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Rainbow numbers in planar host graphs

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Abstract. If $\mathcal{F}$ is a nonempty set of graphs that contain H as a subgraph, then the rainbow number $\mathrm{rb}(\mathcal{F}, H)$ is the least $t \in \mathbb{N}$ such that every t-coloring of $F \in \mathcal{F}$ that uses all t colors contains a rainbow subgraph isomorphic to H. In this paper, we consider rainbow numbers when $\mathcal{F} = \{F\}$ where F is a wheel, a sunflower, or a double-hubbed wheel. Several exact evaluations are determined for various small subgraphs. Implications involving the case where $\mathcal{F}$ consists of all plane triangulations of order n are also discussed.

1. Introduction

Graphs are assumed to be finite, simple (no loops or multiedges), and undirected throughout this article. Readers are referred to [3] for basic terminology and definitions. For any set S, we denote the cardinality of S by $|S|$. If $G = (V(G), E(G))$ is a graph, then its *order* is $|V(G)|$ and its *size* is $|E(G)|$. For any vertex $x \in V(G)$, the *(open) neighborhood of* x is the set

$$N_G(x) := \{y \in V(G) \mid xy \in E(G)\}$$

and the *degree of* x is $d_G(x) := |N_G(x)|$. The *closed neighborhood of* x is given by $\overline{N}_G(x) := N_G(x) \cup \{x\}$. The *minimum degree* of G is given by

$$\delta(G) := \min\{d_G(x) \mid x \in V(G)\}.$$

For any subset $S \subseteq V(G)$, the *subgraph of* G *induced by* S, denoted $G[S]$, has vertex set S and edge set

$$E(G[S]) := \{xy \mid x, y \in S \text{ and } xy \in E(G)\}.$$

If $n \in \mathbb{N}$, we write $[n] := \{1, 2, \ldots, n\}$.

The graphs at the focus of this paper are all *planar* in that they can be drawn in the plane without overlapping edges. A *plane triangulation* is a planar graph with a plane graph depiction in which every face is a triangle. For example, a plane triangulation of order 8 is given in Figure 1.

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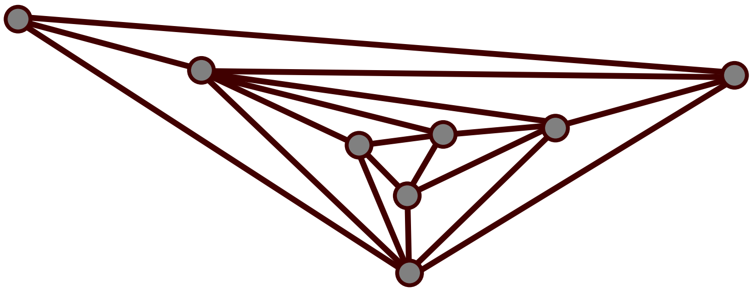

Figure 1: A plane triangulation of order 8.

Denote by $\mathcal{T}_n$ the collection of all plane triangulations of order $n \geq 3$. If $T \in \mathcal{T}_n$, then Euler's formula implies that $|E(T)| = 3n - 6$.

A complete graph of order n, a path of order n, and a cycle of order n are denoted by K_n, P_n, and C_n, respectively. The disjoint union of m copies of a graph G will be denoted by mG. If G and H are any two graphs, then the *join* $G + H$ has vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G) \text{ and } y \in V(H)\}.$$

For $n \geq 3$, the *wheel* is then defined by $W_n := K_1 + C_n$. The single vertex in the K_1 of the wheel is called the *hub* of the wheel. (for example, the first image in Figure 2 is the wheel graph W_5).

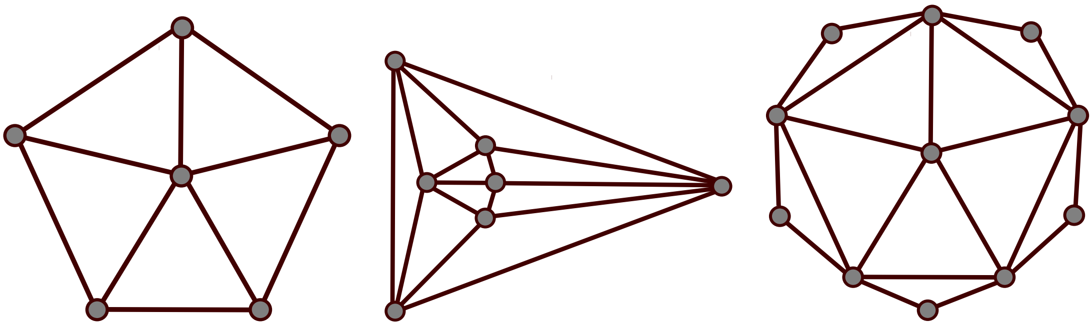

Figure 2: The wheel W_5, the double-hubbed wheel W_5^*, and the sunflower SF_5.

Thus, $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$ (note that some authors define the wheel W_n to have order n).

Define the *double-hubbed wheel* $W_n^* := 2K_1 + C_n$ (for example, the second image in Figure 2 shows W_5^*). The two vertices that make up the $2K_1$-subgraph are called the *hubs* of W_n^*. Note that a double-hubbed wheel is a plane triangulation of order $n + 2$ and has size $3n$. Let SF_n represent the *sunflower* of order $2n + 1$, formed by combining a wheel W_n, with hub x and n-vertex cycle $v_1 v_2 \cdots v_n v_1$, and n additional vertices $w_1, w_2, \ldots, w_n$ such that each vertex w_i is connected by edges to both v_i and v_{i+1} (where $i = 1, 2, \ldots, n$, and $i + 1$ is reduced modulo n). The hub of the W_n-subgraph is called the *hub* of SF_n. The sunflower SF_5 is shown in the third image in Figure 2.

For $n \geq 4$, the *theta graph* θ_n consists of the cycle C_n, along with a single edge joining two nonadjacent vertices in the cycle. For example, the theta graphs θ_4 and θ_5 are shown in the first two images in Figure 3, respectively.

We will also consider the graph θ_4^+, formed by joining a pendant edge to one of the vertices in θ_4 of degree 3 (see the third image in Figure 3), and the graph C_3^+, formed by joining a pendant edge to one of the vertices in C_3 (see the fourth image in Figure 3). In general, for any $k \geq 3$, the graph C_k^+ consists of the cycle C_k with a pendant edge joined to one of its vertices.

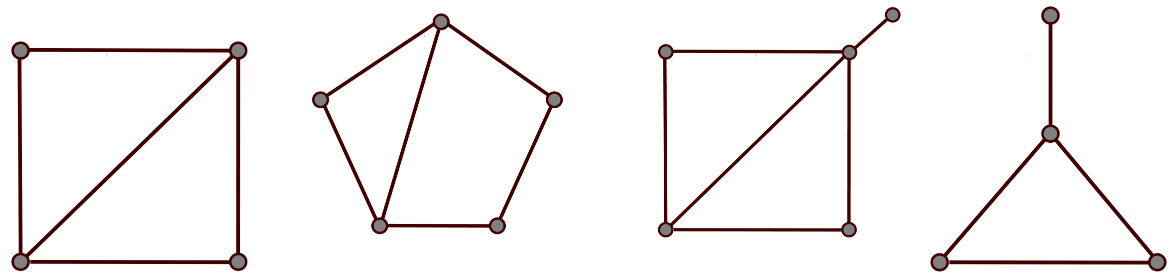

Figure 3: The graphs θ_4, θ_5, θ_4^+, and C_3^+.

A *t-coloring* of a graph G is a map $\psi : E(G) \longrightarrow [t]$. We say that ψ is an *exact t-coloring* if it is surjective. A subgraph H of G is a *rainbow subgraph* for a t-coloring ψ if

$$|\{\psi(e) \mid e \in E(H)\}| = |E(H)|.$$

In other words, the edges of a rainbow subgraph receive distinct colors under the map ψ.

Let $\mathcal{F}$ denote a nonempty set of graphs that all contain a graph H as a subgraph. Then the *rainbow number of H in $\mathcal{F}$*, denoted $\mathrm{rb}(\mathcal{F}, H)$, is defined to be the least $t \in \mathbb{N}$ such that every exact t-coloring of a graph $F \in \mathcal{F}$ contains a rainbow subgraph isomorphic to H. The graphs in $\mathcal{F}$ are called the *host graphs* for $\mathrm{rb}(\mathcal{F}, H)$. Note that if H' is a subgraph of H and H is a subgraph of every element in $\mathcal{F}$, then

$$\mathrm{rb}(\mathcal{F}, H') \leq \mathrm{rb}(\mathcal{F}, H). \tag{1}$$

When $\mathcal{F}$ contains only one graph F, we use the simpler notation $\mathrm{rb}(F, H)$ in place of $\mathrm{rb}(\{F\}, H)$.

In 1975, Erdős, Simonovits, and Sós [7] introduced the *anti-Ramsey number $f(n, H)$*, representing the maximum t such that there exists an exact t-coloring of K_n that lacks a rainbow subgraph isomorphic to H. From this definition, it is clear that

$$\mathrm{rb}(K_n, H) = f(n, H) + 1.$$

Anti-Ramsey numbers (or the equivalent rainbow numbers) for various classes for graphs have been extensively investigated in the literature [1, 6, 7, 9, 12, 15, 16, 17, 20, 27, 32]. Additional work has been carried out when the host graphs are bipartite [2, 26], planar [13, 14, 19, 21, 22, 23, 24, 25, 29, 30, 31], and Kneser graphs [18], among others. The analogous problem has also been considered within the setting of r-uniform hypergraphs [4, 8, 11, 28]. For an overview of such problems, readers are encouraged to consult the dynamic survey by Fujita, Magnant, Mao, and Ozeki [10].

Jendrol', Schiermeyer, and Tu [14] were the first to consider rainbow numbers in plane triangulations, focusing on matchings. Letting M_k denote a matching of size k, the rainbow number $\mathrm{rb}(\mathcal{T}_n, M_k)$ was evaluated for all $k \geq 3$ and $n \geq 9k + 3$ (see [5, 14, 29, 30]). In the case of cycles in plane triangulations, Horňák, Jendrol', Schiermeyer, and Soták [13] proved that

$$\mathrm{rb}(\mathcal{T}_n, C_3) = \left\lfloor \frac{3n}{2} \right\rfloor - 2, \quad \text{for all } n \geq 4. \tag{2}$$

For general $\mathrm{rb}(\mathcal{T}_n, C_k)$, they also offered lower bounds for all $4 \leq k \leq n$ and upper bounds when $k \in \{4, 5\}$. Lan, Shi, and Song [24] improved upon these lower bounds when $k \geq 5$ and $n \geq k^2 - k$, and determined upper bounds when $k \in \{6, 7\}$, framing their work in terms of planar Turán numbers. Applying Proposition 1.1 of [24] and Theorem 2.2 of [23], it follows that

$$\mathrm{rb}(\mathcal{T}_n, \theta_4) \leq \frac{12(n-2)}{5} + 1 = \frac{12n - 19}{5}, \quad \text{for all } n \geq 4. \tag{3}$$

In the case of wheel host graphs, Qin, Lei, and Li [31] proved that $\mathrm{rb}(W_n, C_3) = n + 1$ and Horňák, Jendrol', Schiermeyer, and Soták [13] proved that

$$\mathrm{rb}(W_n, C_4) = \left\lfloor \frac{4n}{3} \right\rfloor + 1. \tag{4}$$

Our investigation begins in Section 2, where we show that

$$\begin{aligned} \mathrm{rb}(W_n, \theta_4) &= \left\lfloor \frac{3n}{2} \right\rfloor + 1, \text{ for all } n \geq 3, \\ \mathrm{rb}(W_n, \theta_4^+) &= \left\lfloor \frac{3n}{2} \right\rfloor + 1, \text{ for all } n \geq 6, \text{ and} \\ \mathrm{rb}(W_n, \theta_5) &= \left\lfloor \frac{3n}{2} \right\rfloor + 1, \text{ for all } n \geq 5. \end{aligned}$$

In Section 3, we turn our attention to rainbow numbers in sunflower graphs. We show that

$$\begin{aligned} rb(SF_n, C_3) &= \begin{cases} 7 & \text{if } n = 3 \\ 2n + 2 & \text{if } n \geq 4, \end{cases} \\ \mathrm{rb}(SF_3, C_3^+) &= 7, \\ \mathrm{rb}(SF_n, C_3^+) &= 2n + 2, \quad \text{for } n \in \{4, 5\}, \\ \mathrm{rb}(SF_3, C_4) &= 9, \\ \mathrm{rb}(SF_4, C_4) &= 13, \\ \mathrm{rb}(SF_n, \theta_4) &= \left\lfloor \frac{7n}{2} \right\rfloor + 1, \quad \text{for all } n \geq 3, \text{ and} \end{aligned}$$

$$3n + 2 \leq \mathrm{rb}(SF_n, C_4) \leq 3n + 1 + \left\lfloor \frac{n}{4} \right\rfloor, \quad \text{for all } n \geq 5.$$

In particular, it follows that

$$\mathrm{rb}(SF_5, C_4) = 17, \quad \mathrm{rb}(SF_6, C_4) = 20, \quad \text{and} \quad \mathrm{rb}(SF_7, C_4) = 23.$$

Finally, in Section 4, we consider rainbow numbers for double-hubbed wheels, proving that for all $n \geq 3$,

$$rb(W_n^*, C_3) = \left\lfloor \frac{3n}{2} \right\rfloor + 1,$$

$$\left\lfloor \frac{4n}{3} \right\rfloor + 2 \leq rb(W_n^*, C_4) \leq 2n + 1, \quad \text{and}$$

$$\mathrm{rb}(W_n^*, \theta_4) \geq \begin{cases} 2n & \text{if } n \text{ is odd} \\ 2n + 1 & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad \mathrm{rb}(W_n^*, \theta_4) < \frac{7n}{3}.$$

When $n = 3k$ for some $k \in \mathbb{N}$, it is also shown that $\mathrm{rb}(W_n^*, \theta_4) \leq 2n + 2$. As W_n^* is a plane triangulation, implications of our work on rainbow numbers of the form $\mathrm{rb}(\mathcal{T}_n, H)$ is also considered. We conclude by discussing some directions for future work on this topic.

2. Rainbow Numbers in Wheels

Let W_n be a wheel graph with hub x and n-vertex-cycle $v_1 v_2 \cdots v_n v_1$. The edges $s_i = xv_i$ are called the *spokes* and the edges $r_i = v_i v_{i+1}$ are called *rim edges*, where $i \in \{1, 2, \ldots, n\}$ and the indices are reduced modulo n. A subgraph of W_n is called *central* if it includes the hub x.

In the following theorem, we determine the value of $\mathrm{rb}(W_n, \theta_4)$. Our proof of the upper bound follows the approach used in Lemma 3 of [13].

Theorem 2.1. *For all $n \geq 3$,* $\mathrm{rb}(W_n, \theta_4) = \left\lfloor \frac{3n}{2} \right\rfloor + 1$.

Proof. First, we provide an exact $\left\lfloor \frac{3n}{2} \right\rfloor$-coloring ψ of W_n that avoids a rainbow θ_4. For each $i \in \{1, \ldots, n\}$, let $\psi(r_i) = i$ and

$$\psi(s_i) = \begin{cases} n + \frac{i}{2}, & \text{if } i \text{ is even,} \\ \psi(s_{i+1}), & \text{if } i \text{ is odd,} \end{cases}$$

where the indices are reduced modulo n. For example, Figure 4 shows the coloring ψ for the cases $n = 5$ and $n = 6$.

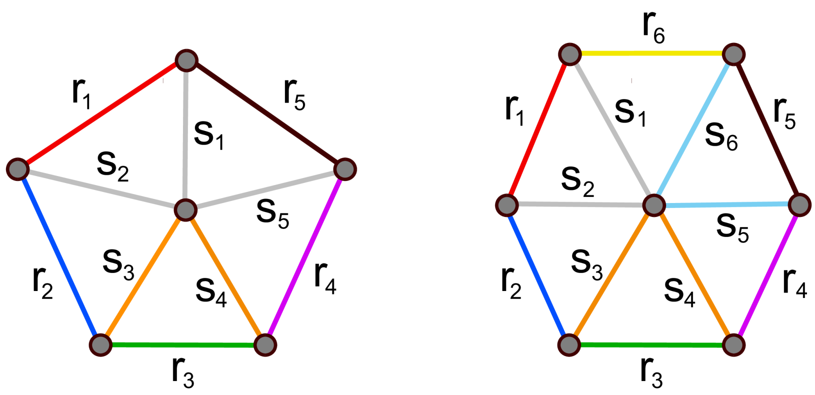

Figure 4: An exact 7-coloring of W_5 and an exact 9-coloring of W_6 that avoid rainbow θ_4-subgraphs.

Every θ_4-subgraph includes the hub x and three consecutive vertices in the cycle $v_1v_2 \cdots v_nv_1$, so it must include both s_i and s_{i-1} for some i. Hence, no rainbow θ_4 exists. Since ψ uses $n + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{3n}{2} \right\rfloor$ colors, it follows that $\mathrm{rb}(W_n, \theta_4) \geq \left\lfloor \frac{3n}{2} \right\rfloor + 1$.

To see that the reverse inequality is also true, let ψ be an exact $\left(\left\lfloor \frac{3n}{2} \right\rfloor + 1\right)$-coloring of W_n that avoids a rainbow θ_4-subgraph. Each spoke is in exactly three central θ_4-subgraphs and each rim edge is in exactly two central θ_4-subgraphs. One can obtain a contradiction using the same argument used in the proof of Lemma 3 of [13], so we leave the details to the reader. It follows that every exact $\left(\left\lfloor \frac{3n}{2} \right\rfloor + 1\right)$-coloring of W_n contains a rainbow θ_4-subgraph, and hence, $\mathrm{rb}(W_n, \theta_4) \leq \left\lfloor \frac{3n}{2} \right\rfloor + 1$. $\square$

When considering the rainbow number $\mathrm{rb}(W_n, \theta_4^+)$, we start with $n = 4$ since θ_4^+ is not a subgraph of W_3.

Theorem 2.2. *For $n \geq 4$,*

$$rb(W_n, \theta_4^+) = \begin{cases} 8, & \text{if } n = 4, \\ 9, & \text{if } n = 5, \\ \left\lfloor \frac{3n}{2} \right\rfloor + 1, & \text{if } n \geq 6. \end{cases}$$

Proof. We divide the proof into cases based on the values of n considered in the statement of the theorem.

Case 1 Assume that $n = 4$. Every θ_4^+-subgraph of W_4 is central and includes all four spokes. If two spokes receive the same color, and all other edges receive distinct colors, then we have produced an exact 7-coloring of W_4 that avoids a rainbow θ_4^+-subgraph. Hence, $\mathrm{rb}(W_4, \theta_4^+) \geq 8$. On the other hand, $|E(W_4)| = 8$, so every exact 8-coloring of W_4 necessarily contains a rainbow θ_4^+-subgraph. Hence, $\mathrm{rb}(W_4, \theta_4^+) \leq 8$.

Case 2 Assume that $n = 5$. Since every θ_4^+-subgraph of W_5 contains at least four of the spokes, coloring three spokes the same color, and giving all other edges distinct colors, results in an exact 8-coloring of W_5 that avoids a rainbow θ_4^+-subgraph. Hence, $\mathrm{rb}(W_5, \theta_4^+) \geq 9$. Now consider an exact 9-coloring of W_5. As $|E(W_5)| = 10$, only a single color is repeated one time. Delete one of the edges in this repeated color.

Regardless of whether the deleted edge is a spoke or a rim edge, the resulting graph contains a θ_4^+-subgraph, which is rainbow. Hence, $\mathrm{rb}(W_5, \theta_4^+) \le 9$.

Case 3 Assume that $n \ge 6$. Since θ_4 is a subgraph of θ_4^+, Inequality (1) and Theorem 2.1 imply that

$$rb(W_n, \theta_4^+) \ge rb(W_n, \theta_4) = \left\lfloor \frac{3n}{2} \right\rfloor + 1.$$

Now consider an exact $\left(\left\lfloor \frac{3n}{2} \right\rfloor + 1\right)$-coloring of W_n. By Theorem 2.1, there exists a rainbow θ_4. Such a subgraph necessarily includes the hub x, three spokes, and two rim edges. If a rainbow θ_4^+-subgraph does not exist, then the other $n-3$ spokes must receive colors that already appear in the θ_4-subgraph. In this case, at most $n+3$ edges can receive distinct colors. As $\left\lfloor \frac{3n}{2} \right\rfloor + 1 > n+3$ whenever $n \ge 6$, it follows that there must exist a rainbow θ_4^+-subgraph. Hence,

$$rb(W_n, \theta_4^+) \le rb(W_n, \theta_4) = \left\lfloor \frac{3n}{2} \right\rfloor + 1,$$

completing the proof. □

Theorem 2.3. *If $n \ge 4$, $rb(W_n, \theta_5) = \left\lfloor \frac{3n}{2} \right\rfloor + 1$.*

Proof. We start by providing an exact $\left\lfloor \frac{3n}{2} \right\rfloor$-coloring ψ of W_n that avoids a rainbow θ_5. For each $i \in \{1, \ldots, n\}$, let $\psi(s_i) = i$ and

$$\psi(r_i) = \begin{cases} n + \frac{i}{2}, & \text{if } i \text{ is even,} \\ \psi(r_{i+1}), & \text{if } i \text{ is odd,} \end{cases}$$

where the indices are reduced modulo n. For example, Figure 4 shows the coloring ψ for the cases $n = 5$ and $n = 6$.

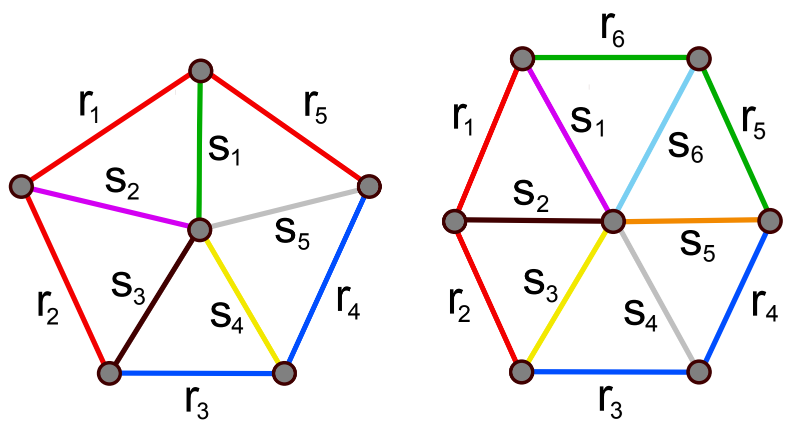

Figure 5: An exact 7-coloring of W_5 and an exact 9-coloring of W_6 that avoid rainbow θ_5-subgraphs.

Every θ_5-subgraph includes the hub x and three consecutive rim edges. Hence, no rainbow θ_5 exists. Since ψ uses $n + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{3n}{2} \right\rfloor$ colors, it follows that $\mathrm{rb}(W_n, \theta_5) \ge \left\lfloor \frac{3n}{2} \right\rfloor + 1$.

To see that the reverse inequality is true, let ψ be an exact t-coloring of W_n that avoids a rainbow θ_5-subgraph. For each $k \in \mathbb{N}$, let $\mathcal{A}_k$ denote the set of colors that appear on exactly k of the edges in W_n under the map ψ. When $j \ge 1$ and $\gamma \in [t]$, define $\beta_j(\gamma)$ to be the number of central θ_5-subgraphs of W_n that have exactly j edges in color γ. Then

$$\beta(\gamma) := \sum_{j \ge 2} \beta_j(\gamma)$$

is the number of central θ_5-subgraphs of W_n that have two or more edges in color γ. Also define the numbers

$$\eta(\gamma) := |\{i \in [n] \mid \psi(xv_i) = \gamma\}| \text{ and } \eta'(\gamma) := |\{i \in [n] \mid \psi(v_i v_{i+1}) = \gamma\}|,$$

representing the number of spokes and the number of rim edges in color γ, respectively. Note that for any $\gamma \in \mathcal{A}_k$, $\eta(\gamma) + \eta'(\gamma) = k$.

The graph W_n contains a total of $2n$ θ_5-subgraphs, all of which are central, and each spoke and each rim edge occur in exactly six of the θ_5-subgraphs. For any $\gamma \in \mathcal{A}_k$, it follows that

$$2\beta(\gamma) \leq 2\beta(\gamma) + \beta_1(\gamma) \leq \sum_{j\geq 1} j\beta_j(\gamma) = 6\eta(\gamma) + 6\eta'(\gamma) \leq 6k.$$

Equivalently, $\beta(\gamma) \leq 3k$. Note that for any $\gamma \in \mathcal{A}_2$, two edges of W_n being colored with γ can prevent at most four central θ_5-subgraphs from being rainbow. Similarly, for any $\gamma \in \mathcal{A}_3$, three edges of W_n being colored with γ can prevent at most six central θ_5-subgraphs from being rainbow. So, $\beta_2(\gamma) \leq 4$ and $\beta_3(\gamma) \leq 6$. Since every one of the $2n$ central θ_5-subgraphs is assumed to include a color γ such that $\beta(\gamma) \geq 1$, it follows that

$$2n \leq \sum_{k\geq 2} \sum_{\gamma \in \mathcal{A}_k} \beta(\gamma) \leq 4|\mathcal{A}_2| + 6|\mathcal{A}_3| + \sum_{k\geq 4} 3k|\mathcal{A}_k|.$$

Combining this inequality with $2n = |E(W_n)| = \sum_{k\geq 1} k|\mathcal{A}_k|$, we see that

$$3|\mathcal{A}_1| + 2|\mathcal{A}_2| + 3|\mathcal{A}_3| \leq 4n.$$

Using the weaker inequality

$$3|\mathcal{A}_1| + 2|\mathcal{A}_2| + |\mathcal{A}_3| \leq 4n,$$

we find the following upper bound for the number of colors used by ψ:

$$\begin{aligned} t &= |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| + \sum_{k\geq 4} |\mathcal{A}_k| \\ &\leq |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| + \sum_{k\geq 4} \frac{k}{4}|\mathcal{A}_k| \\ &\leq \frac{1}{4}(3|\mathcal{A}_1| + 2|\mathcal{A}_2| + |\mathcal{A}_3|) + \frac{1}{4}\sum_{k\geq 1} k|\mathcal{A}_k| \\ &\leq \frac{4n}{4} + \frac{2n}{4} = \frac{3n}{2}. \end{aligned}$$

It follows that every exact t-coloring of W_n contains a rainbow θ_5-subgraph whenever $t \geq \left\lfloor \frac{3n}{2} \right\rfloor + 1$, resulting in the inequality $\mathrm{rb}(W_n, \theta_5) \leq \left\lfloor \frac{3n}{2} \right\rfloor + 1$. □

3. Rainbow Numbers in Sunflowers

Recall that SF_n is formed by combining a wheel W_n, with hub x and n-vertex cycle $v_1v_2\cdots v_nv_1$, with n additional vertices $w_1, w_2, \ldots, w_n$ such that each vertex w_i is connected by edges to both v_i and v_{i+1} (where $i \in [n]$, and $i+1$ is reduced modulo n). The edges $s_i = xv_i$ are called the *spokes*, the edges $r_i = v_iv_{i+1}$ are called the *rim edges*, and the edges $p_i = w_iv_i$ and $p_i' = w_iv_{i+1}$ are called the *petal edges*.

The following lemma will be useful for the rainbow numbers we consider in this section.

Lemma 3.1. *[13] Let C_k be a rainbow cycle of length $k \geq 4$ in an edge-colored graph G. If $G[C_k]$ has a diagonal, then there exists a rainbow cycle in G of length smaller than k. Furthermore, all but one of the edges in the smaller rainbow cycle are included in the rainbow C_k.*

Theorem 3.2. *The rainbow number for C_3 in SF_n satisfies*

$$rb(SF_n, C_3) = \begin{cases} 7, & \text{if } n = 3, \\ 2n+2, & \text{if } n \geq 4. \end{cases}$$

Proof. Regardless of the value of n, SF_n contains n central C_3-subgraphs and n non-central C_3-subgraphs that each consists of a single rim edge and two consecutive petal edges. In the special case where $n = 3$, SF_3 also contains a non-central C_3-subgraph consisting of the three rim edges. Hence, the $n = 3$ case must be handled separately.

Case 1 Assume that $n = 3$ and consider the exact 6-coloring of SF_3 given by $\psi(v_1v_2) = \psi(v_2v_3) = 1$, $\psi(v_1v_3) = 2$, $\psi(xv_i) = 3$, and $\psi(w_iv_i) = \psi(w_iv_{i+1}) = 3 + i$, for all $i \in [3]$ (with the indices reduced modulo 3). See the first image in Figure 6.

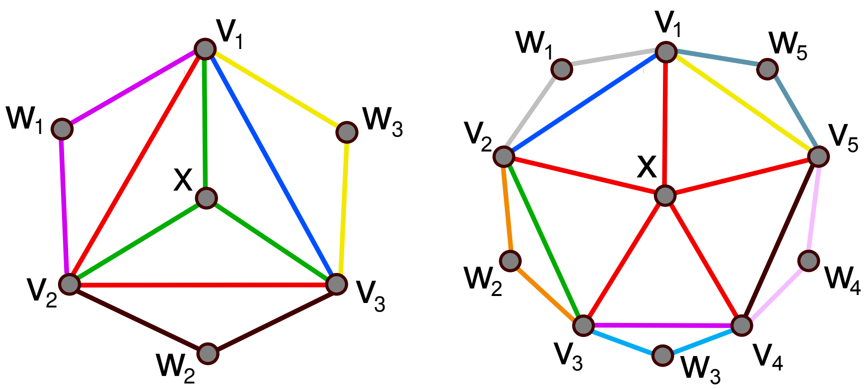

Figure 6: An exact 6-coloring of SF_3 and an exact 11-coloring of SF_5 that avoid rainbow C_3-subgraphs.

As this coloring does not contain a rainbow C_3-subgraph, it follows that $\mathrm{rb}(SF_3, C_3) \geq 7$.

To prove the reverse inequality, consider an exact 7-coloring of SF_3. We complete the proof of the upper bound by considering subcases based on the number of colors used on the spokes.

Subcase 1.1 Assume that the spokes receive three distinct colors, then the rim edges must use the same three colors in order to avoid a rainbow C_3-subgraph. The four remaining colors must all appear on the petal edges, from which it follows that there is some $i \in [3]$ such that w_iv_i and w_iv_{i+1} receive distinct colors, different from that of v_iv_{i+1}, producing a rainbow C_3-subgraph.

Subcase 1.2 Assume that the spokes receive exactly two colors. Without loss of generality, assume that xv_1 and xv_2 receive color 1 and xv_3 receives color 2. Then v_2v_3 and v_1v_3 must also receive one of the colors 1 or 2. Regardless of whether or not v_1v_2 receives a new color, at least four colors are then used on the petal edges. Once again it follows that there is some $i \in [3]$ such that w_iv_i and w_iv_{i+1} receive distinct colors, different from that of v_iv_{i+1}, producing a rainbow C_3-subgraph.

Subcase 1.3 Assume that the spokes all receive the same color. Then the rim edges can use at most two new colors, leaving at least four petal edges in distinct new colors. It follows that there is some $i \in [3]$ such that w_iv_i and w_iv_{i+1} receive distinct colors, different from that of v_iv_{i+1}, producing a rainbow C_3-subgraph.

In all three subcases, we see that an exact 7-coloring of SF_3 contains a rainbow C_3-subgraph. The inequality $\mathrm{rb}(SF_3, C_3) \leq 7$ then follows.

Case 2 Assume that $n \geq 4$ and consider the following exact $(2n + 1)$-coloring of SF_3: $\psi(xv_i) = 1$, $\psi(v_iv_{i+1}) = 1 + i$, and $\psi(w_iv_i) = \psi(w_iv_{i+1}) = n + 1 + i$, for all $i \in [n]$ (with the indices reduced modulo n). See the second image in Figure 6 for the case $n = 5$. Since every central C_3-subgraph contains at least two spokes and every non-central C_3-subgraph contains two petal edges w_iv_i and w_iv_{i+1}, for some $i \in [n]$, it follows that $\mathrm{rb}(SF_n, C_3) \geq 2n + 2$.

To prove the reverse inequality, consider an exact $(2n + 2)$-coloring of SF_n. Let F be a subgraph of SF_n such that $|E(F)| = 2n + 2$ and $\psi(e_i) \neq \psi(e_j)$ for all distinct $e_i, e_j \in E(F)$ (so, F is a rainbow subgraph of SF_n). Since $|V(F)| \leq |V(SF_n)| = 2n + 1$, the graph F is not a tree, and hence, must contain a cycle. So, SF_n contains a rainbow cycle. Let C_k be a rainbow cycle of minimum order k in SF_n. If $k = 3$, then the proof is complete. So, assume that $k \geq 4$ and note that if C_k is central, then it contains some diagonal in SF_n. By Lemma 3.1, there exists a rainbow cycle C_ℓ such that $\ell \leq k$. However, this contradicts the assumption that C_k is a rainbow cycle of minimum order.

It is also possible that the rainbow cycle C_k is not central. It could have any of the orders $n, n+1, \ldots, 2n$ using rim edges and some collection of consecutive petal edges. If any consecutive pair of petal edges are used, then C_k has a diagonal in SF_n, and once again using Lemma 3.1, there is a rainbow C_ℓ, where $\ell \leq k$, giving a contradiction. So, assume that the rainbow C_k consists of only rim edges, in which case $k = n$. Since F has size $2n + 2$, at least two spokes must be in F, which along with some of the edges in C_k, can form a central cycle of order less than n, once again leading to a contradiction. It follows that there exists a rainbow C_3, which implies $\mathrm{rb}(SF_n, C_3) \leq 2n + 2$. □

Theorem 3.3. *The rainbow number for C_3^+ in SF_n satisfies*

$$\mathrm{rb}(SF_3, C_3^+) = 7 \quad \textit{and} \quad \mathrm{rb}(SF_n, C_3^+) = 2n + 2, \quad \textit{for } n \in \{4, 5\}.$$

Proof. Theorem 3.2 implies that

$$rb(SF_n, C_3^+) \geq rb(SF_n, C_3) = \begin{cases} 7, & \text{if } n = 3, \\ 2n + 2, & \text{if } n \geq 4. \end{cases}$$

To prove the reverse inequality, consider the cases $n = 3$ and $n \in \{4, 5\}$ separately.

Case 1 Suppose that $n = 3$ and consider an exact 7-coloring of SF_3. Since $\mathrm{rb}(SF_3, C_3) = 7$, there exists a rainbow C_3 subgraph. If the rainbow C_3-subgraph consists of the three rim edges, then none of the other edges can receive a new color without producing a rainbow C_3^+-subgraph. If the rainbow C_3-subgraph is central (i.e., it includes two spokes and a rim edge), then only one spoke and two petal edges can receive colors not contained in the C_3 without forming a rainbow C_3^+-subgraph. In this case, only six colors can be used. Finally, if the rainbow C_3-subgraph includes two petal edges and a rim edge, then again, only a single spoke and two petal edges can receive colors not included in the C_3. In all cases, we find that the SF_3 must contain a rainbow C_3^+-subgraph.

Case 2 Suppose that $n \in \{4, 5\}$ and consider an exact $(2n + 2)$-coloring of SF_n and note that $\mathrm{rb}(SF_n, C_3) = 2n + 2$ implies that there exists a rainbow C_3-subgraph. Consider two subcases.

Subcase 2.1 Suppose that there exists a rainbow C_3-subgraph that is central. That is, for some $i \in [n]$, the cycle $xv_iv_{i+1}x$ is rainbow. If a rainbow C_3^+ is avoided, then all of the spokes, two rim edges ($v_{i-1}v_i$ and $v_{i+1}v_{i+2}$), and four petal edges ($w_{i-1}v_i$, w_iv_i, w_iv_{i+1}, and $w_{i+1}v_{i+1}$) must use the same colors as the rainbow C_3-subgraph. This leaves only $n - 3$ rim edges and $2n - 4$ petal edges to use the remaining $2n - 1$ colors. However, $3n - 7 < 2n - 1$ whenever $n < 6$, so every such coloring must contain a rainbow C_3^+-subgraph.

Subcase 2.2 Suppose that no central rainbow C_3-subgraph exists. Since $\mathrm{rb}(W_n, C_3) = n + 1$ [13], the W_n-subgraph uses at most n colors. Hence, the petal edges must use at least $n + 2$ colors that do not appear on the rim edges or spokes. Iteratively applying the pigeonhole principle twice, it follows that there exist $i, j \in [n]$ with $i \neq j$ such that $w_iv_iv_{i+1}w_i$ and $w_jv_jv_{j+1}w_j$ are both rainbow C_3-subgraphs that do not share colors with one another. Note that the two rainbow C_3-subgraphs cannot have any vertices in common without producing a rainbow C_3^+-subgraph. So, if a rainbow C_3^+-subgraph is avoided, then there are at most $n - 4$ spokes, $n - 4$ rim edges, and $2n - 8$ petal edges that can receive colors other than the six colors used in the two rainbow C_3-subgraphs. Since $4n - 16 < 2n - 4$ whenever $n < 6$, every such coloring must contain a rainbow C_3^+-subgraph. □

Theorem 3.4. *The rainbow number for C_4 in SF_n satisfies*

$$\mathrm{rb}(SF_3, C_4) = 9, \quad \mathrm{rb}(SF_4, C_4) = 13, \quad \textit{and}$$

$$3n + 2 \leq \mathrm{rb}(SF_n, C_4) \leq 3n + 1 + \left\lfloor \frac{n}{4} \right\rfloor,$$

for all $n \geq 5$.

Proof. We separate the proof into cases based upon the value of n.

Case 1 Assume that $n = 3$. In this case, the sunflower SF_3 contains three C_4-subgraphs that are not central, each of which contains exactly two rim edges and two petal edges. All other C_4-subgraphs are central, and hence include at least two spokes. Consider the following exact 8-coloring of SF_3: $\psi(xv_i) = 1$, $\psi(v_iv_{i+1}) = 2$, $\psi(w_iv_i) = i + 3$, and $\psi(w_iv_{i+1}) = i + 6$, where $i \in [3]$ and the indices are reduced modulo 3. As every C_4-subgraph includes at least two edges that are the same color, it follows that $\mathrm{rb}(SF_3, C_4) \geq 9$.

To prove the reverse inequality, consider an exact 9-coloring of SF_3 that avoids a rainbow C_4-subgraph. By Inequality (4), the W_3-subgraph uses at most four colors. Since there are six petal edges, the W_3-subgraph must use at least three colors. If it uses exactly three colors, then each petal edge receives its own unique color, and regardless of whether there are two colors used for the spokes or two colors used for the rim edges, combining distinctly colored consecutive spokes or rim edges with two corresponding petal edges forms a rainbow C_4. So, assume that the W_3-subgraph uses exactly four colors and consider three subcases.

Subcase 1.1 Suppose that the spokes all receive distinct colors and the rim edges all receive the fourth color. The petal edges use five different colors and there exists some $i \in [3]$ such that w_iv_i and w_i, v_{i+1} receive distinct colors that do not appear in the W_3-subgraph. Then $xv_iw_iv_{i+1}x$ is a rainbow C_4-subgraph.

Subcase 1.2 Suppose that two colors appear on the spokes and two different colors appear on the rim edges. Without loss of generality, assume that xv_1 and xv_2 are red and xv_3 is blue. In order to avoid a rainbow central C_4-subgraph, edges v_1v_2 and v_2v_3 must be the same color. However, v_1v_3 and v_1v_2 must also be the some color, contradicting the assumption that the W_3-subgraph uses four colors.

Subcase 1.3 Suppose that the rim edges all receive distinct colors and the spokes receive the fourth color. The petal edges use five different colors and there exists some $i \in [3]$ such that w_iv_i and w_i, v_{i+1} receive distinct colors that do not appear in the W_3-subgraph. Then $w_iv_{i+1}v_{i+2}v_iw_i$ is a rainbow C_4-subgraph.

Thus, every exact 9-coloring of SF_3 contains a rainbow C_4-subgraph. It follows that $\mathrm{rb}(SF_3, C_4) \leq 9$.

Case 2 Assume that $n = 4$. In this case, the sunflower SF_4 contains one C_4-subgraph that is not central, consisting of the four rim edges. All other C_4-subgraphs are central, and hence, include at least two spokes. Consider the following exact 12-coloring of SF_4: $\psi(xv_i) = 1$, $\psi(v_1v_2) = 2 = \psi(v_2v_3)$, $\psi(v_3v_4) = 3$, $\psi(v_4v_1) = 4$, $\psi(w_iv_i) = i + 4$, and $\psi(w_iv_{i+1}) = i + 8$, where $i \in [4]$ and the indices are reduced modulo 4. As every C_4-subgraph includes at least two edges that are the same color, it follows that $\mathrm{rb}(SF_4, C_4) \geq 13$.

To prove the reverse inequality, consider an exact 13-coloring of SF_4 that avoids a rainbow C_4-subgraph. By Inequality (4), the W_4-subgraph uses at most 5 colors. As there are exactly 8 petal edges, the W_4-subgraph uses exactly 5 colors and each of the 8 petal edges receives its own distinct color. Since the rim edges form a C_4-subgraph, they use at most 3 colors, from which it follows that the spokes use at least 2 of the colors. For some $i \in [n]$, there exist consecutive spokes xv_i and xv_{i+1} that are different colors. Then one of the petals w_iv_i or w_iv_{i+1} must repeat a color, leading to a contradiction. It follows that $\mathrm{rb}(SF_4, C_4) \leq 13$.

Case 3 Assume that $n \geq 5$. In this case, every C_4-subgraph of SF_n is central, and hence, includes at least two spokes. Consider the following exact $(3n+1)$-coloring of SF_n: $\psi(xv_i) = 1$, $\psi(v_iv_{i+1}) = i+1$, $\psi(w_iv_i) = 2i+1$, and $\psi(w_iv_{i+1}) = 3i + 1$, for all $i \in [n]$, and where the indices are reduced modulo n. As every C_4-subgraph includes at least two edges that are the same color, it follows that $\mathrm{rb}(SF_n, C_4) \geq 3n + 2$, for all $n \geq 5$.

To prove the upper bound stated in the theorem, let ψ be an exact t-coloring of SF_n that avoids a rainbow C_4-subgraph. For each $k \in \mathbb{N}$, let $\mathcal{A}_k$ denote the set of colors that appear on exactly k of the edges in SF_n under the map ψ. When $j \geq 1$ and $\gamma \in [t]$, define $\beta_j(\gamma)$ to be the number of central C_4-subgraphs of SF_n that have exactly j edges in color γ. Then

$$\beta(\gamma) := \sum_{j \geq 2} \beta_j(\gamma)$$

is the number of central C_4-subgraphs of SF_n that have two or more edges in color γ. Also define the numbers

$$\begin{aligned}
\eta(\gamma) &:= |\{i \in [n] \mid \psi(xv_i) = \gamma\}|, \\
\eta'(\gamma) &:= |\{i \in [n] \mid \psi(v_iv_{i+1}) = \gamma\}|, \text{ and} \\
\eta''(\gamma) &:= |\{i \in [n] \mid \psi(w_iv_i) = \gamma\}| + |\{i \in [n] \mid \psi(w_iv_{i+1}) = \gamma\}|.
\end{aligned}$$

Note that for any $\gamma \in \mathcal{A}_k$, $\eta(\gamma) + \eta'(\gamma) + \eta''(\gamma) = k$.

The graph SF_n contains a total of $2n$ C_4-subgraphs, all of which are central. Each petal edge is included in exactly one C_4-subgraph, each rim edge is included in exactly two C_4-subgraphs, and each spoke is included in exactly four C_4-subgraphs. For each $\gamma \in \mathcal{A}_k$, we see that

$$2\beta(\gamma) \le 2\beta(\gamma) + \beta_1(\gamma) \le \sum_{j\ge 1} j\beta_j(\gamma) = 4\eta(\gamma) + 2\eta'(\gamma) + \eta''(\gamma) \le 4k,$$

from which it follows that $\beta(\gamma) \le 2k$. Notice that for any $\gamma \in \mathcal{A}_2$, two edges of SF_n colored by γ can prevent at most one central C_4-subgraph from being rainbow and three edges of SF_n colored γ can prevent at most three central C_4-subgraphs from being rainbow. So, $\beta_2(\gamma) \le 1$ and $\beta_3(\gamma) \le 3$. Given that every one of the $2n$ central C_4-subgraphs includes at least one color γ such that $\beta(\gamma) \ge 1$, it follows that

$$2n \le \sum_{k\ge 2} \sum_{\gamma\in\mathcal{A}_k} \beta(\gamma) \le |\mathcal{A}_2| + 3|\mathcal{A}_3| + \sum_{k\ge 4} 2k|\mathcal{A}_k|.$$

Combining this with $4n = |E(SF_n)| = \sum\limits_{k\ge 1} k|\mathcal{A}_k|$, we can conclude that

$$2|\mathcal{A}_1| + 3|\mathcal{A}_2| + 3|\mathcal{A}_3| \le 6n \quad \Longrightarrow \quad |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \le 3n.$$

Then

$$\begin{aligned}
t &= |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| + \sum_{k\ge 4} |\mathcal{A}_k| \\
&\le |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| + \sum_{k\ge 4} \frac{k}{4}|\mathcal{A}_k| \\
&\le \frac{1}{4}(3|\mathcal{A}_1| + 2|\mathcal{A}_2| + |\mathcal{A}_3|) + \frac{1}{4}\sum_{k\ge 1} k|\mathcal{A}_k| \\
&\le \frac{3}{4}(|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3|) + \frac{4n}{4} \\
&\le \frac{3}{4}(3n) + \frac{4n}{4} = 3n + \frac{n}{4}.
\end{aligned}$$

Thus, $t \le 3n + \left\lfloor \frac{n}{4} \right\rfloor$ whenever ψ avoids a rainbow C_4-subgraph. It follows that every exact t-coloring of SF_n contains a rainbow C_4-subgraph whenever $t \ge 3n + 1 + \left\lfloor \frac{n}{4} \right\rfloor$, and hence, $\mathrm{rb}(SF_n, C_4) \le 3n + 1 + \left\lfloor \frac{n}{4} \right\rfloor$, for all $n \ge 5$. □

Note that besides the cases $n = 3, 4$, Theorem 3.4 also implies the following exact evaluations:

$$\mathrm{rb}(SF_5, C_4) = 17, \quad \mathrm{rb}(SF_6, C_4) = 20, \quad \text{and} \quad \mathrm{rb}(SF_7, C_4) = 23.$$

Theorem 3.5. *For all $n \ge 3$, $\mathrm{rb}(SF_n, \theta_4) = \left\lfloor \frac{7n}{2} \right\rfloor + 1$.*

Proof. First, we provide an exact $(\left\lfloor \frac{7n}{2} \right\rfloor + 1)$-coloring of SF_n that avoids a rainbow θ_4-subgraph. For each $i \in [n]$, let $\psi(v_iv_{i+1}) = i$, $\psi(w_iv_i) = n + i$, $\psi(w_iv_{i+1}) = 2n + i$, and

$$\psi(xv_i) = \begin{cases} 3n + \frac{i}{2}, & \text{if } i \text{ is even,} \\ \psi(xv_{i+1}), & \text{if } i \text{ is odd.} \end{cases}$$

Since every θ_4-subgraph is central, no rainbow θ_4-subgraph exists. It follows that $\mathrm{rb}(SF_n, \theta_4) \ge \left\lfloor \frac{7n}{2} \right\rfloor + 1$.

To prove the reverse inequality, consider an exact $(\left\lfloor \frac{7n}{2} \right\rfloor + 1)$-coloring of SF_n. Even if all $2n$ petal edges receive their own distinct colors, at least $\left\lfloor \frac{7n}{2} \right\rfloor + 1 - 2n = \left\lfloor \frac{3n}{2} \right\rfloor + 1$ colors appear on the edges of the W_n-subgraph. By Theorem 2.1, $\mathrm{rb}(W_n, \theta_4) = \left\lfloor \frac{3n}{2} \right\rfloor + 1$, from which it follows that there exists a rainbow θ_4-subgraph. □

4. Rainbow Numbers in Double-Hubbed Wheels

Now we consider the rainbow number when the host graph is the double-hubbed wheel W_n^*, so we begin by formalizing the notation for the various components of W_n^*. Denote the hubs by x and y and assume that the cycle in the definition of W_n^* is given by $v_1v_2\cdots v_nv_1$. Denote the spokes by $s_i = xv_i$ and $t_i = yv_i$, and the rim edges by $r_i = v_iv_{i+1}$, with the indices reduced modulo n.

Theorem 4.1. *For all $n \geq 3$, $rb(W_n^*, C_3) = \left\lfloor \frac{3n}{2} \right\rfloor + 1$.*

Proof. When n is odd, consider the exact $\left\lfloor \frac{3n}{2} \right\rfloor$-coloring of W_n^* given by

$$\psi(s_i) = i, \quad \psi(r_i) = \begin{cases} \psi(s_{i+1}), & \text{if } i \text{ is odd and } i < n, \\ \psi(s_n), & \text{if } i = n, \\ \psi(s_i), & \text{if } i \text{ is even,} \end{cases}$$

$$\text{and } \psi(t_i) = \begin{cases} \psi(s_i), & \text{if } i \text{ is even,} \\ n + 1 + \left\lfloor \frac{i}{2} \right\rfloor, & \text{if } i \text{ is odd and } 3 \leq i \leq n-2, \\ n+1, & \text{if } i = 1, \\ \psi(s_n), & \text{if } i = n. \end{cases}$$

For example, the second image in Figure 7 shows the case $n = 5$.

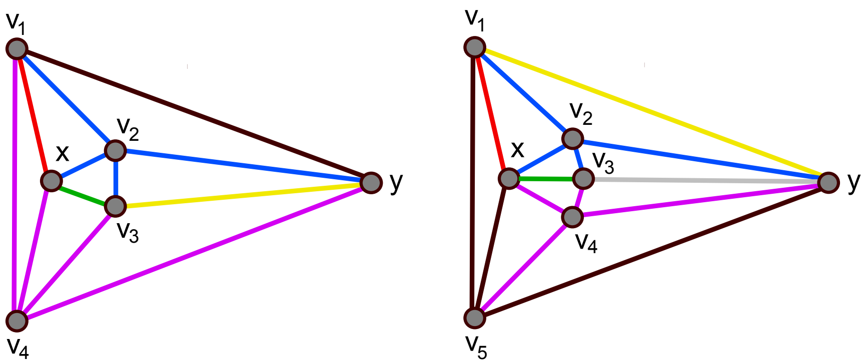

Figure 7: An exact 6-coloring of W_4^* and an exact 7-coloring of W_5^* that avoid rainbow C_3-subgraphs.

Note that all of the C_3-subgraphs that contain one of the hubs use two colors. In the special case where $n = 3$, the rim edges also form a C_3-subgraph, which has edges r_1 and r_2 receiving the same color. It follows that $\text{rb}(W_n^*, C_3) \geq \left\lfloor \frac{3n}{2} \right\rfloor + 1$, for all odd $n \geq 3$.

When n is even, consider the exact $\left\lfloor \frac{3n}{2} \right\rfloor$-coloring of W_n^* given by

$$\psi(s_i) = i, \quad \psi(r_i) = \begin{cases} \psi(s_{i+1}), & \text{if } i \text{ is odd,} \\ \psi(s_i), & \text{if } i \text{ is even,} \end{cases}$$

$$\text{and } \psi(t_i) = \begin{cases} \psi(s_i), & \text{if } i \text{ is even,} \\ n+1, & \text{if } i = 1, \\ n + 1 + \left\lfloor \frac{i}{2} \right\rfloor, & \text{if } i \text{ is odd and } i \geq 3. \end{cases}$$

The first image in Figure 7 shows the case $n = 4$. Again, every C_3-subgraph uses at most two colors, from which it follows that $\text{rb}(W_n^*, C_3) \geq \left\lfloor \frac{3n}{2} \right\rfloor + 1$, for all even $n \geq 4$.

For the upper bound, recall that by (2), $\mathrm{rb}(\mathcal{T}_n, C_3) = \left\lfloor \frac{3n}{2} \right\rfloor - 2$. Since W_n^* is a plane triangulation of order $n+2$, it follows that $\mathrm{rb}(W_n^*, C_3) \le \left\lfloor \frac{3(n+2)}{2} \right\rfloor - 2 = \left\lfloor \frac{3n}{2} \right\rfloor + 1$. $\square$

Theorem 4.2. *For all $n \ge 3$, $\left\lfloor \frac{4n}{3} \right\rfloor + 2 \le rb(W_n^*, C_4) \le 2n + 1$.*

Proof. Consider the exact $(\left\lfloor \frac{4n}{3} \right\rfloor + 1)$-coloring of W_n^* given by

$$\psi(s_i) = i, \quad \psi(r_i) = \begin{cases} \psi(s_{i+1}), & \text{if } i \equiv 1 \pmod 3, \\ \psi(s_i), & \text{if } i \equiv 2 \pmod 3, \\ n + \frac{i}{3}, & \text{if } i \equiv 0 \pmod 3, \end{cases}$$

$$\text{and } \psi(t_i) = \left\lfloor \frac{4n}{3} \right\rfloor + 1.$$

The cases $n = 4$ and $n = 5$ are shown in Figure 8.

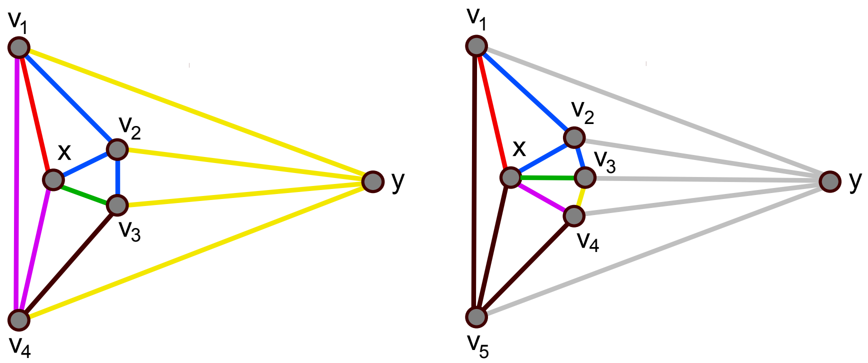

Figure 8: An exact 6-coloring of W_4^* and an exact 7-coloring of W_5^* that avoid rainbow C_4-subgraphs.

Any C_4-subgraph that is entirely contained in the W_n-subgraph with hub x uses at most three colors and any C_4-subgraph that includes y has color $\left\lfloor \frac{4n}{3} \right\rfloor + 1$ appear on two of its edges. It follows that $\mathrm{rb}(W_n^*, C_4) \ge \left\lfloor \frac{4n}{3} \right\rfloor + 2$.

The upper bound follows from Theorem 4 of [13], where it was shown that

$$\mathrm{rb}(\mathcal{T}_n, C_4) \le 2(n-2) + 1, \quad \text{for all } n \ge 4.$$

Since W_n^* is a plane triangulation of order $n+2$, it follows that

$$\mathrm{rb}(W_n^*, C_4) \le 2n + 1 \quad \text{for all } n \ge 3,$$

completing the proof of the theorem. $\square$

Now we consider the rainbow number for θ_4 in the set $\mathcal{T}_n$ of plane triangulations of order n. For each $T \in \mathcal{T}_n$ of order at least four and each $v \in V(T)$, the subgraph of T induced by the closed neighborhood $\overline{N}_T(v)$ contains a subgraph $W(v)$ that is isomorphic to W_d, where $d = d_T(v)$. If ψ is an edge coloring of T, denote by $C_\psi(v)$ the set of colors used by ψ on the edges of the wheel $W(v)$. In the next theorem, our proof of the upper bound for $\mathrm{rb}(\mathcal{T}_n, \theta_4)$ follows an approach similar to Theorem 4 of [13] and improves upon the bound in (3) whenever $n \ge 47$.

Theorem 4.3. *For all $n \ge 3$, $\mathrm{rb}(\mathcal{T}_n, \theta_4) < \frac{7(n-2)}{3}$.*

Proof. Let $T \in \mathcal{T}_n$ and suppose that $\psi : E(T) \longrightarrow \left[\frac{7(n-2)}{3}\right]$ is an exact edge coloring that avoids a rainbow θ_4-subgraph. By Lemma 1 of [13], it follows that

$$4\left(\frac{7(n-2)}{3}\right) \leq \sum_{v \in V(T)} |C_\psi(v)| \leq \sum_{v \in V(T)} (\mathrm{rb}(W(v), \theta_4) - 1).$$

By Theorem 2.1, it follows that

$$4\left(\frac{7(n-2)}{3}\right) \leq \sum_{v \in V(T)} \frac{3}{2} \cdot d_T(v) \leq \frac{3}{2} \cdot 2|E(T)| = 3(3n-6) = 9(n-2).$$

This inequality is equivalent to $28(n-2) \leq 27(n-2)$, which gives a contradiction. Hence, every exact t-coloring of T that avoids a rainbow θ_4-subgraph uses fewer than $\frac{7(n-2)}{3}$ colors. □

Since W_n^* is a plane triangulation of order $n+2$, Theorem 4.3 implies that $\mathrm{rb}(W_n^*, \theta_4) < \frac{7n}{3}$. In Theorem 4.5, we provide a lower bound, and in the case where $n = 3k$, an improvement of this bound is given. Before we state and prove this theorem, we need the following lemma, concerning the case $n = 3k$. For such values of n, W_n^* contains k edge-disjoint wheels $W(1), W(2), \ldots, W(k)$ that are isomorphic to W_4, with hubs v_{3j-1} and cycles $xv_{3j-2}yv_{3j}x$, for $1 \leq j \leq k$. The only edges in W_n^* that are not in these W_4-subgraphs are the edges $v_{3j}v_{3j+1}$ that join them. For any exact coloring ψ of W_n^*, we denote by $\psi(W(j))$ the set of colors used on the edges of $W(j)$.

Lemma 4.4. *Suppose that $n = 3k$ for some $k \in \mathbb{N}$, and let ψ be an exact coloring of W_n^* that avoids a rainbow θ_4-subgraph. For each j such that $1 \leq j \leq k$,*

$$|\psi(W(j)) \cup \psi(W(j+1)) \cup \{\psi(v_{3j}v_{3j+1})\}| \leq 12.$$

Proof. By Theorem 2.1, $|\psi(W(j))| \leq 6$ and $|\psi(W(j+1))| \leq 6$. So, if $\psi(W(j)) \cap \psi(W(j+1)) \neq \emptyset$ or $\psi(v_{3j}v_{3j+3}) \in \psi(W(j)) \cup \psi(W(j+1))$, then the statement in the lemma clearly follows. Assume that $W(j)$ and $W(j+1)$ do not have any colors in common and

$$\psi(v_{3j}v_{3j+1}) \notin \psi(W(j)) \cup \psi(W(j+1)).$$

To prevent the subgraph induced by $\{x, y, v_{3j}, v_{3j+1}\}$ from being a rainbow θ_4-subgraph, either $\psi(xv_{3j}) = \psi(yv_{3j})$ or $\psi(xv_{3j+1}) = \psi(yv_{3j+1})$. Without loss of generality, assume that $\psi(xv_{3j}) = \psi(yv_{3j})$ and consider the following cases.

<u>Case 1</u> Suppose that $\psi(v_{3j-1}v_{3j}) \neq \psi(xv_{3j}) = \psi(yv_{3j})$. In order for the subgraph induced by $\{x, v_{3j-1}, v_{3j}, v_{3j+1}\}$ to not be a rainbow θ_4-subgraph, edge xv_{3j-1} must use one of the colors $\psi(xv_{3j})$ or $\psi(v_{3j-1}v_{3j})$. In order for the subgraph induced by $\{y, v_{3j-1}, v_{3j}, v_{3j+1}\}$ to avoid being a rainbow θ_4-subgraph, edge yv_{3j-1} must use one of the colors $\psi(yv_{3j})$ or $\psi(v_{3j-1}v_{3j})$. At this point, five of the edges in $W(j)$ use two colors and only three other edges exist. Hence, $|\psi(W(j))| \leq 5$.

<u>Case 2</u> Suppose that $\psi(v_{3j-1}v_{3j}) = \psi(xv_{3j}) = \psi(yv_{3j})$. The five other edges of $W(j)$ span a θ_4-subgraph, and hence, use at most four colors. It follows that $|\psi(W(j))| \leq 5$.

In both cases, $|\psi(W(j))| \leq 5$ and $|\psi(W(j+1))| \leq 6$, from which the lemma follows. □

Theorem 4.5. *For all $n \geq 3$,*

$$\mathrm{rb}(W_n^*, \theta_4) \geq \begin{cases} 2n, & \text{if } n \text{ is odd}, \\ 2n+1, & \text{if } n \text{ is even}, \end{cases} \quad \text{and} \quad \mathrm{rb}(W_n^*, \theta_4) < \frac{7n}{3}.$$

Furthermore, if $n = 3k$ for some $k \in \mathbb{N}$, then $\mathrm{rb}(W_n^*, \theta_4) \leq 2n + 2$.

Proof. When n is odd, consider the following exact $(2n-1)$-coloring of W_n^*:

$$\psi(s_i) = i,\ \psi(t_1) = 1,\ \psi(t_i) = n - 1 + i \text{ when } i \geq 2, \text{ and}$$

$$\psi(r_i) = \begin{cases} \psi(s_{i+1}), & \text{if } i \text{ is odd,} \\ \psi(t_{i+1}), & \text{if } i \text{ is even.} \end{cases}$$

When n is even, consider the following exact $(2n)$-coloring of W_n^*:

$$\varphi(s_i) = i, \quad \varphi(t_i) = n + i, \text{ and}$$

$$\varphi(r_i) = \begin{cases} \varphi(s_{i+1}), & \text{if } i \text{ is odd,} \\ \varphi(t_{i+1}), & \text{if } i \text{ is even.} \end{cases}$$

Figure 9 shows the cases $n = 4, 5$.

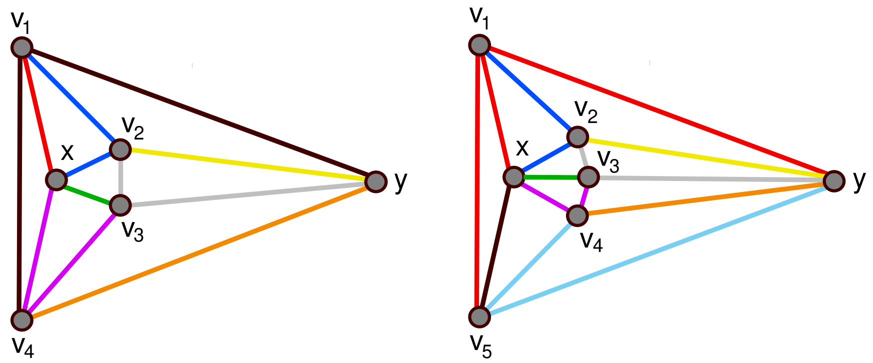

Figure 9: An exact 8-coloring of W_4^* and an exact 9-coloring of W_5^* that avoid rainbow θ_4-subgraphs.

In either coloring, every θ_4-subgraph either includes two adjacent triangles that contain the vertex x, two adjacent triangles that contain the vertex y, or have a diagonal given by one of the edges r_i. In all cases, some color is repeated in one of the triangles, leading to the lower bound stated in the theorem.

For the upper bound, we use the fact that W_n^* is a plane triangulation of order $n+2$. So, by Theorem 4.3, it follows that $\text{rb}(W_n^*, \theta_4) < \frac{7n}{3}$. In the special case where $n = 3k$, first note that $W(1)$ uses at most six colors (Theorem 2.1). Then for each j such that $1 \leq j \leq k-1$, $|\{\psi(v_{3j}v_{3j+1})\} \cup \psi(W(j+1))| \leq 6$ by Lemma 4.4. Finally, we have two cases: $\psi(v_n v_1) \in \psi(W(1)) \cup \psi(W(k))$ or $W(1)$ uses at most six colors. Either way, W_n^* uses at most $6k + 1 = 2n + 1$, leading to the bound $\text{rb}(W_n^*, \theta_4) \leq 2n + 2$. □

Since W_n^* is a plane triangulation of order $n+2$, the following corollary to Theorem 4.5 follows immediately.

Corollary 4.6.

$$\text{rb}(\mathcal{T}_n, \theta_4) \geq \begin{cases} 2n - 4, & \textit{if } n \textit{ is odd,} \\ 2n - 3, & \textit{if } n \textit{ is even,} \end{cases} \qquad \textit{for all } n \geq 5.$$

5. Conclusion

We conclude by mentioning some directions for future research. Rainbow numbers in wheels, sunflowers, and double-hubbed wheels can be considered for larger cycles and theta graphs than the ones considered here. Since double-hubbed wheels are plane triangulations, such work may lead to lower bounds for the corresponding rainbow numbers in plane triangulations. Another host graph worth considering is the plane triangulation $K_1 + SFn$, but we reserve this topic for future study.

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