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Sufficient conditions for fractional k-extendable graphs

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Abstract. Let k and n be two positive integers. A graph G is said to be fractional k-extendable for $0 \leq k \leq \frac{n-2}{2}$ if every k-matching M in G is contained in a fractional perfect matching $G[F_h]$ of G such that $h(e) = 1$ for all $e \in M$, where $h : E(G) \to [0, 1]$ be a function. Let $e(G)$ denote the size of G and $\rho(G)$ denote the spectral radius of G. In this paper, we first provide a tight size condition to ensure that a connected graph is fractional k-extendable. Then, we determine a lower bound on the spectral radius of a connected graph G to guarantee that G is fractional k-extendable. Finally, we construct some extremal graphs to show that all the bounds are sharp.

1. The first section

All graphs considered in this paper are simple, undirected and connected. Let $G = (V(G), E(G))$ denote a graph, where $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set. The order of G is the number $n = |V(G)|$ of its vertices and its size is the number $e(G) = |E(G)|$ of its edges. A graph G is called trivial if its order $n = 1$. For $v \in V(G)$, the neighborhood of v in G, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. Then $d_G(v) = |N_G(v)|$ is the degree of v in G. For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph formed from G by deleting the vertices in S and their incident edges. Given two vertex-disjoint graphs G_1 and G_2, the union $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and the join $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by adding all the edges joining a vertex of G_1 to a vertex of G_2. Let K_n denote the complete graph of order n.

Let a and b be two positive integers with $a \leq b$. Let F be a spanning subgraph of G. We call F an $[a, b]$-factor of G if $a \leq d_F(v) \leq b$ holds for any $v \in V(G)$. If $a = b = r$, then an $[a, b]$-factor is called an r-factor. A set M of edges in a graph is called a matching if no two edges of M share a vertex. A k-matching is a matching of size k. If a matching covers all the vertices of a graph G, then it is called a perfect matching (or 1-factor) of G. Let G be a graph of order n with a perfect matching. Then G is said to be k-extendable for

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$0 \le k \le \frac{n-2}{2}$ if every k-matching in G can be extended to a perfect matching. In particular, G is 0-extendable if and only if G contains a perfect matching.

Let $h : E(G) \to [0,1]$ be a function. If $a \le \sum_{e \in E_G(v)} h(e) \le b$ holds for any $v \in V(G)$, then the subgraph of G with vertex set $V(G)$ and edge set F_h, denoted by $G[F_h]$, is called a fractional $[a,b]$-factor of G with indicator function h, where $E_G(v)$ denotes the set of edges incident with v in G and $F_h = \{e \in E(G) : h(e) > 0\}$. If $a = b = r$, then a fractional $[a,b]$-factor is called a fractional r-factor. A fractional 1-factor is also called a fractional perfect matching. Let G be a graph of order n with a k-matching. Then G is said to be fractional k-extendable for $0 \le k \le \frac{n-2}{2}$ if every k-matching M in G is contained in a fractional perfect matching $G[F_h]$ of G such that $h(e) = 1$ for all $e \in M$. We also say that M can be extended to a fractional perfect matching of G. Especially, G is fractional 0-extendable if and only if G contains a fractional perfect matching.

The perfect matching and k-extendable graph attracted much attention. Tutte [26] derived a characterization for a graph with a perfect matching. Anderson [2, 3] studied the connection between binding number and a perfect matching in a graph and showed two binding number conditions for graphs to possess perfect matchings. Sumner [25] investigated the existence of perfect matchings in graphs. Niessen [19] established a relationship between neighborhood union and a perfect matching in a graph. Enomoto [6] gave a toughness condition for the existence of a perfect matching in a graph. Plummer [23] first introduced the concept of k-extendable graph and provided some properties of k-extendable graphs. Up to now, much attention has been paid on various graphic parameters of k-extendable graphs, such as connectivity [13, 21], binding number [24], minimum degree [1], genus [22], independence number [17], distance-regular graph [5]. Much effort has been devoted to finding sufficient conditions for the existence of $[1,2]$-factors (see [8, 11, 34, 38]) and $[a,b]$-factors (see [18, 27, 33, 35, 37]) in graphs.

The fractional perfect matching and fractional k-extendable graph also attracted much attention. Lovász and Plummer [14] gave a necessary and sufficient condition for a graph to have a fractional perfect matching. Liu and Zhang [10] presented a toughness condition for the existence of a fractional perfect matching in a graph. Yang, Ma and Liu [28] established a connection between isolated toughness and a fractional perfect matching in a graph. Ma and Liu [16] obtained a characterization of fractional k-extendable graphs and gave some sufficient conditions for fractional k-extendable graphs. Zhu and Liu [39] provided a binding number condition for fractional k-extendable graphs. Some results on fractional $[a,b]$-factors in graphs can be found in [7, 15, 31, 32, 36].

For a graph G of order n, the adjacency matrix $A(G)$ of G is the $n \times n$ matrix in which entry a_{ij} is 1 or 0 according to whether v_i and v_j are adjacent or not. The eigenvalues of the adjacency matrix $A(G)$ are also called the eigenvalues of G. Clearly, $A(G)$ is a real symmetric nonnegative matrix. Consequently, its eigenvalues are real, which can be arranged in non-increasing order as $\lambda_1(G) \ge \lambda_2(G) \ge \cdots \ge \lambda_n(G)$. Note that the adjacency spectral radius (or spectral radius, for short) of G, denoted by $\rho(G)$, is equal to $\lambda_1(G)$.

Recently, O [20] obtained a spectral radius condition to ensure that a connected graph contains a perfect matching. By imposing the minimum degree of a graph as a parameter, Liu, Liu and Feng [12] extended O's result [20]. Zhang and Lin [30] showed a distance spectral condition to ensure the existence of a perfect matching in a connected graph. Li, Miao and Zhang [9] claimed a relationship between spectral radius and a fractional perfect matching in a connected graph.

Motivated by [16, 20] directly, it is natural and interesting to give some sufficient conditions to guarantee that a graph is fractional k-extendable. Here, we focus on the sufficient conditions including structure graph condition or spectral graph condition, which are shown in the following.

Theorem 1.1. Let k and n be two positive integers, and let G be a connected graph of order n with $n \ge 2k+3$. Assume that G satisfies

$$e(G) > \begin{cases} \frac{1}{8}(n+2k-1)(3n-2k-1), & if\ n \in \{2k+3, 2k+5, 2k+7, 2k+9, 2k+11\}, \\ \frac{1}{8}(n+2k-2)(3n-2k), & if\ n \in \{2k+4, 2k+6, 2k+8, 2k+10\}, \\ \binom{n-2}{2} + 2(2k+1), & if\ n \ge 2k+12. \end{cases}$$

Then G is fractional k-extendable unless $G = K_{2k} \vee (K_{n-2k-1} \cup K_1)$.

Theorem 1.2. Let k and n be two positive integers, and let G be a connected graph of order n with $n \geq 2k+3$. Assume that one of the following four conditions holds:

(i) $\rho(G) > \theta(k,n)$ for $n = 2k+10$ or $n \geq 2k+12$ or $(k,n) = (1, 2k+11)$, where $\theta(k,n)$ is the largest root of $x^3 + (4-n)x^2 + (1-4k-n)x + 2(2k+1)(n-2k-4) = 0$;

(ii) $\rho(G) > k + 2 + \sqrt{(k+2)^2 + 6(2k+5)}$ for $k \geq 2$ and $n = 2k+11$;

(iii) $\rho(G) > \frac{n+2k-3+\sqrt{(n+2k-3)^2+4(n-2k+1)(n+2k-1)}}{4}$ for $n \in \{2k+3, 2k+5, 2k+7, 2k+9\}$;

(iv) $\rho(G) > \frac{n+2k-4+\sqrt{(n+2k-4)^2+4(n-2k+2)(n+2k-2)}}{4}$ for $n \in \{2k+4, 2k+6, 2k+8\}$.

Then G is fractional k-extendable unless $G = K_{2k} \vee (K_{n-2k-1} \cup K_1)$.

The proofs of Theorems 1.1 and 1.2 will be provided in Sections 3 and 4, respectively.

2. Preliminary lemmas

In this section, we put forward some necessary preliminary lemmas, which are very important to the proofs of our main results.

Ma and Liu [16] gave a necessary and sufficient condition for the existence of fractional k-extendable graphs.

Lemma 2.1 ([16]). Let $k \geq 1$ be an integer, and let G be a graph with a k-matching. Then G is fractional k-extendable if and only if

$$i(G-S) \leq |S| - 2k$$

holds for any $S \subseteq V(G)$ such that $G[S]$ contains a k-matching, where $i(G-S)$ denotes the number of isolated vertices in $G-S$.

Lemma 2.2 ([4]). Let G be a connected graph, and let H be a proper subgraph of G. Then $\rho(G) > \rho(H)$.

Let M be a real symmetric matrix whose rows and columns are indexed by $V = \{1, 2, \cdots, n\}$. Suppose that M can be written as

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1s} \\ \vdots & \ddots & \vdots \\ M_{s1} & \cdots & M_{ss} \end{pmatrix}$$

in terms of partition $\pi : V = V_1 \cup V_2 \cup \cdots \cup V_s$, wherein M_{ij} is the submatrix (block) of M obtained by rows in V_i and columns in V_j. The average row sum of M_{ij} is denoted by q_{ij}. Then matrix $M_\pi = (q_{ij})$ is said to be the quotient matrix of M. If the row sum of every block M_{ij} is a constant, then the partition is equitable.

Lemma 2.3 ([29]). Let M be a real matrix with an equitable partition π, and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative, then the largest eigenvalues of M and M_π are equal.

3. The proof of Theorem 1.1

In this section, we verify Theorem 1.1, which poses a sufficient condition via the size of a connected graph to ensure that the graph is fractional k-extendable.

Proof of Theorem 1.1. Suppose, to the contrary, that $G \neq K_{2k} \vee (K_{n-2k-1} \cup K_1)$ and G is not fractional k-extendable. Then by Lemma 2.1, there exists some nonempty subset S of $V(G)$ such that $G[S]$ contains a k-matching and $i(G-S) \geq |S| - 2k + 1$. Choose a connected graph G of order n such that its size is as large as possible. In light of the choice of G, the induced subgraph $G[S]$ and all connected components in $G-S$ are complete graphs. Furthermore, $G = G[S] \vee (G-S)$.

Note that there is at most one non-trivial connected component in $G-S$. Otherwise, we can construct a new graph G' by adding edges among all non-trivial connected components to obtain a bigger non-trivial connected component. Clearly, $e(G) < e(G')$, which contradicts the choice of G. We denote by m the number of non-trivial connected components in $G - S$. Then $m = 0$ or 1. For convenience, let $|S| = s$ and $i(G - S) = i$. Then

$$i \geq s - 2k + 1. \tag{1}$$

The following proof will be divided into two cases by the value of m.

Case 1. $m = 1$.

In this case, we see $G = K_s \vee (K_{n_1} \cup iK_1)$, where $n_1 = n - s - i \geq 2$. We are to verify $i = s - 2k + 1$. Assume that $i \geq s - 2k + 2$. Then we create a new graph G'' derived from G by joining every vertex of K_{n_1} with one vertex in iK_1 by an edge. Then we possess $e(G'') = e(G) + n_1 > e(G)$ and $i(G'' - S) = i - 1 \geq (s - 2k + 2) - 1 = s - 2k + 1$, which is a contradiction to the choice of G. Therefore, $i \leq s - 2k + 1$. Together with (1), we derive $i = s - 2k + 1$. And so $n_1 = n - s - i = n - 2s + 2k - 1$ and $G = K_s \vee (K_{n_1} \cup iK_1) = K_s \vee (K_{n-2s+2k-1} \cup (s - 2k + 1)K_1)$.

We easily see $s \geq 2k + 1$. Otherwise $s = 2k$ and $G = K_{2k} \vee (K_{n-2k-1} \cup K_1)$, which is a contradiction to $G \neq K_{2k} \vee (K_{n-2k-1} \cup K_1)$. Note that $G = K_s \vee (K_{n-2s+2k-1} \cup (s - 2k + 1)K_1)$, and so $e(G) = \binom{n-s+2k-1}{2} + s(s - 2k + 1)$. Together with $n = 2s - 2k + 1 + n_1 \geq 2s - 2k + 3 \geq 2k + 5$, we infer

$$\begin{aligned}\binom{n-2}{2} + 2(2k+1) - e(G) &= \binom{n-2}{2} + 2(2k+1) - \binom{n-s+2k-1}{2} - s(s-2k+1)\\ &= \frac{(s-2k-1)(2n-3s+2k-8)}{2}\\ &\geq \frac{(s-2k-1)(2(2s-2k+3)-3s+2k-8)}{2}\\ &= \frac{(s-2k-1)(s-2k-2)}{2}\\ &\geq 0,\end{aligned}$$

which yields that

$$e(G) \leq \binom{n-2}{2} + 2(2k+1) \tag{2}$$

for $n \geq 2k + 5$. This is a contradiction for $n \geq 2k + 12$.

As for $n \in \{2k + 5, 2k + 7, 2k + 9, 2k + 11\}$, we get

$$\frac{1}{8}(n + 2k - 1)(3n - 2k - 1) - \binom{n-2}{2} - 2(2k + 1) = -\frac{(n - 2k - 3)(n - 2k - 13)}{8} > 0.$$

Combining this with (2), we possess $e(G) \leq \binom{n-2}{2} + 2(2k+1) < \frac{1}{8}(n+2k-1)(3n-2k-1)$, which is a contradiction.

As for $n \in \{2k + 6, 2k + 8, 2k + 10\}$, we obtain

$$\frac{1}{8}(n + 2k - 2)(3n - 2k) - \binom{n-2}{2} - 2(2k + 1) = -\frac{(n - 2k - 4)(n - 2k - 10)}{8} \geq 0.$$

Together with (2), we infer $e(G) \leq \binom{n-2}{2} + 2(2k + 1) \leq \frac{1}{8}(n + 2k - 2)(3n - 2k)$, which is a contradiction.

Case 2. $m = 0$.

In this case, we possess $G = K_s \vee iK_1$. If $i \geq s - 2k + 3$, we can construct a new graph G^* by adding an edge in $V(iK_1)$. Then one has $i(G^* - S) \geq s - 2k + 1$. Combining this with $e(G) < e(G^*)$, we derive a contradiction to the choice of G. Hence, we deduce $i \leq s - 2k + 2$. Together with (1), we only need to consider $i = s - 2k + 1$ and $i = s - 2k + 2$.

Subcase 2.1. $i = s - 2k + 1$.

For $i = s - 2k + 1$, we possess $n = s + i = 2s - 2k + 1$, $G = K_s \vee (s - 2k + 1)K_1$ and $e(G) = \binom{s}{2} + s(s - 2k + 1)$. If $s = 2k$, then $n = 2k + 1$, which contradicts $n \geq 2k + 3$. Next, we consider $s \geq 2k + 1$. By a direct computation, we have

$$\begin{aligned}\binom{n-2}{2} + 2(2k+1) - e(G) =&\binom{2s-2k-1}{2} + 2(2k+1) - \binom{s}{2} - s(s-2k+1)\\ =&\frac{(s-2k-1)(s-2k-6)}{2}\\ \geq&0\end{aligned}$$

for $s \geq 2k + 6$, which yields that

$$e(G) \leq \binom{n-2}{2} + 2(2k+1)$$

for $s \geq 2k + 6$. This is a contradiction for $n \geq 2k + 13$.

Note that $n = 2s - 2k + 1$ is odd. By a simple computation, we derive

$$e(G) = \binom{s}{2} + s(s - 2k + 1) = \frac{1}{2}(3s^2 - 4ks + s) = \frac{1}{8}(n + 2k - 1)(3n - 2k - 1),$$

which leads to a contradiction when $n \in \{2k + 3, 2k + 5, 2k + 7, 2k + 9, 2k + 11\}$.

Subcase 2.2. $i = s - 2k + 2$.

For $i = s - 2k + 2$, we obtain $n = s + i = 2s - 2k + 2$, $G = K_s \vee (s - 2k + 2)K_1$ and $e(G) = \binom{s}{2} + s(s - 2k + 2)$. If $s = 2k$, then $n = 2k + 2$, which is a contradiction to $n \geq 2k + 3$. In what follows, we deal with $s \geq 2k + 1$. By a simple computation, we derive

$$\begin{aligned}\binom{n-2}{2} + 2(2k+1) - e(G) =&\binom{2s-2k}{2} + 2(2k+1) - \binom{s}{2} - s(s-2k+2)\\ =&\frac{(s-2k-1)(s-2k-4)}{2}\\ \geq&0\end{aligned}$$

for $s \geq 2k + 4$. Thus, we deduce

$$e(G) \leq \binom{n-2}{2} + 2(2k+1)$$

for $s \geq 2k + 4$. This is a contradiction for $n \geq 2k + 12$.

Recall that $n = 2s - 2k + 2$ is even. By a direct computation, we possess

$$e(G) = \binom{s}{2} + s(s - 2k + 2) = \frac{1}{2}(3s^2 - 4ks + 3s) = \frac{1}{8}(n + 2k - 2)(3n - 2k),$$

which leads to a contradiction when $n \in \{2k + 4, 2k + 6, 2k + 8, 2k + 10\}$. This completes the proof of Theorem 1.1. □

4. The proof of Theorem 1.2

In this section, we prove Theorem 1.2, which puts forward an adjacency spectral radius condition for a connected graph to be fractional k-extendable.

Proof of Theorem 1.2. Let $\varphi(x) = x^3 + (4 - n)x^2 + (1 - 4k - n)x + 2(2k + 1)(n - 2k - 4)$ and let $\theta(k, n)$ be the largest root of $\varphi(x) = 0$. Suppose to the contrary that $G \neq K_{2k} \vee (K_{n-2k-1} \cup K_1)$ and G is not fractional k-extendable. In view of Lemma 2.1, there exists some nonempty subset S of $V(G)$ such that $G[S]$ contains a k-matching and $i(G-S) \geq |S| - 2k + 1$. Choose such a connected graph G of order n so that its spectral radius is as large as possible. Together with Lemma 2.2 and the choice of G, the induced subgraph $G[S]$ and every connected component of $G-S$ are complete graphs, respectively. Furthermore, $G = G[S] \vee (G-S)$. We easily see that there exists at most one non-trivial connected component in $G-S$. Otherwise, we can add edges among all non-trivial connected components to get a non-trivial connected component of larger size. Then Lemma 2.2 deduces a contradiction to the choice of G. We denote by m the number of non-trivial connected components in $G-S$. Then $m = 0$ or 1. For convenience, let $|S| = s$ and $i(G-S) = i$. Then

$$i \geq s - 2k + 1. \tag{3}$$

The following proof will be divided into two cases by the value of m.

Case 1. $m = 1$.

In this case, we possess $G = K_s \vee (K_{n_1} \cup iK_1)$, where $n_1 = n - s - i \geq 2$. We are to claim $i = s - 2k + 1$. Assume that $i \geq s - 2k + 2$. Then we create a new graph G' formed from G by joining every vertex of K_{n_1} with one vertex in iK_1 by an edge. Then $i(G'-S) = i - 1 \geq (s-2k+2) - 1 = s - 2k + 1$ and G is a proper subgraph of G'. According to Lemma 2.2, we infer $\rho(G) < \rho(G')$, which is a contradiction to the choice of G. Consequently, $i \leq s - 2k + 1$. Combining this with (3), we deduce $i = s - 2k + 1$. And so $n_1 = n - s - i = n - 2s + 2k - 1$ and $G = K_s \vee (K_{n_1} \cup iK_1) = K_s \vee (K_{n-2s+2k-1} \cup (s-2k+1)K_1)$.

If $s = 2k$, then we have $G = K_{2k} \vee (K_{n-2k-1} \cup K_1)$, which leads to a contradiction to $G \neq K_{2k} \vee (K_{n-2k-1} \cup K_1)$. Hence, we infer $s \geq 2k + 1$.

Consider the partition $V(G) = V(K_s) \cup V(K_{n-2s+2k-1}) \cup V((s-2k+1)K_1)$. The corresponding quotient matrix of $A(G)$ is equal to

$$B_1 = \begin{pmatrix} s-1 & n-2s+2k-1 & s-2k+1 \\ s & n-2s+2k-2 & 0 \\ s & 0 & 0 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix B_1 equals

$$f_1(x) = x^3 + (s-2k+3-n)x^2 + (2ks - s^2 - 2k + 2 - n)x + s(s-2k+1)(n-2s+2k-2).$$

Note that the partition $V(G) = V(K_s) \cup V(K_{n-2s+2k-1}) \cup V((s-2k+1)K_1)$ is equitable. Then in terms of Lemma 2.3, the largest root, say ρ_1, of $f_1(x) = 0$ equals the spectral radius of G. Thus, we have $f_1(\rho_1) = 0$ and $\rho(G) = \rho_1$.

Note that $K_s \vee (n-s)K_1$ is a proper subgraph of G. According to Lemma 2.2, we deduce

$$\rho_1 = \rho(G) > \rho(K_s \vee (n-s)K_1). \tag{4}$$

Consider the partition $V(K_s \vee (n-s)K_1) = V(K_s) \cup V((n-s)K_1)$. The corresponding quotient matrix of $A(K_s \vee (n-s)K_1)$ equals

$$B_2 = \begin{pmatrix} s-1 & n-s \\ s & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$f_2(x) = x^2 - (s-1)x - s(n-s).$$

It is easy to see that the partition $V(K_s \vee (n-s)K_1) = V(K_s) \cup V((n-s)K_1)$ is equitable. In light of Lemma 2.3, the largest root, say ρ_2, of $f_2(x) = 0$ equals the spectral radius of $K_s \vee (n-s)K_1$. Thus, we obtain

$$\rho(K_s \vee (n-s)K_1) = \rho_2 = \frac{s-1+\sqrt{(s-1)^2 + 4s(n-s)}}{2}. \tag{5}$$

It follows from (4) and (5) that

$$\rho_1 > \frac{s-1+\sqrt{(s-1)^2+4s(n-s)}}{2}. \tag{6}$$

In what follows, we aim to show $\varphi(\rho_1) < 0$. Bear in mind that $f_1(\rho_1) = 0$. By plugging the value ρ_1 into x of $\varphi(x) - f_1(x)$, we derive

$$\begin{aligned}\varphi(\rho_1) =&\varphi(\rho_1) - f_1(\rho_1)\\ =&(s-2k-1)(-\rho_1^2 + (s+1)\rho_1 + 2s^2 - sn - 2ks + 6s - 2n + 4k + 8)\\ =&(s-2k-1)g_1(\rho_1),\end{aligned} \tag{7}$$

where $g_1(\rho_1) = -\rho_1^2 + (s+1)\rho_1 + 2s^2 - sn - 2ks + 6s - 2n + 4k + 8$. According to (6), $s \geq 2k+1$ and $n = 2s - 2k + 1 + n_1 \geq 2s - 2k + 3$, we possess

$$\frac{s+1}{2} < \frac{s-1+\sqrt{(s-1)^2+4s(n-s)}}{2} < \rho_1.$$

Consequently, we deduce

$$\begin{aligned}g_1(\rho_1) <&g_1\left(\frac{s-1+\sqrt{(s-1)^2+4s(n-s)}}{2}\right)\\ =&-(2s+2)n + 3s^2 - (2k-7)s + 4k + 7 + \sqrt{(s-1)^2+4s(n-s)}.\end{aligned} \tag{8}$$

Claim 1. If $s \geq 2k+3$ and $n \geq 2s-2k+3$, or $s = 2k+2$ and $n \geq 2s-2k+4$, then $(2s+2)n - 3s^2 + (2k-7)s - 4k - 7 > \sqrt{(s-1)^2+4s(n-s)}$.

Proof. By a simple calculation, we derive

$$\begin{aligned}&((2s+2)n - 3s^2 + (2k-7)s - 4k - 7)^2 - ((s-1)^2 + 4s(n-s))\\ =&(2s+2)^2n^2 - (12s^3 + (40-8k)s^2 + (60+8k)s + 16k + 28)n\\ &+ 9s^4 + 3(14-4k)s^3 + (4k^2 - 4k + 94)s^2\\ &+ (-16k^2 + 28k + 100)s + 16k^2 + 56k + 48\\ :=&h_1(n),\end{aligned} \tag{9}$$

where $h_1(n) = (2s+2)^2n^2 - (12s^3 + (40-8k)s^2 + (60+8k)s + 16k + 28)n + 9s^4 + 3(14-4k)s^3 + (4k^2 - 4k + 94)s^2 + (-16k^2 + 28k + 100)s + 16k^2 + 56k + 48$. For $s \geq 2k+3$ and $n \geq 2s - 2k + 3 \geq 2k + 9$, we deduce

$$\frac{12s^3 + (40-8k)s^2 + (60+8k)s + 16k + 28}{2(2s+2)^2} < 2s - 2k + 3 \leq n.$$

Consequently, we obtain

$$\begin{aligned}h_1(n) \geq&h_1(2s-2k+3)\\ =&s^4 + (6-4k)s^3 + (4k^2 - 28k + 2)s^2 + (32k^2 - 36k - 16)s + 64k^2 + 16k\\ :=&l_1(s).\end{aligned} \tag{10}$$

Let $l_1(x) = x^4 + (6-4k)x^3 + (4k^2 - 28k + 2)x^2 + (32k^2 - 36k - 16)x + 64k^2 + 16k$ be a real function in x with $x \in [2k+1, +\infty)$. The derivative function of $l_1(x)$ is

$$l_1'(x) = 4x^3 + 3(6-4k)x^2 + 2(4k^2 - 28k + 2)x + 32k^2 - 36k - 16.$$

Furthermore, we possess

$$l_1''(x) = 12x^2 + 6(6-4k)x + 2(4k^2 - 28k + 2).$$

Note that

$$-\frac{6(6-4k)}{24} = \frac{2k-3}{2} < 2k+1 < s.$$

Then $l_1''(x)$ is increasing in the interval $[2k+1, +\infty)$. Thus $l_1''(x) \geq l_1''(2k+1) = 8k^2 + 40k + 52 > 0$, which yields that $l_1'(x)$ is increasing in the interval $[2k+1, +\infty)$ and so $l_1'(x) \geq l_1'(2k+1) = 10 > 0$. Thus, we infer that $l_1(x)$ is increasing in the interval $[2k+1, +\infty)$. Combining this with $s \geq 2k+3$, we have

$$l_1(s) \geq l_1(2k+3) = 12k^2 + 80k + 213 > 0.$$

Together with (9) and (10), we infer $((2s+2)n - 3s^2 + (2k-7)s - 4k - 7)^2 > (s-1)^2 + 4s(n-s)$, that is, $(2s+2)n - 3s^2 + (2k-7)s - 4k - 7 > \sqrt{(s-1)^2 + 4s(n-s)}$.

For $s = 2k+2$ and $n \geq 2s - 2k + 4 = 2k + 8$, we infer

$$\frac{12s^3 + (40-8k)s^2 + (60+8k)s + 16k + 28}{2(2s+2)^2} < 2s - 2k + 4 \leq n.$$

Thus, we derive

$$\begin{aligned}
h_1(n) \geq& h_1(2s-2k+4)\\
=& s^4 + (10-4k)s^3 + (4k^2 - 36k + 22)s^2 + (32k^2 - 76k - 4)s + 64k^2 - 16k\\
=& (2k+2)^4 + (10-4k)(2k+2)^3 + (4k^2 - 36k + 22)(2k+2)^2\\
&+ (32k^2 - 76k - 4)(2k+2) + 64k^2 - 16k\\
=& 32k^2 + 128k + 176\\
>& 0.
\end{aligned}$$

Combining this with (9), we get $(2s+2)n - 3s^2 + (2k-7)s - 4k - 7 > \sqrt{(s-1)^2 + 4s(n-s)}$. This completes the proof of Claim 1. □

It follows from (7), (8), $s \geq 2k+1$ and Claim 1 that

$$\varphi(\rho_1) = (s - 2k - 1)g_1(\rho_1) \leq 0,$$

which gives $\rho(G) = \rho_1 \leq \theta(k,n)$ when $n \geq 2k+5$ and $n \neq 2k+7$, which contradicts $\rho(G) > \theta(k,n)$ for $n = 2k+10$ or $n \geq 2k+12$ or $(k,n) = (1, 2k+11)$.

As for $k \geq 2$ and $n = 2k+11$, one has $\varphi(x) = x^3 - (2k+7)x^2 - (6k+10)x + 14(2k+1)$ and $\varphi'(x) = 3x^2 - 2(2k+7)x - 6k - 10$. By a direct computation, we derive $\varphi(k+2+\sqrt{(k+2)^2 + 6(2k+5)}) = -60 + 8\sqrt{(k+2)^2 + 6(2k+5)} > 0$ and $\varphi'(k+2+\sqrt{(k+2)^2 + 6(2k+5)}) = 2k^2 + 32k + 76 + 2(k-1)\sqrt{(k+2)^2 + 6(2k+5)} > 0$, and so $\rho(G) = \rho_1 \leq \theta(k,n) < k+2+\sqrt{(k+2)^2 + 6(2k+5)}$, which is a contradiction to $\rho(G) > k+2+\sqrt{(k+2)^2 + 6(2k+5)}$ for $k \geq 2$ and $n = 2k+11$.

As for $n \in \{2k+3, 2k+5, 2k+7, 2k+9\}$, one has $\varphi(x) = x^3 + (4-n)x^2 + (1-4k-n)x + 2(2k+1)(n-2k-4)$ and $\varphi'(x) = 3x^2 + 2(4-n)x + 1 - 4k - n$. By a direct computation, we can deduce $\varphi\left(\frac{n+2k-3+\sqrt{(n+2k-3)^2+4(n-2k+1)(n+2k-1)}}{4}\right) > 0$ and $\varphi'\left(\frac{n+2k-3+\sqrt{(n+2k-3)^2+4(n-2k+1)(n+2k-1)}}{4}\right) > 0$, and so $\rho(G) = \rho_1 \leq \theta(k,n) < \frac{n+2k-3+\sqrt{(n+2k-3)^2+4(n-2k+1)(n+2k-1)}}{4}$, which is a contradiction to $\rho(G) > \frac{n+2k-3+\sqrt{(n+2k-3)^2+4(n-2k+1)(n+2k-1)}}{4}$ for $n \in \{2k+3, 2k+5, 2k+7, 2k+9\}$.

As for $n \in \{2k+4, 2k+6, 2k+8\}$, one has $\varphi(x) = x^3 + (4-n)x^2 + (1-4k-n)x + 2(2k+1)(n-2k-4)$ and $\varphi'(x) = 3x^2 + 2(4-n)x + 1 - 4k - n$. By a direct computation, we can deduce $\varphi\left(\frac{n+2k-4+\sqrt{(n+2k-4)^2+4(n-2k+2)(n+2k-2)}}{4}\right) > 0$

and $\varphi'\left(\frac{n+2k-4+\sqrt{(n+2k-4)^2+4(n-2k+2)(n+2k-2)}}{4}\right) > 0$, and so $\rho(G) = \rho_1 \leq \theta(k,n) < \frac{n+2k-4+\sqrt{(n+2k-4)^2+4(n-2k+2)(n+2k-2)}}{4}$, which is a contradiction to $\rho(G) > \frac{n+2k-4+\sqrt{(n+2k-4)^2+4(n-2k+2)(n+2k-2)}}{4}$ for $n \in \{2k+4, 2k+6, 2k+8\}$.

Case 2. $m = 0$.

In this case, we get $G = K_s \vee iK_1$. If $i \geq s-2k+3$, we can create a new graph G'' by adding an edge in $V(iK_1)$. Then one has $i(G''-S) \geq s-2k+1$ and G is a proper subgraph of G''. Combining these with Lemma 2.2, we obtain $\rho(G) < \rho(G'')$, we get a contradiction to the choice of G. Consequently, we infer $i \leq s-2k+2$. Combining this with (3), we possess $i = s-2k+1$ or $i = s-2k+2$.

Subcase 2.1. $i = s-2k+1$.

Obviously, $n = s+i = 2s-2k+1$ and $G = K_s \vee (s-2k+1)K_1 = K_s \vee (n-s)K_1$. If $s = 2k$, then $n = 2k+1$, which is a contradiction to $n \geq 2k+3$. Hence, we infer $s \geq 2k+1$.

If $(s,n) \in \{(2k+1, 2k+3), (2k+2, 2k+5), (2k+3, 2k+7), (2k+4, 2k+9)\}$, then it follows from (5) and $n = 2s-2k+1$ that

$$\begin{aligned}\rho(G) =&\rho_2 = \frac{s-1+\sqrt{(s-1)^2+4s(n-s)}}{2}\\ =&\frac{n+2k-3+\sqrt{(n+2k-3)^2+4(n-2k+1)(n+2k-1)}}{4},\end{aligned}$$

which is a contradiction. If $k \geq 2$ and $s = 2k+5$, then $n = 2k+11$ and $\rho(G) = \rho_2 = k+2+\sqrt{(k+2)^2+6(2k+5)}$, a contradiction. Next, we consider $s \geq 2k+6$, or $k = 1$ and $s = 2k+5$. Note that $f_2(\rho_2) = 0$. By plugging the value ρ_2 into x of $\varphi(x) - xf_2(x)$, we get

$$\begin{aligned}\varphi(\rho_2) =&\varphi(\rho_2) - \rho_2 f_2(\rho_2)\\ =&-(n-s-3)\rho_2^2 + ((s-1)n - s^2 - 4k + 1)\rho_2 + 2(2k+1)(n-2k-4)\\ =&-(n-s-3)\left(\frac{s-1+\sqrt{(s-1)^2+4s(n-s)}}{2}\right)^2\\ &+((s-1)n - s^2 - 4k + 1)\left(\frac{s-1+\sqrt{(s-1)^2+4s(n-s)}}{2}\right)\\ &+2(2k+1)(n-2k-4)\\ =&-(s-2k-2)\left(\frac{s-1+\sqrt{5s^2-(8k-2)s+1}}{2}\right)^2\\ &+(s^2-2ks-s-2k)\left(\frac{s-1+\sqrt{5s^2-(8k-2)s+1}}{2}\right)\\ &+2(2k+1)(2s-4k-3)\\ =&-s^3+(4k+2)s^2-(4k^2-4k-4)s-(2k+1)(8k+5)\\ &+(s-2k-1)\sqrt{5s^2-(8k-2)s+1}\\ =&(s-2k-1)(-s^2+(2k+1)s+8k+5+\sqrt{5s^2-(8k-2)s+1}).\end{aligned} \tag{11}$$

Claim 2. If $s \geq 2k+6$, or $(k,s) = (1, 2k+5)$, then $s^2-(2k+1)s-8k-5 > \sqrt{5s^2-(8k-2)s+1}$.

Proof. By a direct calculation, we derive

$$\begin{aligned}(s^2-&(2k+1)s-8k-5)^2-(5s^2-(8k-2)s+1)\\ =&s^4-2(2k+1)s^3+(4k^2-12k-14)s^2+(32k^2+44k+8)s+(8k+4)(8k+6)\\ :=&g_2(s).\end{aligned} \tag{12}$$

Let $g_2(x) = x^4 - 2(2k+1)x^3 + (4k^2 - 12k - 14)x^2 + (32k^2 + 44k + 8)x + (8k+4)(8k+6)$ be a real function in x with $x \in [2k+6, +\infty)$. The derivative function of $g_2(x)$ is

$$g_2'(x) = 4x^3 - 6(2k+1)x^2 + 2(4k^2 - 12k - 14)x + 32k^2 + 44k + 8$$

and

$$g_2''(x) = 12x^2 - 12(2k+1)x + 2(4k^2 - 12k - 14).$$

Note that

$$-\frac{-12(2k+1)}{24} = \frac{2k+1}{2} < 2k+6 \le x.$$

Consequently, $g_2''(x)$ is increasing in the interval $[2k+6, +\infty)$. Thus $g_2''(x) \ge g_2''(2k+6) = 8k^2 + 96k + 332 > 0$, and so $g_2'(x)$ is increasing in the interval $[2k+6, +\infty)$. Then we have $g_2'(x) \ge g_2'(2k+6) = 8k^2 + 132k + 488 > 0$, which yields that $g_2(x)$ is increasing in the interval $[2k+6, +\infty)$. Combining this with $s \ge 2k+6$, we possess

$$g_2(s) \ge g_2(2k+6) = 24k + 432 > 0.$$

Together with (12), we deduce

$$s^2 - (2k+1)s - 8k - 5 > \sqrt{5s^2 - (8k-2)s + 1}.$$

For $(k, s) = (1, 2k+5)$, then $n = 2k+11$ and $g_2(2k+5) = -4k^2 - 64k + 89 = 21 > 0$. Thus, we infer

$$s^2 - (2k+1)s - 8k - 5 > \sqrt{5s^2 - (8k-2)s + 1}.$$

This completes the proof of Claim 2. □

According to (11) and Claim 2, we possess $\varphi(\rho_2) < 0$ for $s \ge 2k+6$ or $(k, s) = (1, 2k+5)$, and so $\rho(G) = \rho_2 < \theta(k, n)$, a contradiction.

Subcase 2.2. $i = s - 2k + 2$.

Clearly, $n = s + i = 2s - 2k + 2$ and $G = K_s \vee (s - 2k + 2)K_1 = K_s \vee (n-s)K_1$. If $s = 2k$, then $n = 2k+2$, which is a contradiction to $n \ge 2k+3$. Therefore, we deduce $s \ge 2k+1$.

If $(s, n) \in \{(2k+1, 2k+4), (2k+2, 2k+6), (2k+3, 2k+8)\}$, then it follows from (5) and $n = 2s - 2k + 2$ that

$$\begin{aligned}\rho(G) =& \rho_2 = \frac{s - 1 + \sqrt{(s-1)^2 + 4s(n-s)}}{2}\\ =& \frac{n + 2k - 4 + \sqrt{(n + 2k - 4)^2 + 4(n - 2k + 2)(n + 2k - 2)}}{4},\end{aligned}$$

which is a contradiction. In what follows, we consider $s \ge 2k+4$. Note that $f_2(\rho_2) = 0$. By plugging the value ρ_2 into x of $\varphi(x) - x f_2(x)$, we possess

$$\begin{aligned}\varphi(\rho_2) =& \varphi(\rho_2) - \rho_2 f_2(\rho_2)\\ =& -(n - s - 3)\rho_2^2 + ((s-1)n - s^2 - 4k + 1)\rho_2 + 2(2k+1)(n - 2k - 4).\end{aligned} \tag{13}$$

By virtue of (5), (13) and $n = 2s - 2k + 2$, we derive

$$\begin{aligned}
\varphi(\rho_2) =& -(n-s-3)\rho_2^2 + ((s-1)n - s^2 - 4k + 1)\rho_2 + 2(2k+1)(n-2k-4)\\
=& -(n-s-3)\left(\frac{s-1+\sqrt{(s-1)^2+4s(n-s)}}{2}\right)^2\\
&+ ((s-1)n - s^2 - 4k + 1)\left(\frac{s-1+\sqrt{(s-1)^2+4s(n-s)}}{2}\right)\\
&+ 2(2k+1)(n-2k-4)\\
=& -(s-2k-1)\left(\frac{s-1+\sqrt{5s^2-(8k-6)s+1}}{2}\right)^2\\
&+ (s^2-2ks-2k-1)\left(\frac{s-1+\sqrt{5s^2-(8k-6)s+1}}{2}\right)\\
&+ 2(2k+1)(2s-4k-2)\\
=&(s-2k-1)(-s^2+(2k-1)s+8k+3+\sqrt{5s^2-(8k-6)s+1}).
\end{aligned} \tag{14}$$

Claim 3. If $s \geq 2k+4$, then $s^2 - (2k-1)s - 8k - 3 > \sqrt{5s^2-(8k-6)s+1}$.
Proof. By a direct computation, we have

$$\begin{aligned}
&(s^2-(2k-1)s-8k-3)^2 - (5s^2-(8k-6)s+1)\\
&=s^4 - (4k-2)s^3 + (4k^2-20k-10)s^2 + (32k^2+4k-12)s + 64k^2 + 48k + 8\\
&:=g_3(s).
\end{aligned} \tag{15}$$

Let $g_3(x) = x^4 - (4k-2)x^3 + (4k^2-20k-10)x^2 + (32k^2+4k-12)x + 64k^2 + 48k + 8$ be a real function in x with $x \in [2k+4, +\infty)$. We may obtain the derivative function of $g_3(x)$ as

$$g_3'(x) = 4x^3 - 3(4k-2)x^2 + 2(4k^2-20k-10)x + 32k^2 + 4k - 12.$$

Furthermore, we possess

$$g_3''(x) = 12x^2 - 6(4k-2)x + 2(4k^2-20k-10).$$

Note that

$$-\frac{-6(4k-2)}{24} = \frac{2k-1}{2} < 2k+4 \leq x.$$

Then $g_3''(x)$ is increasing in the interval $[2k+4, +\infty)$. Thus $g_3''(x) \geq g_3''(2k+4) = 8k^2 + 80k + 220 > 0$, and so $g_3'(x)$ is increasing in the interval $[2k+4, +\infty)$. Hence, we infer $g_3'(x) \geq g_3'(2k+4) = 8k^2 + 92k + 260 > 0$, which implies that $g_3(x)$ is increasing in the interval $[2k+4, +\infty)$. Combining this with $s \geq 2k+4$, we have

$$g_3(s) \geq g_3(2k+4) = 8k + 184 > 0.$$

Together with (15) and $s \geq 2k+4$, we deduce

$$s^2 - (2k-1)s - 8k - 3 > \sqrt{5s^2-(8k-6)s+1}.$$

This completes the proof of Claim 3. □

In terms of (14) and Claim 3, we deduce $\varphi(\rho_2) < 0$ for $s \geq 2k+4$, and so $\rho(G) = \rho_2 < \theta(k,n)$, a contradiction. This completes the proof of Theorem 1.2. □

5. Extremal graphs

In this section, we claim that the bounds established in Theorems 1.1 and 1.2 are best possible, respectively.

Theorem 5.1. Let k and n be two positive integers.

(i) For $n \in \{2k+3, 2k+5, 2k+7, 2k+9, 2k+11\}$, we possess $e\Big(K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1\Big) = \frac{1}{8}(n+2k-1)(3n-2k-1)$ and the graph $K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1$ is not fractional k-extendable.

(ii) For $n \in \{2k+4, 2k+6, 2k+8, 2k+10\}$, we possess $e\Big(K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1\Big) = \frac{1}{8}(n+2k-2)(3n-2k)$ and the graph $K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1$ is not fractional k-extendable.

(iii) For $n \geq 2k+12$, we possess $e(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)) = \binom{n-2}{2} + 2(2k+1)$ and the graph $K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)$ is not fractional k-extendable.

Proof. It is straightforward to check the sizes of the graphs $K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1$, $K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1$ and $K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)$, respectively.

(i) For the graph $K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1$, set $S = V\Big(K_{\frac{n+2k-1}{2}}\Big)$, then $|S| = \frac{n+2k-1}{2}$ and $i\Big(K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1 - S\Big) = \frac{n-2k+1}{2} = \frac{n+2k-1}{2} - 2k + 1 = |S| - 2k + 1 > |S| - 2k$. In terms of Lemma 2.1, the graph $K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1$ is not fractional k-extendable.

(ii) For the graph $K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1$, set $S = V\Big(K_{\frac{n+2k-2}{2}}\Big)$, then $|S| = \frac{n+2k-2}{2}$ and $i\Big(K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1 - S\Big) = \frac{n-2k+2}{2} = \frac{n+2k-2}{2} - 2k + 2 = |S| - 2k + 2 > |S| - 2k$. By virtue of Lemma 2.1, the graph $K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1$ is not fractional k-extendable.

(iii) For the graph $K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)$, set $S = V(K_{2k+1})$, then $|S| = 2k+1$ and $i(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1) - S) = 2 = 2k+1-2k+1 = |S| - 2k + 1 > |S| - 2k$. According to Lemma 2.1, the graph $K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)$ is not fractional k-extendable. □

Theorem 5.2. Let k and n be two positive integers, and let $\theta(k,n)$ be the largest root of $x^3 + (4-n)x^2 + (1-4k-n)x + 2(2k+1)(n-2k-4) = 0$.

(i) For $n = 2k+10$ or $n \geq 2k+12$ or $(k,n) = (1, 2k+11)$, we possess $\rho(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)) = \theta(k,n)$ and the graph $K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)$ is not fractional k-extendable.

(ii) For $k \geq 2$ and $n = 2k+11$, we possess $\rho(K_{2k+5} \vee 6K_1) = k+2+\sqrt{(k+2)^2 + 6(2k+5)}$ and the graph $K_{2k+5} \vee 6K_1$ is not fractional k-extendable.

(iii) For $n \in \{2k+3, 2k+5, 2k+7, 2k+9\}$, we possess

$$\rho\Big(K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1\Big) = \frac{n+2k-3+\sqrt{(n+2k-3)^2 + 4(n-2k+1)(n+2k-1)}}{4}$$

and the graph $K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1$ is not fractional k-extendable.

(iv) For $n \in \{2k+4, 2k+6, 2k+8\}$, we possess

$$\rho\Big(K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1\Big) = \frac{n+2k-4+\sqrt{(n+2k-4)^2 + 4(n-2k+2)(n+2k-2)}}{4}$$

and the graph $K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1$ is not fractional k-extendable.

Proof. (i) Consider the partition $V(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)) = V(K_{2k+1}) \cup V(K_{n-2k-3}) \cup V(2K_1)$. The corresponding quotient matrix of $A(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1))$ equals

$$B_1 = \begin{pmatrix} 2k & n-2k-3 & 2 \\ 2k+1 & n-2k-4 & 0 \\ 2k+1 & 0 & 0 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix B_1 is $x^3 + (4-n)x^2 + (1-4k-n)x + 2(2k+1)(n-2k-4)$. By virtue of Lemma 2.3, the largest root $\theta(k,n)$ of $x^3 + (4-n)x^2 + (1-4k-n)x + 2(2k+1)(n-2k-4) = 0$ equals $\rho(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1))$. That is, $\rho(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)) = \theta(k,n)$. Set $S = V(K_{2k+1})$. Then $|S| = 2k+1$ and $i(K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1) - S) = 2 > 1 = (2k+1) - 2k = |S| - 2k$. According to Lemma 2.1, the graph $K_{2k+1} \vee (K_{n-2k-3} \cup 2K_1)$ is not fractional k-extendable.

(ii) Consider the partition $V(K_{2k+5} \vee 6K_1) = V(K_{2k+5}) \cup V(6K_1)$. The corresponding quotient matrix of $A(K_{2k+5} \vee 6K_1)$ equals

$$B_2 = \begin{pmatrix} 2k+4 & 6 \\ 2k+5 & 0 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix B_2 is $x^2 - (2k+4)x - 6(2k+5)$. Using Lemma 2.3, the largest root of $x^2-(2k+4)x-6(2k+5) = 0$ equals $\rho(K_{2k+5} \vee 6K_1)$. Namely, $\rho(K_{2k+5} \vee 6K_1) = k+2+\sqrt{(k+2)^2 + 6(2k+5)}$. Set $S = V(K_{2k+5})$. Then $|S| = 2k+5$ and $i(K_{2k+5} \vee 6K_1 - S) = 6 > 5 = (2k+5) - 2k = |S| - 2k$. It follows from Lemma 2.1 that the graph $K_{2k+5} \vee 6K_1$ is not fractional k-extendable.

(iii) Consider the partition $V\Big(K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1\Big) = V\Big(K_{\frac{n+2k-1}{2}}\Big) \cup V\Big(\frac{n-2k+1}{2}K_1\Big)$. The corresponding quotient matrix of $A\Big(K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1\Big)$ equals

$$B_3 = \begin{pmatrix} \frac{n+2k-3}{2} & \frac{n-2k+1}{2} \\ \frac{n+2k-1}{2} & 0 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix B_3 is $x^2 - \frac{n+2k-3}{2}x - \frac{(n-2k+1)(n+2k-1)}{4}$. From Lemma 2.3, the largest root of $x^2 - \frac{n+2k-3}{2}x - \frac{(n-2k+1)(n+2k-1)}{4} = 0$ equals $\rho\Big(K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1\Big)$. That is, $\rho\Big(K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1\Big) = \frac{n+2k-3+\sqrt{(n+2k-3)^2+4(n-2k+1)(n+2k-1)}}{4}$. Let $S = V\Big(K_{\frac{n+2k-1}{2}}\Big)$. Then $|S| = \frac{n+2k-1}{2}$ and $i\Big(K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1 - S\Big) = \frac{n-2k+1}{2} > \frac{n-2k-1}{2} = \frac{n+2k-1}{2} - 2k = |S| - 2k$. In light of Lemma 2.1, the graph $K_{\frac{n+2k-1}{2}} \vee \frac{n-2k+1}{2}K_1$ is not fractional k-extendable.

(iv) Consider the partition $V\Big(K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1\Big) = V\Big(K_{\frac{n+2k-2}{2}}\Big) \cup V\Big(\frac{n-2k+2}{2}K_1\Big)$. The corresponding quotient matrix of $A\Big(K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1\Big)$ equals

$$B_4 = \begin{pmatrix} \frac{n+2k-4}{2} & \frac{n-2k+2}{2} \\ \frac{n+2k-2}{2} & 0 \end{pmatrix}.$$

Then the characteristic polynomial of the matrix B_3 is $x^2 - \frac{n+2k-4}{2}x - \frac{(n-2k+2)(n+2k-2)}{4}$. According to Lemma 2.3, the largest root of $x^2 - \frac{n+2k-4}{2}x - \frac{(n-2k+2)(n+2k-2)}{4} = 0$ equals $\rho\Big(K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1\Big)$. Namely, $\rho\Big(K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1\Big) = \frac{n+2k-4+\sqrt{(n+2k-4)^2+4(n-2k+2)(n+2k-2)}}{4}$. Let $S = V\Big(K_{\frac{n+2k-2}{2}}\Big)$. Then $|S| = \frac{n+2k-2}{2}$ and $i\Big(K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1 - S\Big) = \frac{n-2k+2}{2} > \frac{n-2k-2}{2} = \frac{n+2k-2}{2} - 2k = |S| - 2k$. By Lemma 2.1, the graph $K_{\frac{n+2k-2}{2}} \vee \frac{n-2k+2}{2}K_1$ is not fractional k-extendable. □

6. Conclusions

In this paper, we establish tight lower bounds on the size and the spectral radius of G to ensure that G is fractional k-extendable, respectively. Furthermore, all the corresponding extremal graphs are characterized completely. On the other hand, from the proofs of Theorems 1.1 and 1.2 we easily see that the extremal graph under every condition is unique.

There are still some other interesting problems to be considered along the above line. For example, if we focus on the graph G with minimum degree δ, how to establish tight lower bounds on the size and the spectral radius of G to guarantee that G is fractional k-extendable, respectively. Similarly, if we focus on the graph G with minimum degree δ, how to establish some other spectral conditions to guarantee that G is fractional k-extendable, respectively. We will do them in the near future.

Data availability statement

My manuscript has no associated data.

Declaration of competing interest

The authors declare that they have no conflicts of interest to this work.

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