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Power graphs of full transformation semigroups

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Abstract. In this paper we characterize completely the connected components of the power graph of the full transformation semigroup $\mathcal{T}_X$ on a finite set X and provide the necessary and sufficient conditions for which the connected component is a semigroup. We also demonstrate that the power graph of $\mathcal{T}_X$ is planar if and only if $|X| \leq 4$.

1. Introduction

There are a number of constructions of graphs from groups or semigroups in the literature, including the intersection graphs [5],[40], commuting graphs [4], Cayley graphs[32], power graphs [29],[33],[35] and so on.

The most natural source of problems in semigroup theory is the study of full transformation semigroups (see[13], [15], [16], [22], [11], [14]). With the continuous development of research on full transformation semigroups, the investigations of graphs associated to full transformation semigroups have attracted attention of many researchers. Based on the results from [27], transitivity of Cayley graphs of symmetric inverse semigroups which are subsemigroups of full transformation semigroups are studied in [12], and this is generalized to the case of homogeneous (graded) inverse semigroups in [19]. In 2015, Araújo et al. [2] calculated the clique numbers and the diameters of the commuting graphs of symmetric inverse semigroups. In 2019, Riyas and Geetha [37] studied Cayley graphs of symmetric inverse semigroups relative to Green's R-classes. In 2021, Tisklang and Panma [38] characterized Cayley graphs of the finite transformation semigroups with restricted range and determined its conditions to be undirected. Moreover, they described the induced subgraphs which can be embedded into Cayley graphs. In the same year, Riyas et al. [36] concentrated on Cayley graphs of full transformation semigroups relative to Green's L-classes. They characterized the connectivity, strong connectivity and Hamiltonian connectivity of the induced subgraphs of Cayley graphs.

The notion of the directed power graph of a group first appeared in [28], and the directed power graphs of a semigroup are first defined and studied in [29–31]. In all of these papers, directed power graphs

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are simply called power graphs. Given a semigroup S, the *directed power graph* of S, denoted by $\vec{\mathcal{G}}(S)$ is defined as the directed graph with vertex set S, in which there is an arc from x to y if and only if $x \neq y$ and $y = x^m$, for some $m \in \mathbb{N}^+$ ($\mathbb{N}^+$ is the set of positive integers). The notion of the directed power graph of a semigroup covers that of the undirected power graph as the underlying undirected graph. Hence, the *undirected power graph* of a semigroup S, denoted by $\mathcal{G}(S)$, is the graph with vertex set S, in which distinct vertices x and y are adjacent, denoted by $x \sim y$, if either $x = y^m$ or $y = x^m$, for some $m \in \mathbb{N}^+$. Undirected power graphs of semigroups, simply called *power graphs of semigroups*, are studied in [8] by Chakrabarty et al., where they, among other things, described the connected components of $\mathcal{G}(S)$ and characterized the class of semigroups for which their power graphs are connected or complete. Based on these, Cameron and Ghosh [7] explored power graphs of finite groups, obtained many profound results and promoted the research of related problems of power graphs of finite groups. Cameron [6] showed that two finite groups which have isomorphic power graphs are conformal. In 2020, Jain and Kumar [23] showed that the converse of Cameron's result holds for two classes of groups: finite nilpotent groups of odd order with class two, and finite p-groups of class less than p. We refer the reader to comprehensive surveys [1, 33] for more results on the power graphs of groups. However, although interesting problems are raised in [1], since [8], it seems there was no research on the power graphs of semigroups up until recently, with two directions of investigation standing out: the one related to isomorphism problem which started in [7] for finite groups, and the one related to Problems 11 and 12 in [1] on the power graphs of semigroups of homogeneous elements of graded rings. As for the isomorphism problem, in 2022, Ashegh Bonabi and Khosravi [3] characterized the structure of the power graph of a completely simple semigroup and described the order of the automorphism group of its power graph. In 2024, Cheng et al. [9] determined the structure of the power graph of a completely 0-simple semigroup and gave a characterization that G^0-normal completely 0-simple orthodox semigroups with abelian group $\mathcal{H}$-classes are isomorphic, based on their power graphs. As for the Problems 11 and 12 from [1], Ilić-Georgijević [18, 20, 21] studied the question of connectedness of the power graphs of semigroups of homogeneous elements of general $\triangle$-*graded rings inducing* $\triangle$ [24–26], depending on the properties of the rings in question.

We also refer the reader to the survey [34] for the results on the related *enhanced power graphs* of groups.

Motivated by the above works, we shall study power graphs of full transformation semigroups in this paper. In Section 2 we introduce some preliminaries and notations about full transformation semigroups. In Section 3 we characterize the connectivity of power graphs of full transformation semigroups and provide the necessary and sufficient conditions for the connected component to be a semigroup. In Section 4 we prove that the power graph of full transformation semigroup $\mathcal{T}_X$ is planar if and only if $|X| \leq 4$.

2. Preliminaries

In this section we review some of the necessary terminology and notations on full transformation semigroups (see [10]). Let $X = \{i_1, i_2, ..., i_n\}$ be a finite set of n elements. A *transformation* of X is an array of the following form:

$$\alpha = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}, \tag{2.1}$$

where all $a_l \in X$ and $1 \leq l \leq n$. The *value* of the transformation α at the element x is denoted by $\alpha(x)$.

The map of the form $\alpha : A \to X$, where $A = \{j_1, j_2, \ldots, j_k\}$ is a subset of X, is called a *partial transformation* of X. Note that the set A could be empty. Again, α is written in the following form:

$$\alpha = \begin{pmatrix} j_1 & j_2 & \cdots & j_k \\ \alpha(j_1) & \alpha(j_2) & \cdots & \alpha(j_k) \end{pmatrix}. \tag{2.2}$$

Let ε be the identity transformation.

For a (partial) transformation α we associate the following standard notions:

- The *domain* of α : $\mathrm{dom}(\alpha) = A$;

- The *image* of α : $\mathrm{im}(\alpha) = \{\alpha(x) \mid x \in A\}$;
- The *kernel* of α : $\ker(\alpha) = \{(x, y) \in A \times A \mid \alpha(x) = \alpha(y)\}$;
- The *fixed point* of α : $\mathrm{fix}(\alpha) = \{x \in A \mid \alpha(x) = x\}$.

If $\mathrm{dom}(\alpha) = X$, then α is called *full* or *total*. The set of all full transformations of X is denoted by $\mathcal{T}_X$, and the set of all partial transformations of X is denoted by $\mathcal{PT}_X$. Obviously, $\mathcal{T}_X \subset \mathcal{PT}_X$.

Let $\alpha \in \mathcal{PT}_X$ and $B \subseteq X$. The restriction $\alpha|_B$ of α to B is defined as follows: $\alpha|_B$ is a partial transformation on B such that $\mathrm{dom}(\alpha|_B) = B \cap \mathrm{dom}(\alpha)$ and $\alpha|_B(x) = \alpha(x)$ for $x \in \mathrm{dom}(\alpha|_B)$.

Let $\alpha : X \to Y$ and $\beta : Y \to Z$ be two partial transformations. We define their *product* or *composition* $\beta \cdot \alpha$ as partial transformations by $\beta \cdot \alpha(x) = \beta(\alpha(x)), x \in X$. We abbreviate $\beta \cdot \alpha$ to $\beta\alpha$.

We know that both $\mathcal{T}_X$ and $\mathcal{PT}_X$ are semigroups with respect to the composition of (partial) transformations (see [10, Proposition 2.1.3]). The semigroup $\mathcal{T}_X$ is called the *full transformation semigroup* on X or the *symmetric semigroup* of X. The semigroup $\mathcal{PT}_X$ is called the *semigroup of all partial transformations* on X. Let S_X denote the *symmetric group* on X. It is easy to see that both $\mathcal{T}_X$ and S_X are subsemigroups of $\mathcal{PT}_X$.

3. Power graphs of full transformation semigroups

In this section we shall characterize the connectivity and the connected components of the power graph $\mathcal{G}(\mathcal{T}_X)$ with $|X| = n$. For each $\alpha \in \mathcal{T}_X$ let

$$\alpha = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}. \tag{3.1}$$

Suppose that $\mathrm{im}\alpha = \{b_1, b_2, \ldots, b_m\}$. We claim that there exists a subset $B \subseteq \mathrm{im}\alpha$ such that $\alpha|_B$ is a cycle in S_X. We set $\alpha^0(x) = x$ for all $x \in X$. Since $\{\alpha^k(b_1)\}_{k\geq 0} \subseteq X$ and X is finite, it follows that the set

$$\{m_1 \in \mathbb{N} \mid (\exists m_2 \in \mathbb{N})\alpha^{m_1}(b_1) = \alpha^{m_2}(b_1), m_1 \neq m_2\}$$

is non-empty and so has a least element. Let us denote this least element by r. Then the set

$$\{m_2 \in \mathbb{N} \mid \alpha^r(b_1) = \alpha^{m_2}(b_1)\}$$

is non-empty and has the least element l. Thus $(\alpha^r(b_1)\ \ \alpha^{r+1}(b_1)\ \ \cdots\ \ \alpha^{l-1}(b_1))$ forms a cycle in S_X. Let $B = \{\alpha^r(b_1), \alpha^{r+1}(b_1), \ldots, \alpha^{l-1}(b_1)\}$. Then

$$\alpha|_B = (\alpha^r(b_1)\ \ \alpha^{r+1}(b_1)\ \ \cdots\ \ \alpha^{l-1}(b_1))$$

is a cycle in S_X.

Following the above method, which resembles that of investigating monogenic semigroups from [17], we can find all disjoint subsets $B_i (1 \leq i \leq s)$ of $\mathrm{im}\alpha$ such that $\alpha|_{B_i}$ are cycles in S_X. Without loss of generality, suppose that

$$B_1 = \{j_1^{(1)}, j_2^{(1)}, \ldots, j_{t_1}^{(1)}\}, \ldots, B_s = \{j_1^{(s)}, j_2^{(s)}, \ldots, j_{t_s}^{(s)}\}.$$

Let $t = t_1 + \cdots + t_s$. Thus $|X \setminus \bigcup_{i=1}^{s} B_i| = n - t$. Assume that $X \setminus \bigcup_{i=1}^{s} B_i = \{k_1, k_2, \ldots, k_{n-t}\}$. Then α can be decomposed by the following factorization formula:

$$\begin{aligned} \alpha &= (j_1^{(1)}\ \ j_2^{(1)}\ \ \cdots\ \ j_{t_1}^{(1)}) \cdots (j_1^{(s)}\ \ j_2^{(s)}\ \ \cdots\ \ j_{t_s}^{(s)}) \begin{pmatrix} k_1 & k_2 & \cdots & k_{n-t} \\ \alpha(k_1) & \alpha(k_2) & \cdots & \alpha(k_{n-t}) \end{pmatrix} \\ &= \widehat{\alpha} \cdot \alpha', \end{aligned} \tag{3.2}$$

where $\widehat{\alpha} = (j_1^{(1)}\ \ j_2^{(1)}\ \ \cdots\ \ j_{t_1}^{(1)}) \cdots (j_1^{(s)}\ \ j_2^{(s)}\ \ \cdots\ \ j_{t_s}^{(s)}) \in S_X$ are the products of disjoint cycles and $\alpha' = \begin{pmatrix} k_1 & k_2 & \cdots & k_{n-t} \\ \alpha(k_1) & \alpha(k_2) & \cdots & \alpha(k_{n-t}) \end{pmatrix} \in \mathcal{PT}_X$ is a partial transformation that has no cycles.

Let $X_{\widehat{\alpha}} := \mathrm{im}\widehat{\alpha} = \{j_1^{(1)}, j_2^{(1)}, \ldots, j_{t_1}^{(1)}, \ldots, j_1^{(s)}, j_2^{(s)}, \ldots, j_{t_s}^{(s)}\}$ and $[n - t] := \{1, 2, ..., n - t\}$.

Lemma 3.1. *Let $\alpha \in \mathcal{T}_X$. In the Factorization formula (3.2) of α, for each $k_i (i \in [n-t])$ there exists $l \in \mathbb{N}^+$ such that $\alpha^l(k_i) \in X_{\widehat{\alpha}}$.*

Proof. Suppose by way of contradiction that there exists $i \in [n-t]$ such that $\alpha^l(k_i) \notin X_{\widehat{\alpha}}$ for all $l \in \mathbb{N}^+$. Then $\{\alpha^l(k_i)\}_{l\geq 1} \subseteq \{k_1, k_2, \ldots, k_{n-t}\}$. Since $\{k_1, k_2, \ldots, k_{n-t}\}$ is a finite set, it follows that there exist $l_1, l_2 \in \mathbb{N}^+$ such that $l_1 \neq l_2$ and $\alpha^{l_1}(k_i) = \alpha^{l_2}(k_i)$. Hence there exists a cycle in α', a contradiction, as required. □

For each $k_i (i \in [n-t])$ let

$$d(k_i) = \min\{l \mid \alpha^l(k_i) \in X_{\widehat{\alpha}}, l \in \mathbb{N}^+\},$$

and

$$d(\alpha') = \max\{d(k_i) \mid i \in [n-t]\}.$$

It is a matter of routine to verify that $\operatorname{im}((\alpha')^{d(\alpha')}) \subseteq X_{\widehat{\alpha}}$.

It is easy to check that $\widehat{\alpha}$ and α' in the factorization formula (3.2) are commutative as elements in $\mathcal{PT}_X$. By Lemma 3.1, we have

$$\alpha^{d(\alpha')} = \widehat{\widehat{\alpha}} \cdot \begin{pmatrix} k_1 & k_2 & \cdots & k_{n-t} \\ h_1 & h_2 & \cdots & h_{n-t} \end{pmatrix} = \widehat{\widehat{\alpha}} \cdot \alpha'', \tag{3.3}$$

where $\widehat{\widehat{\alpha}} \in S_{X_{\widehat{\alpha}}}$ and $\alpha'' = \begin{pmatrix} k_1 & k_2 & \cdots & k_{n-t} \\ h_1 & h_2 & \cdots & h_{n-t} \end{pmatrix}$ with $h_i \in X_{\widehat{\alpha}}$ for all $i \in [n-t]$.

It is well known that idempotent elements and Green's relations play important roles in the study of algebraic structures of semigroup. Let S be a semigroup. An element $e \in S$ is called an idempotent if $e^2 = e$. We denote the set of all idempotents of S by $E(S)$.

Lemma 3.2. *[10, Theorem 2.7.2] $\alpha \in E(\mathcal{PT}_X)$ if and only if $\operatorname{im}(\alpha) \subseteq \operatorname{dom}(\alpha)$ and $\alpha|_{\operatorname{im}(\alpha)} = \varepsilon|_{\operatorname{im}(\alpha)}$.*

The *Green's relations* $\mathcal{L}$ and $\mathcal{R}$ on S are defined by

$$a\mathcal{L}b \Leftrightarrow S^1a = S^1b \text{ and } a\mathcal{R}b \Leftrightarrow aS^1 = bS^1,$$

where $a, b \in S$ and S^1 is the monoid obtained from S by adjoining an identity if necessary. Note that $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ (see [17]).

Lemma 3.3. *[10, Theorem 4.5.1] Let $\alpha, \beta \in \mathcal{T}_X$. Then*

(i) *$\alpha\mathcal{L}\beta$ if and only if $\operatorname{im}(\alpha) = \operatorname{im}(\beta)$;*

(ii) *$\alpha\mathcal{R}\beta$ if and only if $\ker(\alpha) = \ker(\beta)$;*

(iii) *$\alpha\mathcal{H}\beta$ if and only if $\operatorname{im}(\alpha) = \operatorname{im}(\beta)$ and $\ker(\alpha) = \ker(\beta)$.*

Now we shall characterize the connectivity of $\mathcal{G}(\mathcal{T}_X)$.

Lemma 3.4. *Let $\alpha, \beta \in \mathcal{T}_X$ and $\alpha^m = \beta$ for some $m \in \mathbb{N}^+$. Then the following statements hold:*

(i) *$\operatorname{im}(\beta) \subseteq \operatorname{im}(\alpha)$;*

(ii) *$\ker(\alpha) \subseteq \ker(\beta)$;*

(iii) *$\operatorname{fix}(\alpha) \subseteq \operatorname{fix}(\beta)$.*

Proof. (i) For any $y \in \operatorname{im}(\beta)$ there exists $x \in X$ such that $\beta(x) = y$. That is to say, $y = \beta(x) = \alpha^m(x) = \alpha \cdot \alpha^{m-1}(x)$ and so $y \in \operatorname{im}(\alpha)$. It follows that $\operatorname{im}(\beta) \subseteq \operatorname{im}(\alpha)$.

(ii) If $(x, y) \in \ker(\alpha)$, then $\alpha(x) = \alpha(y)$. This implies that $\alpha^m(x) = \alpha^m(y)$ and so $\beta(x) = \beta(y)$. Thus $(x, y) \in \ker(\beta)$ and so $\ker(\alpha) \subseteq \ker(\beta)$.

(iii) If $x \in \operatorname{fix}(\alpha)$, then $\alpha(x) = x$. It follows that $\beta(x) = \alpha^m(x) = x$ and so $x \in \operatorname{fix}(\beta)$. This shows that $\operatorname{fix}(\alpha) \subseteq \operatorname{fix}(\beta)$. □

Lemma 3.5. *Let $\alpha \in \mathcal{T}_X$. Then $\alpha \sim \varepsilon$ in $\mathcal{G}(\mathcal{T}_X)$ if and only if* $\mathrm{im}(\alpha) = X$.

Proof. Suppose that $\mathrm{im}(\alpha) = X$. Then α is a permutation on X, i.e., $\alpha \in S_X$. Since S_X is finite, it follows that $\alpha^m = \varepsilon$ for some $m \in \mathbb{N}^+$ and so $\alpha \sim \varepsilon$. Conversely, if $\alpha \sim \varepsilon$, then $\alpha^m = \varepsilon$ for some $m \in \mathbb{N}^+$. By Lemma 3.4, $X = \mathrm{im}(\varepsilon) \subseteq \mathrm{im}(\alpha)$ and so $\mathrm{im}(\alpha) = X$. □

Proposition 3.6. *Let $e \in E(\mathcal{T}_X)$ and $\alpha \in \mathcal{T}_X$. Then the following statements are equivalent:*

(i) *$\alpha \sim e$ in $\mathcal{G}(\mathcal{T}_X)$;*

(ii) $\mathrm{im}(\alpha^{d(\alpha')}) = \mathrm{im}(e)$ *and* $\ker(\alpha^{d(\alpha')}) = \ker(e)$;

(iii) $\alpha^{d(\alpha')}\mathcal{H}e$.

Proof. By Lemma 3.3, it is easy to see that (ii) and (iii) are equivalent. We need only to prove that (i) and (ii) are equivalent. Suppose that $\alpha \sim e$. Then $\alpha^m = e$ for some $m \in \mathbb{N}^+$ and so $(\alpha^{d(\alpha')})^m = e$. Therefore

$$X_{\widehat{\alpha}} = \mathrm{im}((\alpha^{d(\alpha')})^m) = \mathrm{im}(e) \subseteq \mathrm{im}(\alpha^{d(\alpha')}) = X_{\widehat{\alpha}}.$$

Thus $\mathrm{im}(e) = X_{\widehat{\alpha}} = \mathrm{im}(\alpha^{d(\alpha')})$. Since $(\alpha^{d(\alpha')})^m = e$, it follows by Lemma 3.4 that $\ker(\alpha^{d(\alpha')}) \subseteq \ker(e)$. If $(x, y) \in \ker(e)$, then $e(x) = e(y)$ and so $\alpha^m(x) = \alpha^m(y)$. This implies that $\alpha^{d(\alpha')} \cdot \alpha^m(x) = \alpha^{d(\alpha')} \cdot \alpha^m(y)$. Thus $\alpha^m \cdot \alpha^{d(\alpha')}(x) = \alpha^m \cdot \alpha^{d(\alpha')}(y)$. Since $\alpha^{d(\alpha')}(x), \alpha^{d(\alpha')}(y) \in X_{\widehat{\alpha}}$ and $\alpha|_{X_{\widehat{\alpha}}}$ is a permutation on $X_{\widehat{\alpha}}$, it follows that $\alpha^{d(\alpha')}(x) = \alpha^{d(\alpha')}(y)$ and so $\ker(e) \subseteq \ker(\alpha^{d(\alpha')})$. This shows $\ker(e) = \ker(\alpha^{d(\alpha')})$.

Conversely, suppose that $\mathrm{im}(\alpha^{d(\alpha')}) = \mathrm{im}(e)$ and $\ker(\alpha^{d(\alpha')}) = \ker(e)$. Then $\mathrm{im}(e) = \mathrm{im}(\alpha^{d(\alpha')}) = X_{\widehat{\alpha}}$ and so by Lemma 3.2, $e|_{X_{\widehat{\alpha}}} = \varepsilon|_{X_{\widehat{\alpha}}}$. Now we consider the factorization formula (3.3)

$$\alpha^{d(\alpha')} = \widehat{\widehat{\alpha}} \cdot \alpha'' = \widehat{\widehat{\alpha}} \cdot \begin{pmatrix} k_1 & k_2 & \cdots & k_{n-t} \\ h_1 & h_2 & \cdots & h_{n-t} \end{pmatrix},$$

where $\widehat{\widehat{\alpha}} \in S_{X_{\widehat{\alpha}}}, h_i \in X_{\widehat{\alpha}}$ for each $i \in [n-t]$. Since $\widehat{\widehat{\alpha}} \in S_{X_{\widehat{\alpha}}}$, it follows that $\widehat{\widehat{\alpha}}^l = \varepsilon|_{X_{\widehat{\alpha}}}$ for some $l \in \mathbb{N}^+$. Thus $\widehat{\widehat{\alpha}}^l = \varepsilon|_{X_{\widehat{\alpha}}} = e|_{X_{\widehat{\alpha}}}$, i.e.,

$$(\alpha^{d(\alpha')})^l(x) = e(x) = x \text{ for all } x \in X_{\widehat{\alpha}}. \tag{3.4}$$

For each $k_j (j \in [n-t])$ we have $\alpha^{d(\alpha')}(k_j) = h_j \in X_{\widehat{\alpha}}$. Then there exists $i \in X_{\widehat{\alpha}}$ such that $\alpha^{d(\alpha')}(i) = h_j$ and so $(k_j, i) \in \ker(\alpha^{d(\alpha')}) = \ker(e)$. Therefore

$$\begin{aligned}(\alpha^{d(\alpha')})^l(k_j) &= (\alpha^{d(\alpha')})^{l-1} \cdot \alpha^{d(\alpha')}(k_j) \\ &= (\alpha^{d(\alpha')})^{l-1} \cdot \alpha^{d(\alpha')}(i) = (\alpha^{d(\alpha')})^l(i) = i.\end{aligned}$$

Since $(k_j, i) \in ker(e)$, it follows that $e(k_j) = e(i) = i$ and so

$$(\alpha^{d(\alpha')})^l(k_j) = e(k_j) \text{ for all } j \in [n-t]. \tag{3.5}$$

Then by (3.4) and (3.5) we have that $(\alpha^{d(\alpha')})^l = e$. This implies that $\alpha \sim e$. □

Let S be a finite semigroup. We review the structure of connected components of power graph $\mathcal{G}(S)$ from [8]. Define an equivalence relation ρ on S by

$$a\rho b \Leftrightarrow a^m = b^m \text{ for some } m \in \mathbb{N},$$

and denote the equivalence class of $e \in E(S)$ under ρ by C_e, i.e.,

$$C_e = \{a \in S \mid a\rho e\} = \{a \in S \mid a^m = e\} \text{ for some } m \in \mathbb{N}.$$

Lemma 3.7. *[8, Theorem 2.3] Let S be a finite semigroup and $a, b \in S$ such that $a \neq b$. Then a and b are connected by a path in $\mathcal{G}(S)$ if and only if $a\rho b$.*

Lemma 3.8. *[8, Corollary 2.4] Let S be a finite semigroup. Then the connected components of $\mathcal{G}(S)$ are precisely $\{C_e \mid e \in E(S)\}$. Each connected component C_e contains the unique idempotent e.*

By the above lemmas we have the following result.

Proposition 3.9. *Let $\alpha, \beta \in \mathcal{T}_X$. Then α and β are connected by a path in $\mathcal{G}(\mathcal{T}_X)$ if and only if $\alpha^{d(\alpha')}\mathcal{H}\beta^{d(\beta')}$.*

Proof. Suppose that $\alpha \in C_e$ for some $e \in E(\mathcal{T}_X)$ by Lemma 3.8. Then α and β are connected by a path in $\mathcal{G}(\mathcal{T}_X)$ if and only if $\alpha \sim e$ and $\beta \sim e$. By Proposition 3.6, we can show that

$$\mathrm{im}(\alpha^{d(\alpha')}) = \mathrm{im}(e) = \mathrm{im}(\beta^{d(\beta')})$$

and

$$\ker(\alpha^{d(\alpha')}) = \ker(e) = \ker(\beta^{d(\beta')}).$$

This implies by Lemma 3.3 that $\alpha^{d(\alpha')}\mathcal{H}\beta^{d(\beta')}$. □

Next we shall characterize the structures of the connected components of $\mathcal{G}(\mathcal{T}_X)$.

Lemma 3.10. *Let $e \in E(\mathcal{T}_X)$. If $|\mathrm{im}(e)| = n$, then $e = \varepsilon$ and $C_e \cong S_n$.*

Proof. Suppose that $|\mathrm{im}(e)| = n$. Then $e = \varepsilon$. If $\alpha \in C_e$, then there exists $m \in \mathbb{N}^+$ such that $\alpha^m = e$. Thus $\mathrm{im}(e) \subseteq \mathrm{im}(\alpha)$ and so $|\mathrm{im}(\alpha)| = n$. Hence α is a permutation over X, i.e., $\alpha \in S_X$. Conversely, if $\alpha \in S_X$, then there exists $m \in \mathbb{N}^+$ such that $\alpha^m = \varepsilon = e$ and so $\alpha \in C_e$. This implies $C_e = S_X \cong S_n$. □

Lemma 3.11. *Let $e \in E(\mathcal{T}_X)$. If $|\mathrm{im}(e)| = n-1$, then $C_e \cong S_{n-1}$.*

Proof. Suppose that $|\mathrm{im}(e)| = n-1$. Without loss of generality, assume that

$$e = (i_1)(i_2)\cdots(i_k)\cdots(i_{n-1})\begin{pmatrix} i_n \\ i_k \end{pmatrix} \text{ for some } k \in [n-1].$$

Let

$$A = \{\alpha \in \mathcal{T}_X \mid \alpha = \widehat{\widehat{\alpha}} \cdot \begin{pmatrix} i_n \\ a_k \end{pmatrix}, \text{ where } \widehat{\widehat{\alpha}} = \begin{pmatrix} i_1 & i_2 & \cdots & i_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix} \in S_{\overline{X}} \text{ is a permutation over } \overline{X} = \{i_1, i_2, \ldots, i_{n-1}\}\}.$$

We claim that $C_e = A$. Let $\alpha \in A$ and $\begin{pmatrix} i_n \\ a_k \end{pmatrix} = \alpha'$. It is easy to see that $\mathrm{im}(\alpha) = \mathrm{im}(e)$, $\ker(\alpha) = \ker(e)$, $d(\alpha') = 1$ and $\alpha^{d(\alpha')} = \alpha$. This shows that $\mathrm{im}(\alpha^{d(\alpha')}) = \mathrm{im}(e)$ and $\ker(\alpha^{d(\alpha')}) = \ker(e)$ and so by Proposition 3.6, $\alpha \sim e$. It follows that $\alpha \in C_e$. Thus $A \subseteq C_e$.

Conversely, suppose $\alpha \in C_e$. Then $\alpha \sim e$. Thus there exists $m \in \mathbb{N}^+$ such that $\alpha^m = e$ and so $\{i_1, i_2, \ldots, i_{n-1}\} = \mathrm{im}(e) \subseteq \mathrm{im}(\alpha) \neq X$. This implies $\mathrm{im}(\alpha) = \mathrm{im}(e)$. By Proposition 3.6, $\mathrm{im}(\alpha^{d(\alpha')}) = \mathrm{im}(e)$ and $\ker(\alpha^{d(\alpha')}) = \ker(e)$. It follows that

$$\alpha^{d(\alpha')} = \begin{pmatrix} i_1 & i_2 & \cdots & i_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix} \cdot \begin{pmatrix} i_n \\ a_k \end{pmatrix}, \text{ where } \{a_1, \ldots, a_{n-1}\} = \{i_1, \ldots, i_{n-1}\}.$$

This shows that

$$\alpha = \begin{pmatrix} i_1 & i_2 & \cdots & i_{n-1} \\ b_1 & b_2 & \cdots & b_{n-1} \end{pmatrix} \cdot \begin{pmatrix} i_n \\ b \end{pmatrix},$$

where $\{b_1, \ldots, b_{n-1}\} = \{i_1, \ldots, i_{n-1}\}, b \in \{i_1, \ldots, i_{n-1}\}$. Let $\begin{pmatrix} i_n \\ b \end{pmatrix} = \alpha'$. Then $d(\alpha') = 1$ and so $\alpha = \alpha^{d(\alpha')}$. Hence $\ker(\alpha) = \ker(e)$. Since $(i_k, i_n) \in \ker(e)$, it follows that $b = \alpha(i_n) = \alpha(i_k) = b_k$ and so $\alpha \in A$. Thus $C_e \subseteq A$. Therefore $C_e = A$.

Now we shall show that A is a group. Firstly, for $\alpha, \beta \in A$, suppose that

$$\alpha = \widehat{\widehat{\alpha}} \cdot \begin{pmatrix} i_n \\ a_k \end{pmatrix}, \beta = \widehat{\widehat{\beta}} \cdot \begin{pmatrix} i_n \\ b_k \end{pmatrix}, \text{ where } \widehat{\widehat{\alpha}}, \widehat{\widehat{\beta}} \in S_{\overline{X}}, a_k = \widehat{\widehat{\alpha}}(i_k), b_k = \widehat{\widehat{\beta}}(i_k).$$

Then $\beta\alpha|_{\overline{X}} = \widehat{\widehat{\beta}}\widehat{\widehat{\alpha}}|_{\overline{X}} \in S_{\overline{X}}$ and

$$(\beta\alpha)(i_n) = \beta(\alpha(i_n)) = \beta(\alpha(i_k)) = (\beta\alpha)(i_k).$$

This implies that

$$\beta\alpha = \widehat{\widehat{\beta}}\widehat{\widehat{\alpha}} \begin{pmatrix} i_n \\ (\beta\alpha)(i_k) \end{pmatrix} \in A.$$

Next, if $\alpha = \widehat{\widehat{\alpha}} \begin{pmatrix} i_n \\ a_k \end{pmatrix} \in A$, $e = \widehat{\widehat{e}} \begin{pmatrix} i_n \\ i_k \end{pmatrix}$, it is a matter of routine to verify that

$$e\alpha = \widehat{\widehat{e}} \begin{pmatrix} i_n \\ i_k \end{pmatrix} \cdot \widehat{\widehat{\alpha}} \begin{pmatrix} i_n \\ a_k \end{pmatrix} = \widehat{\widehat{e}}\,\widehat{\widehat{\alpha}}(\begin{pmatrix} i_n \\ a_k \end{pmatrix}) = \alpha, \alpha e = \widehat{\widehat{\alpha}}\,\widehat{\widehat{e}}(\begin{pmatrix} i_n \\ a_k \end{pmatrix}) = \alpha.$$

This shows that e is an identity element in A.

Finally, let $\alpha = \widehat{\widehat{\alpha}} \cdot \begin{pmatrix} i_n \\ a_k \end{pmatrix} \in A$. Since $\widehat{\widehat{\alpha}} \in S_{\overline{X}}$, it follows that there exists $\widehat{\widehat{\beta}} \in S_{\overline{X}}$ such that $\widehat{\widehat{\beta}}\,\widehat{\widehat{\alpha}} = \varepsilon\mid_{\overline{X}}$. Suppose that $\beta = \widehat{\widehat{\beta}} \cdot \begin{pmatrix} i_n \\ b_k \end{pmatrix}$, where $b_k = \widehat{\widehat{\beta}}(i_k)$. Then $\beta \in A$ and

$$\beta\alpha = \widehat{\widehat{\beta}} \cdot \begin{pmatrix} i_n \\ b_k \end{pmatrix} \widehat{\widehat{\alpha}} \cdot \begin{pmatrix} i_n \\ a_k \end{pmatrix}.$$

Thus

$$\beta\alpha\mid_{\overline{X}} = \widehat{\widehat{\beta}}\,\widehat{\widehat{\alpha}} = \varepsilon\mid_{\overline{X}} = e_{\overline{X}}$$

and

$$\beta\alpha(i_n) = \beta\alpha(i_k) = i_k = e(i_n).$$

Hence $\beta\alpha = e$.

This shows that $C_e = A$ is a group.

Let $\varphi : A \to S_{\overline{X}}$ be a map such that $\varphi(\alpha) = \widehat{\widehat{\alpha}}$ for all $\alpha \in A$. It is easy to check that φ is an isomorphism. Hence $C_e = A \cong S_{\overline{X}} \cong S_{n-1}$. □

Lemma 3.12. *Let $e \in E(\mathcal{T}_X)$. If $|\mathrm{im}(e)| \leq n-2$, then C_e is not a semigroup.*

Proof. Suppose that

$$e = (i_1)(i_2)\cdots(i_{k-1}) \begin{pmatrix} i_k & i_{k+1} & \cdots & i_n \\ a_k & a_{k+1} & \cdots & a_n \end{pmatrix},$$

where $a_l \in \{i_1, i_2, \ldots, i_{k-1}\}$ for all $l, k \leq l \leq n$. Then $\mathrm{im}(e) = \{i_1, i_2, \ldots, i_{k-1}\}$. Since $|\mathrm{im}(e)| \leq n-2$, it follows that there are at least two elements in the set $\{i_k, i_{k+1}, \ldots, i_n\}$. Without loss of generality, take i_k and i_{k+1}. Assume that $e(i_k) = i_r, e(i_{k+1}) = i_s$ for some $r, s \in [k-1]$. Let

$$\alpha = (i_1)\cdots(i_r i_s)\cdots(i_{k-1}) \begin{pmatrix} i_k & i_{k+1} & \cdots & i_l & \cdots & i_n \\ i_s & i_k & \cdots & \gamma(i_l) & \cdots & \gamma(i_n) \end{pmatrix}$$

and

$$\beta = (i_1)\cdots(i_ri_s)\cdots(i_{k-1})\begin{pmatrix} i_k & i_{k+1} & \cdots & i_l & \cdots & i_n \\ i_{k+1} & i_r & \cdots & \gamma(i_l) & \cdots & \gamma(i_n) \end{pmatrix},$$

where $\gamma(i_l) = \begin{cases} e(i_l), & e(i_l) \neq i_r, i_s; \\ i_s, & e(i_l) = i_r; \\ i_r, & e(i_l) = i_s, \end{cases}$ for all $l \in \mathbb{N}^+$ such that $k+2 \leq l \leq n$.

Then

$$\alpha^2 = (i_1)(i_2)\cdots(i_{k-1})\begin{pmatrix} i_k & i_{k+1} & \cdots & i_l & \cdots & i_n \\ i_r & i_s & \cdots & e(i_l) & \cdots & e(i_n) \end{pmatrix} = e$$

and so $\alpha \in C_e$. Similarly, we have that $\beta^2 = e$ and so $\beta \in C_e$.

It is easy to check that $\beta\alpha(i_{k+1}) = i_{k+1}$. But $e(i_{k+1}) \neq i_{k+1}$, so $\ker(\beta\alpha) \nsubseteq \ker(e)$. It follows by Lemma 3.4 that $\beta\alpha \notin C_e$. Hence, C_e is not a semigroup. □

Lemmas 3.10, 3.11 and 3.12 together imply the main theorem of this section:

Theorem 3.13. *Let $|X| = n \geq 2, e \in E(\mathcal{T}_X)$. Then the following statements are equivalent:*

(i) *C_e is a semigroup;*

(ii) *C_e is a group;*

(iii) *$|\mathrm{im}(e)| = n$ or $n-1$.*

Since every connected component contains a unique idempotent, we give a characterization of the unique idempotent for each connected component in the following proposition.

Proposition 3.14. *Let $\alpha \in \mathcal{T}_X$. Suppose that $\alpha^{d(\alpha')}$ has the Factorization formula (3.4),*

$$\alpha^{d(\alpha')} = \widehat{\widehat{\alpha}} \cdot \alpha'' = \widehat{\widehat{\alpha}} \cdot \begin{pmatrix} k_1 & k_2 & \cdots & k_{n-t} \\ h_1 & h_2 & \cdots & h_{n-t} \end{pmatrix}$$

where $\widehat{\widehat{\alpha}} \in S_{X_{\widehat{\alpha}}}, h_i \in X_{\widehat{\alpha}}$ for all $i \in [n-t]$. Let the order of $\widehat{\widehat{\alpha}}$ be l. Then $\alpha \in C_e$, where

$$e = \varepsilon|_{X_{\widehat{\alpha}}} \cdot \begin{pmatrix} k_1 & k_2 & \cdots & k_{n-t} \\ g_1 & g_2 & \cdots & g_{n-t} \end{pmatrix} \text{with } g_i = \widehat{\widehat{\alpha}}^{\,l-1}(h_i), 1 \leq i \leq n-t. \tag{3.6}$$

Proof. By the preconditions we have that

$$\begin{aligned}\alpha^{d(\alpha')\cdot l} = (\alpha^{d(\alpha')})^l = (\widehat{\widehat{\alpha}} \cdot \alpha'')^l &= \widehat{\widehat{\alpha}}^{\,l} \cdot \alpha''' \\ &= \varepsilon|_{X_{\widehat{\alpha}}} \cdot \alpha''', \text{ where } \mathrm{im}(\alpha''') \subseteq X_{\widehat{\alpha}}.\end{aligned}$$

Then by Lemma 3.2, we deduce that $\alpha^{d(\alpha')\cdot l}$ is idempotent. It can be checked for each $i \in [n-t]$ that

$$\alpha'''(k_i) = (\alpha^{d(\alpha')})^l(k_i) = (\alpha^{d(\alpha')})^{l-1} \cdot \alpha^{d(\alpha')}(k_i) = \widehat{\widehat{\alpha}}^{\,l-1}(h_i).$$

This implies that α is adjacent to $\alpha^{d(\alpha')\cdot l}$ and $\alpha^{d(\alpha')\cdot l} = e$. □

Figure 1: Power graph $\mathcal{G}(\mathcal{T}_X)$ with $|X| = 2$.

4. The planarity of the power graph of $\mathcal{T}_X$

In this section we shall consider the planarity of the power graph $\mathcal{G}(\mathcal{T}_X)$. If $|X| = 1$, then the power graph $\mathcal{G}(\mathcal{T}_X)$ is trivial. Consider the following three cases.

Case 1. $|X| = 2$. Then $|\mathcal{T}_X| = 4$, $|E(\mathcal{T}_X)| = 3$. Let

$$A_1 = (1)(2), A_2 = (1)\begin{pmatrix} 2 \\ 1 \end{pmatrix}, A_3 = (2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}, A_4 = (12).$$

Thus $\mathcal{T}_X = \{A_1, A_2, A_3, A_4\}$, $E(\mathcal{T}_X) = \{A_1, A_2, A_3\}$. The power graph $\mathcal{G}(\mathcal{T}_X)$ is depicted in Figure 1.

Case 2. $|X| = 3$. Then $|\mathcal{T}_X| = 27$, $|E(\mathcal{T}_X)| = 10$. Let

$$A_1 = (1)(2)(3), A_2 = (1)(23), A_3 = (2)(13), A_4 = (3)(12), A_5 = (123), A_6 = (132);$$

$$B_1 = (1)(2)\begin{pmatrix} 3 \\ 1 \end{pmatrix}, B_2 = (1)(2)\begin{pmatrix} 3 \\ 2 \end{pmatrix}, B_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, B_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix},$$

$$B_5 = (2)(3)\begin{pmatrix} 1 \\ 2 \end{pmatrix}, B_6 = (2)(3)\begin{pmatrix} 1 \\ 3 \end{pmatrix}, B_7 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}, B_8 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix},$$

$$B_9 = (1)(3)\begin{pmatrix} 2 \\ 3 \end{pmatrix}, B_{10} = (1)(3)\begin{pmatrix} 2 \\ 1 \end{pmatrix}, B_{11} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, B_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix},$$

$$B_{13} = (1)\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, B_{14} = (1)\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}, B_{15} = (2)\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, B_{16} = (2)\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix},$$

$$B_{17} = (3)\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, B_{18} = (3)\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

Thus

$$\mathcal{T}_X = \{A_1, \ldots, A_6, B_1, \ldots, B_{18}, C_1, C_2, C_3\},$$

$$E(\mathcal{T}_X) = \{A_1, B_1, B_2, B_6, B_7, B_9, B_{10}, C_1, C_2, C_3\}.$$

The power graph $\mathcal{G}(\mathcal{T}_X)$ is depicted in Figure 2.

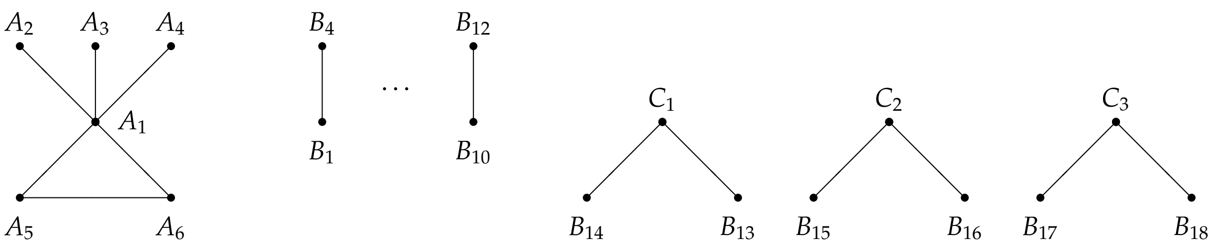

Figure 2: Power graph $\mathcal{G}(\mathcal{T}_X)$ with $|X| = 3$.

Case 3. $|X| = 4$. Then $|\mathcal{T}_X| = 256$, $|E(\mathcal{T}_X)| = 41$. Let

$$A = \{A_i \mid A_i \in \mathcal{T}_X, |\mathrm{im}(A_i)| = 4\}.$$

Thus $A = S_4, |A| = 24$. Let

$$B = \{B_i \mid B_i \in \mathcal{T}_X, \operatorname{im}(B_i) = \{1,2,3\}\};$$
$$E = \{E_i \mid E_i \in \mathcal{T}_X, \operatorname{im}(E_i) = \{1,2,4\}\};$$
$$F = \{F_i \mid F_i \in \mathcal{T}_X, \operatorname{im}(F_i) = \{1,3,4\}\};$$
$$G = \{G_i \mid G_i \in \mathcal{T}_X, \operatorname{im}(G_i) = \{2,3,4\}\};$$
$$C = \{C_i \mid C_i \in \mathcal{T}_X, |\operatorname{im}(C_i)| = 2\};$$
$$D = \{D_i \mid D_i \in \mathcal{T}_X, |\operatorname{im}(D_i)| = 1\}.$$

Hence $|B| = |E| = |F| = |G| = 36, |C| = 84, |D| = 4$.

The power graph $\mathcal{G}(\mathcal{T}_X)$ is depicted in Figure 3.

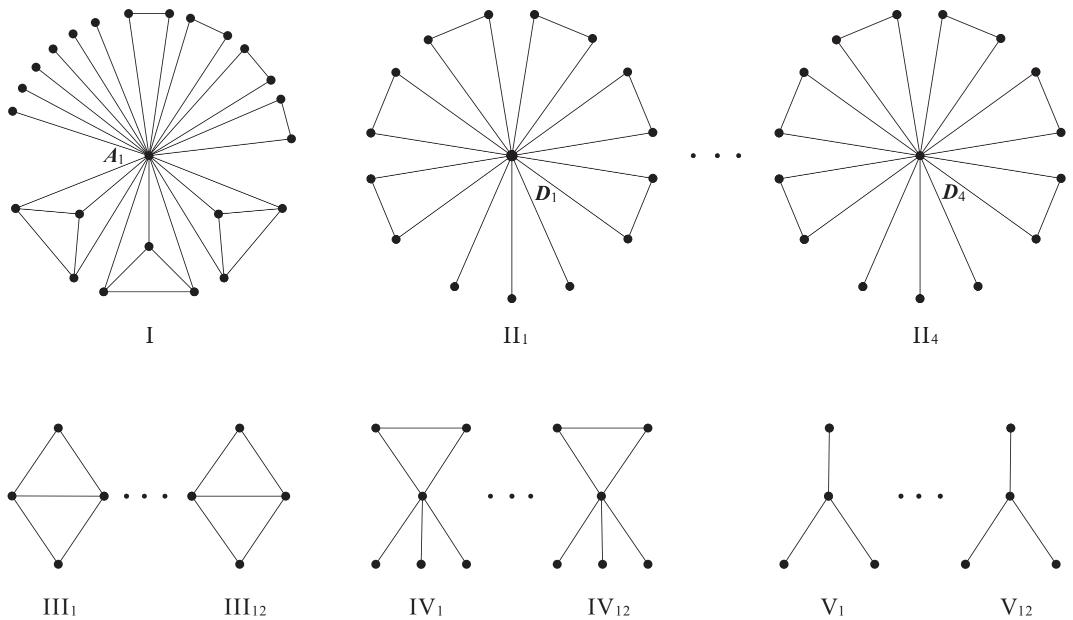

Figure 3: Power graph $\mathcal{G}(\mathcal{T}_X)$ with $|X| = 4$.

Lemma 4.1. *[8, Theorem 2.12] Let G be a finite group. Then $\mathcal{G}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and for some $m \in \mathbb{N}$.*

Lemma 4.2. *[39, Theorem 4.3] A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.*

Based on the above characterization of the power graphs of full transformation semigroups, we have

Theorem 4.3. *The power graph $\mathcal{G}(\mathcal{T}_X)$ on X is planar if and only if $|X| \leq 4$.*

Proof. We can observe by Figures 1, 2 and 3 that $\mathcal{G}(\mathcal{T}_X)$ is planar if $|X| \leq 4$. Suppose that $|X| \geq 5$. Consider the cycle $\alpha = (12345) \in \mathcal{T}_X$. Since $\langle\alpha\rangle$ is the cyclic group of order 5, it follows by Lemma 4.1 that $\mathcal{G}(\langle\alpha\rangle)$ is a complete graph with 5 vertices. Then $\mathcal{G}(\mathcal{T}_X)$ has the subgraph $\mathcal{G}(\langle\alpha\rangle)$ that is isomorphic to K_5. Hence $\mathcal{G}(\mathcal{T}_X)$ is not planar by Lemma 4.2. We conclude that $\mathcal{G}(\mathcal{T}_X)$ is planar if and only if $|X| \leq 4$. □

Declarations

Conflict of interest The authors declare that they have no conflict of interest.
Availability of data materials This manuscript has no associated data.

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