solution defined on $\Lambda$. □

## 4. Examples

### 4.1. Example

Consider the following fractional partial differential equation:

$$\begin{cases}
\mathfrak{D}^{\frac{1}{2}}\left[\dfrac{\chi(\iota,y)-\varpi(\iota,\chi(\iota,y))}{\omega(\iota,\chi(\iota,y))}\right] = \dfrac{\exp(-\iota)}{10}\left[\ln|1+\zeta(\iota,y)| + \arctan\left(\displaystyle\int_0^\iota \frac{1}{4}\exp(\iota\varsigma)\zeta(\varsigma,y)d\varsigma\right)\right], \\
\mathfrak{D}^{\frac{1}{2}}\left[\dfrac{\zeta(\iota,y)-\varpi(\iota,\zeta(\iota,y))}{\omega(\iota,\zeta(\iota,y))}\right] = \dfrac{1}{9+\iota}\left[\arctan\chi(\iota,y) + \ln\left|1+\displaystyle\int_0^\iota \frac{1}{4}\cos(\iota\varsigma)\chi(\varsigma,y)d\varsigma\right|\right], \\
\chi(0,y) = \zeta(0,y) = 0.5,
\end{cases} \tag{16}$$

with

$$\omega(\iota,\chi(\iota,y)) = \frac{\iota}{10+e^\iota}\left(\frac{|\chi(\iota,y)|}{1+|\chi(\iota,y)|}\right) + \frac{1}{8},$$

$$\varpi(\iota,\chi(\iota,y)) = \frac{e^{1-\iota}}{2}\sqrt{\frac{1}{4}|\chi(\iota,y)| + 1},$$

we find $W_0 = \max\limits_{\iota\in\Lambda}|\omega(\iota,0)| = \frac{1}{8}$, $\overline{W}_0 = \max\limits_{\iota\in\Lambda}|\varpi(\iota,0)| = \frac{e}{2}$, and

$$\hbar(\iota,\zeta(\iota,y),\mathfrak{G}'\zeta(\iota,y)) = \frac{\exp(-\iota)}{10}\left[\ln|1+\zeta(\iota,y)| + \arctan\left(\int_0^\iota \frac{1}{4}\exp(\iota\varsigma)\zeta(\varsigma,y)d\varsigma\right)\right],$$

$$\hbar(\iota, \chi(\iota, y), \mathfrak{G}\chi(\iota, y)) = \frac{1}{9+\iota}\left[\arctan \chi(\iota, y) + \ln\left|1 + \int_0^\iota \frac{1}{4}\cos(\iota\varsigma)\chi(\varsigma, y)d\varsigma\right|\right]$$

where $\chi(\iota) = \chi(\iota, y)$, $\zeta(\iota) = \zeta(\iota, y)$, and

$$\mathfrak{G}\chi(\iota, y) = \frac{1}{4}\int_0^\iota \cos(\iota\varsigma)\chi(\varsigma, y)d\varsigma, \ \ \textit{and}\ \mathfrak{G}'\zeta(\iota, y) = \frac{1}{4}\int_0^\iota \exp(\iota\varsigma)\zeta(\varsigma, y)d\varsigma.$$

For arbitrary $\chi, \zeta \in \mathcal{X}$ and $\iota \in \Lambda$, we can obtain,

$$\begin{aligned}|\omega(\iota, \chi(\iota, y)) - \omega(\iota, \zeta(\iota, y))| &\leq \frac{\iota}{11}\left|\frac{\chi(\iota, y) - \zeta(\iota, y)}{(1+|\chi(\iota, y)|)(1+|\zeta(\iota, y)|)}\right. \\ &\leq \frac{\iota}{11}|\chi(\iota, y) - \zeta(\iota, y)|,\end{aligned}$$

$$\begin{aligned}|\varpi(\iota, \chi(\iota, y)) - \varpi(\iota, \zeta(\iota, y))| &\leq \frac{e^{1-\iota}}{2}\left|\sqrt{\frac{1}{4}|\chi(\iota, y)| + 1} - \sqrt{\frac{1}{4}|\zeta(\iota, y)| + 1}\right| \\ &\leq \frac{e^{1-\iota}}{2}\frac{1}{4}\frac{|\chi(\iota, y) - \zeta(\iota, y)|}{\sqrt{\frac{1}{4}|\chi(\iota, y)| + 1}\sqrt{\frac{1}{4}|\zeta(\iota, y)| + 1}} \\ &\leq \frac{e^{1-\iota}}{8}|\chi(\iota, y) - \zeta(\iota, y)|,\end{aligned}$$

so

$$\rho(\iota) = \frac{\iota}{11}, \ \ \textit{and}\ \ \varrho(\iota) = \frac{e^{1-\iota}}{8}, \tag{17}$$

then conditions $(\mathcal{H}_3)$ is verified.
Which imply that from (17):

$$\|\rho\| \simeq 0.09 \ \ \textit{and}\ \ \|\varrho\| \simeq 0.34.$$

Also

$$\begin{aligned}&|\hbar(\iota, \chi(\iota, y), \mathfrak{G}'\chi(\iota, y)) - \hbar(\iota, \zeta(\iota, y), \mathfrak{G}'\zeta(\iota, y))| \\ &\leq \left|\frac{e^{-\iota}}{10}\right|\left[\left\|\chi(\varsigma, y) - \zeta(\iota, y)\right\| + \frac{1}{4}\left\|\int_0^\iota \exp(\iota\varsigma)[\chi(\varsigma, y) - \zeta(\varsigma, y)]d\varsigma\right\|\right] \\ &\leq \frac{1}{10}\left[\|\chi - \zeta\| + \frac{1}{4}\|\chi - \zeta\|\right],\end{aligned}$$

$$\begin{aligned}&|\hbar(\iota, \chi(\iota, y), \mathfrak{G}\chi(\iota, y)) - \hbar(\iota, \zeta(\iota, y), \mathfrak{G}\zeta(\iota, y))| \\ &\leq \left|\frac{1}{9+\iota}\right|\left[\left\|\chi(\varsigma, y) - \overline{\chi}(\iota, y)\right\| + \frac{1}{4}\left\|\int_0^\iota \cos(\iota\varsigma)[\chi(\varsigma, y) - \zeta(\varsigma, y)]d\varsigma\right\|\right] \\ &\leq \frac{1}{10}\left[\|\chi - \zeta\| + \frac{1}{4}\|\chi - \zeta\|\right],\end{aligned}$$

with $\ell_1(\iota) = \ell_2(\iota) = \frac{1}{10}$ and $i^* = j^* = \frac{1}{4}$. then we can deduce that $\||\hbar(\iota, \chi(\iota, y), \mathfrak{G}\chi(\iota, y))\| \leq \|g\| \simeq \frac{1}{8}$, which means that condition $(\mathcal{H}_1)$.

Hence knowing $\mathcal{B}(\frac{1}{2}) = 1.25$:

$$\begin{aligned} &\|\rho\|\|\varrho\|\left(\left|\frac{\chi_0}{\omega(0,\chi_0)}\right| + \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}\right) \\ &\leq 0.0306\left(\frac{0.5}{\frac{1}{8}} + \left[1 - \frac{1}{2} + \frac{1}{1.77}\right] \times \frac{\frac{1}{8}}{\mathcal{B}(\frac{1}{2})} = \right) \\ &= 0.1256436 < 1. \end{aligned}$$

As a result, we have satisfied condition (8). Therefore, according to Theorem 3.4, a solution for the system (1) exists on $\Lambda$.

*4.2. Example*

To prove the result in the section 3.2 we use the same exemple in the exemple 4.1 and we add the term $\mathfrak{A}\chi(\iota) = \dfrac{\cos(-\chi(\iota, y)\exp((\iota, y))}{\iota + 50}$ in the first equation and the term $\mathfrak{A}\zeta(\iota) = \dfrac{\cos(-\zeta(\iota, y)\exp((\iota, y))}{\iota + 50}$ in the second equation of the system (16). We can conclude that

$$\frac{\overline{W}_0 + \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\|g\|}{\mathcal{B}(q)}}{1 - |\varrho\| - \left(1 - q + \frac{\iota^q}{\Gamma(q)}\right)\frac{\gamma}{\mathcal{B}(q)}} < 3.$$

Thus $N' \geq 3$. It follows that certain conditions are necessary $(\mathcal{H}_1)$, $(\mathcal{H}'_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ are satisfied. Thus, by Theorem 3.5 We affirm that there is a viable solution to the problem at hand.

## 5. Conclusion

In this paper, we investigated the existence results for a new class of coupled system of hybrid fractional integro-differential equations involving Atangana-Baleanu-Caputo operator. We established the existence results for the problem using standard fixed point theorem. A numerical example is presented to clarify the obtained result. As a direction for future research, we aim to extend these results to study the $\psi$-Hilfer fractional derivative, along with graphical and numerical examples.


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## Data Availability

The data used to support the finding of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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