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A simple characterization of Wright-convexity and applications

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Abstract. In this work we present simple characterizations of W-convex and W-log-convex functions. As applications, we give short proofs for integral forms of Hermite-Hadamard or Levin-Stečkin inequalities. We also give a characterization theorem for inner product spaces as normed spaces satisfying a certain property.

1. Introduction

Convex functions play a main role in various fields such as optimization, economics, and mathematical analysis.The study of convex functions is not only theoretically important, but also practically significant due to their applications in various optimization problems where they often simplify the analysis and solution processes.

Since the introduction of convex functions, many variants and extensions of convexity were presented.

In 1954, the Englich mathematician E. M. Wright proposed a new type of convexity. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called *Wright-convex* if the following inequality holds true:

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y), \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$. We also say that f is a W-convex *function*.

Remark that any convex function is W-convex and any W-convex function is midconvex. For proofs, more details and further properties, please see, e.g., [12], [2], [6], [8], or [9] and all references therein.

As

$$tx + (1-t)y + (1-t)x + ty = x + y,$$

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the relation (1) becomes:

$$f(u) + f(v) \leq f(x) + f(y), \tag{2}$$

for all $x, y, u, v \in I$, such that u and v lie between x and y with $u + v = x + y$.

Important results about W-convex functions where presented by Ng [5], who proved that any W-convex function is the sum of a convex and an additive function. This is in concordance with the fact that the convex functions are closely related to linear functions, making them a natural extension in the study of nonlinear phenomena from other branches of science.

Our work aims to present an original type of characterizing W-convex functions, more simple than those already known and very useful as well. As direct consequences, we provide a very short proof for two important results involving this class of functions or convex functions, respective Hermite-Hadamard and Levin-Stečkin inequalities.

2. A characterization of W-convex functions

Let $I \subseteq \mathbb{R}$ be an interval and let $a, b \in I$, $a < b$ be arbitrarily fixed. For a function $f : [a, b] \to \mathbb{R}$, we define $S_f : [a, b] \to \mathbb{R}$ by the formula:

$$S_f(x) = \frac{1}{2}\{f(x) + f(a + b - x)\}, \quad x \in [a, b].$$

Remark that $S_f(x) = S_f(a + b - x)$, for all $x \in [a, b]$.

Now we are in a position to give the following characterization of W-convexity:

Theorem 2.1. *The next assertions are equivalent:*
(a) The function $f : I \to \mathbb{R}$ is W-convex;
(b) for any $a, b \in I$, $a < b$, the function S_f is decreasing on $\left[a, \frac{a+b}{2}\right]$;
(c) for any $a, b \in I$, $a < b$, the function S_f is increasing on $\left[\frac{a+b}{2}, b\right]$.

Proof. **(a) ⇒ (b)**. Let $u, v \in \left[a, \frac{a+b}{2}\right]$ with $u < v$. We evaluate $S_f(v) - S_f(u)$:

$$S_f(v) - S_f(u) = \frac{1}{2}\{f(a + b - v) + f(v) - f(a + b - u) - f(u)\}.$$

Furthermore, $a+b-v$ and v are in $(u, a + b - u)$ and $(a + b - v)+v = (a + b - u)+u$. We obtain $S_f(v)-S_f(u) < 0$ due to (2). In consequence, S_f is decreasing on $\left[a, \frac{a+b}{2}\right]$.

(b) ⇒ (c) follows directly using the equality $S_f(x) = S_f(a + b - x)$ (this means the function S_f is symmetric with respect the line $x = \frac{a+b}{2}$).

(c) ⇒ (a). Let $x, y \in I$, $x < y$ and let $t \in [0, 1]$. As

$$tx + (1 - t)y + (1 - t)x + ty = x + y,$$

we can assume that

$$(1 - t)x + ty \in \left[\frac{x + y}{2}, y\right].$$

Since the function $S_f(u) = \frac{1}{2}\{f(u) + f(x + y - u)\}$ is nondecresing, we have:

$$S_f((1 - t)x + ty) \leq S_f(y),$$

or, equivalently, the relation (1). The proof is now completed. □

3. The Hermite-Hadamard inequality

The version of this inequality for W-convex functions is due to Olbryś [9, Theorem 11]. He proved that for every W-convex function $f : I \to \mathbb{R}$ and all $a, b \in I$, $a < b$, the following inequality holds true:

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \{f(x) + f(a+b-x)\}\,dx \leq f(a) + f(b). \tag{3}$$

Śliwińska and Wasowicz [10] provided a different proof of (3) based on Ng Theorem.

We give here a simple proof of (3), using our Theorem 1.

Theorem 3.1. *Let $I \subseteq \mathbb{R}$ be an interval, $a, b \in I$, $a < b$ and let $f : [a, b] \to \mathbb{R}$ be W-convex. Then the following inequalities are valid:*
a)

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \{f(x) + f(a+b-x)\}\,dx \leq \frac{f(a)+f(b)}{2}; \tag{4}$$

b)

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{\frac{a+b}{2}}^b \{f(x) + f(a+b-x)\}\,dx \leq \frac{f(a)+f(b)}{2}. \tag{5}$$

Proof. a) The associated function S_f is monotonically decreasing on $\left[a, \frac{a+b}{2}\right]$, so S_f is integrable on $\left[a, \frac{a+b}{2}\right]$. By using the monotonicity of S_f, we have:

$$S_f\left(\frac{a+b}{2}\right) \leq S_f(x) \leq S_f(a),$$

for all $x \in \left[a, \frac{a+b}{2}\right]$. Moreover, by integration on $\left[a, \frac{a+b}{2}\right]$, we deduce that:

$$\left(\frac{a+b}{2} - a\right) S_f\left(\frac{a+b}{2}\right) \leq \int_a^{\frac{a+b}{2}} S_f(x)\,dx \leq \left(\frac{a+b}{2} - a\right) S_f(a),$$

that is:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \{f(x) + f(a+b-x)\}\,dx \leq \frac{f(a)+f(b)}{2}.$$

b) Analogously, the function S_f is monotonically increasing on $\left[\frac{a+b}{2}, b\right]$, so S_f is integrable on $\left[\frac{a+b}{2}, b\right]$. By using the monotonicity of S_f, we have:

$$S_f\left(\frac{a+b}{2}\right) \leq S_f(x) \leq S_f(b),$$

for all $x \in \left[\frac{a+b}{2}, b\right]$. Moreover, by integration on $\left[\frac{a+b}{2}, b\right]$, we deduce that:

$$\left(b - \frac{a+b}{2}\right) S_f\left(\frac{a+b}{2}\right) \leq \int_{\frac{a+b}{2}}^b S_f(x)\,dx \leq \left(b - \frac{a+b}{2}\right) S_f(a),$$

that is:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{\frac{a+b}{2}}^b \{f(x) + f(a+b-x)\}\,dx \leq \frac{f(a)+f(b)}{2}.$$

□

Now, the inequality (3) follows by summing (4) and (5).

4. The Levin-Stečkin inequality

Another integral inequality involving convex functions is the Levin-Stečkin inequality.

Let $a, b \in \mathbb{R}$, $a < b$. Recall that, for $f : [a, b] \to \mathbb{R}$ convex and $p : [a, b] \to \mathbb{R}$, increasing on $\left[a, \frac{a+b}{2}\right]$, with $p(a+b-x) = p(x)$, for all $x \in [a, b]$, the following inequality (named *the Levin-Stečkin inequality*) holds true:

$$\int_a^b p(x) f(x)\, dx \le \frac{1}{b-a} \int_a^b p(x)\, dx \cdot \int_a^b f(x)\, dx. \tag{6}$$

A recent, but non-elementary proof is due to Mercer [4]. Witkowski [11] used the convexity of f to prove (6).

Here we use our Theorem 1 to prove and to extend (6).

Theorem 4.1. *Let $f : [a, b] \to \mathbb{R}$ be a W-convex function and let $p : [a, b] \to \mathbb{R}$ be a increasing function on $\left[a, \frac{a+b}{2}\right]$ such that $p(a+b-x) = p(x)$, for all $x \in [a, b]$. Then the following inequality is valid:*

$$\int_a^b p(x)\{f(x) + f(a+b-x)\}\, dx \le \frac{1}{b-a} \int_a^b p(x)\, dx \cdot \int_a^b \{f(x) + f(a+b-x)\}\, dx. \tag{7}$$

Proof. The montonicity of the function S_f and integral version of Chebyshev inequality on $\left[a, \frac{a+b}{2}\right]$, respectively $\left[\frac{a+b}{2}, b\right]$, give us

$$\frac{2}{b-a} \int_a^{\frac{a+b}{2}} p(x) S_f(x)\, dx \le \frac{2}{b-a} \int_a^{\frac{a+b}{2}} p(x)\, dx \cdot \frac{2}{b-a} \int_a^{\frac{a+b}{2}} S_f(x)\, dx \tag{8}$$

and

$$\frac{2}{b-a} \int_{\frac{a+b}{2}}^b p(x) S_f(x)\, dx \le \frac{2}{b-a} \int_{\frac{a+b}{2}}^b p(x)\, dx \cdot \frac{2}{b-a} \int_{\frac{a+b}{2}}^b S_f(x)\, dx. \tag{9}$$

As $p(a+b-x) = p(x)$, we deduce that:

$$\int_a^{\frac{a+b}{2}} p(x)\, dx = \int_{\frac{a+b}{2}}^b p(x)\, dx = \frac{1}{2} \int_b^b p(x)\, dx.$$

Now, (7) follows by summation of (8) and (9), and the proof is completed. □

Finally, let $f : [a, b] \to \mathbb{R}$ be a convex function. As

$$\int_a^b f(a+b-x)\, dx = \int_a^b f(x)\, dx,$$

(7) becomes (6).

5. *W*-log-convex functions

Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to (0, \infty)$ is called *W-log-convex* if

$$f((1-t)x + ty) f((1-t)y + tx) \le f(x) f(y),$$

for all $t \in [0, 1]$ and $x, y \in I$. We associate to a W-log-convex function $f : I \to (0, \infty)$ the function $G_f : [a, b] \to \mathbb{R}$, for every $a, b \in I$, $a < b$, by the formula:

$$G_f(x) = \sqrt{f(x) f(a+b-x)}, \quad x \in [a, b].$$

Remark that $f : I \to (0, \infty)$ is W-log-convex if and only if $\log \circ f : I \to \mathbb{R}$ is W-convex. In this way, we can deduce the analogue of Theorem 1 for characterization the W-log-convex functions.

Theorem 5.1. *The next assertions are equivalent:*
(a) The function $f : I \to (0, \infty)$ *is W-log-convex;*
(b) for any $a, b \in I$, $a < b$, *the function* G_f *is decreasing on* $\left[a, \frac{a+b}{2}\right]$;
(c) for any $a, b \in I$, $a < b$, *the function* G_f *is increasing on* $\left[\frac{a+b}{2}, b\right]$.

Proof. As

$$\log G_f(x) = \frac{1}{2}\{\log f(x) + \log f(a+b-x)\} = S_{\overline{f}},$$

where $\overline{f} = \log \circ f$, the conclusion follows from Theorem 1 applied to the function $\overline{f}$. □

We present the following version of Hermite-Hadamard inequality for W-log-convex functions:

Theorem 5.2. *Let* $I \subseteq \mathbb{R}$ *be an interval and* $f : I \to (0, \infty)$ *a W-log-convex function. Then for every* $a, b \in I$, $a < b$, *we have:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b G_f(x)\,dx \le \sqrt{f(a)\,f(b)}. \tag{10}$$

Proof. First, note that from the previous characterization theorem, the function G_f is integrable on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ (being monotonic on these intervals), and consequently, G_f is integrable on the interval $[a, b]$. Using (10), and the monotonicity of G_f on $\left[a, \frac{a+b}{2}\right]$, we get:

$$f\left(\frac{a+b}{2}\right) = G_f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_a^{\frac{a+b}{2}} G_f(x)\,dx \le G_f(a) = \sqrt{f(a)\,f(b)},$$

so

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_a^{\frac{a+b}{2}} G_f(x)\,dx \le \sqrt{f(a)\,f(b)}. \tag{11}$$

Similarly, using (10), and the monotonicity of G_f on $\left[\frac{a+b}{2}, b\right]$, we get:

$$f\left(\frac{a+b}{2}\right) = G_f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_{\frac{a+b}{2}}^b G_f(x)\,dx \le G_f(b) = \sqrt{f(a)\,f(b)},$$

so

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_{\frac{a+b}{2}}^b G_f(x)\,dx \le \sqrt{f(a)\,f(b)}. \tag{12}$$

Now, the conclusion follows by summation the inequalities (11) and (12). □

Remark that our Theorem 5 generalize Theorem 2.1 presented by Dragomir and Mond [1]. Moreover, our proof of Theorem 5 is based on a different idea compared to the similar result given in [1].

The next result is the analogue of Levin-Stečkin inequality (Theorem 3) for W-log-convex functions:

Theorem 5.3. *Let* $f : [a, b] \to (0, \infty)$ *be a W-log-convex function and let* $p : [a, b] \to \mathbb{R}$ *be increasing on* $\left[a, \frac{a+b}{2}\right]$, *with* $p(a+b-x) = p(x)$, *for all* $x \in [a, b]$. *The following inequality holds true:*

$$\int_a^b p(x)\,\sqrt{f(x)\,f(a+b-x)}\,dx \le \frac{1}{b-a}\int_a^b p(x)\,dx \int_a^b \sqrt{f(x)\,f(a+b-x)}\,dx.$$

The proof follows by direct use of the integral form of Chebyshev inequality and Theorem 3.

6. The characterization of inner product spaces by W-convex functions

Some characterizations of inner product spaces as normed spaces satisfying a certain property using the notions of convexity, or strong-convexity are provided, e.g., by Marinescu et al. [3], or Nikodem and Páles [7].

The main result we present here is the following characterization using the W-convexity:

Theorem 6.1. *Let $(X, \|\cdot\|)$ be a normed space (over $\mathbb{R}$, or $\mathbb{C}$). The following assertions are equivalent:*
i) X is a space with scalar product;
ii) for every function $f : [0, \infty) \to \mathbb{R}$ such that $f \circ \sqrt{\cdot}$ is W-convex, we have:

$$f\left(\|x+y\|\right) + f\left(\|x-y\|\right) \le f\left(\|x\| + \|y\|\right) + f\left(\left|\|x\| - \|y\|\right|\right),$$

for all $x, y \in X$.

Proof. First, remark that if $f \circ \sqrt{\cdot}$ is W-convex, then for all $a, d \in [0, \infty)$, $a \le d$, and $b, c \in [a, d]$, with $a+d = b+c$, we have:

$$f\left(\sqrt{b}\right) + f\left(\sqrt{c}\right) \le f\left(\sqrt{a}\right) + f\left(\sqrt{d}\right).$$

i)⇒ii). Let $(\cdot|\cdot)$ be the scalar product that define the norm of X. Let $x, y \in X$. By using the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = \left(\|x\| + \|y\|\right)^2 + \left(\|x\| - \|y\|\right)^2,$$

and the Schwarz inequality, we deduce that:

$$\left|Re\,(x|y)\right| \le \left|(x|y)\right| \le \|x\| \cdot \|y\|.$$

We have:

$$-\|x\|\,\|y\| \le Re(\cdot|\cdot) \le \|x\|\,\|y\|.$$

Thus:

$$\|x-y\|^2 \le \left(\|x\| + \|y\|\right)^2\ ,\quad \|x+y\|^2 \le \left(\|x\| + \|y\|\right)^2,$$

and

$$\left(\|x\| - \|y\|\right)^2 \le \|x+y\|^2\ ,\quad \left(\|x\| - \|y\|\right)^2 \le \|x-y\|^2.$$

Now, the assertion ii) follows by using the W-convexity of $f \circ \sqrt{\cdot}$ for

$$a = \left(\|x\| - \|y\|\right)^2\ ,\quad d = \left(\|x\| + \|y\|\right)^2,$$

and $\{b, c\} = \left\{\|x-y\|^2, \|x+y\|^2\right\}$.

ii)⇒i). Let us consider the function $f : [0, \infty) \to \mathbb{R}$, $f(t) = t^2$. Then the function $f \circ \sqrt{\cdot}$ is convex, and consequently, f is W-convex. Now, for all $x, y \in X$, we have:

$$\|x+y\|^2 + \|x-y\|^2 \le 2\left(\|x\|^2 + \|y\|^2\right),$$

and by replacing x and y with $x+y$ and $x-y$, we obtain the inverse inequality, and using Jordan and von Neumann's theorem the proof is completed. □

7. Conclusions

W-convex functions, although recently introduced in mathematics, have aroused the interest of a large number of mathematicians. This interest is due, among other things, to the connection of these functions to the midconvex and convex functions.

In this article, in Section 2, we have obtained a simple characterization of these functions in the spirit of those obtained by Olbryś, and in Section 3 we have provided a simple proof of the Hermite-Hadamard inequality.

In Section 4 we have obtained a Levin-Stećkin-type inequality for W-convex functions.

In section 5 we dealt with a new class of functions, namely the W-log-convex functions, and the inequalities we obtained create the premises for discovering new results.

Section 6, in which we obtained characterizations of the scalar product from pre-Hilbertian spaces, will be the starting point of a future work of us.

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