Filomat 39:8 (2025), 2491–2499
https://doi.org/10.2298/FIL2508491A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Applications of the Tachibana operator on invariant submanifolds of Lorentzian Trans-Sasakian manifolds

Mehmet Atçeken[a], Tuğba Mert[b,*], Mića S. Stanković[c]

[a]*Department of Mathematics, University of Aksaray, 68100, Aksaray, Turkey*
[b]*Department of Mathematics, University of Sivas Cumhuriyet, 58140, Sivas, Turkey*
[c]*Faculty of Sciences and Mathematics, University of Niš, Serbia*

Abstract. In the present paper, Tachibana operatory is applied to an invariant submanifold of a lorentzian trans-Sasakian manifold by means of through various tensors and the results obtained are discussed in terms of geometry. Finally, we give a non-trivial example in order to our results illustrate.

1. Introduction

A differentiable manifold $\widetilde{M}^{2n+1}$ which carries a field ϕ of endomorphism of the tangent space, a timelike vector ξ, called characteristic vector field, a 1-form η and the lorentzian metric g satisfying

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2}$$

and

$$(\widetilde{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y) - \eta(Y)\phi X\}, \tag{3}$$

for all $X, Y \in \Gamma(T\widetilde{M})$[7], where $\widetilde{\nabla}$ denote the Levi-Civita connection and α, β are smooth functions on $\widetilde{M}$. From (2), we have

$$\widetilde{\nabla}_X \xi = -\alpha\phi X - \beta\phi^2 X, \tag{4}$$

$$(\widetilde{\nabla}_X \eta)Y = \alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{5}$$

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C42.
Keywords. Lorentzian trans-Sasakain manifold, Invariant submanifold and Tachibana tensor.
Received: 20 September 2024; Accepted: 10 January 2025
Communicated by Dragan S. Djordjević
The third author acknowledge the grant of the Ministry of Science, Technological Development and Innovation of Serbia 451-03-137/2025-03/ 200124 for carrying out this research.
* Corresponding author: Tuğba Mert
Email addresses: mehmet.atceken382@gmail.com (Mehmet Atçeken), tmert@cumhuriyet.edu.tr (Tuğba Mert), mica.stankovic@pmf.edu.rs (Mića S. Stanković)
ORCID iDs: https://orcid.org/0000-0002-1242-4359 (Mehmet Atçeken), https://orcid.org/0000-0001-8258-8298 (Tuğba Mert), https://orcid.org/0000-0002-5632-0041 (Mića S. Stanković)

As special cases, if $\alpha = 0$ and $\beta \in \mathbf{R}$ the set of real numbers, then the manifold reduces to a Lorentzian β-Kenmotsu manifold, if $\beta = 0$ and $\alpha \in \mathbf{R}$, then the manifold reduces to a Lorentzian α-Sasakian manifold. On the other hand, $\alpha = 0$ and $\beta = 1$, the manifold reduces Lorentzian Kenmotsu manifold introduced by Mihai Oiaga and Rosca[1]. Furthermore, if $\beta = 0$ and $\alpha = 1$, then the manifold is called a Lorentzian Sasakian manifold. In this sense,the geometry of Invariant submanifold has been studied and continues to be studied by many geometers [1–3, 5].

In the present paper, we have searched the conditions $Q(S,\sigma) = 0$, $Q(g,\sigma) = 0$ $Q(g, R\cdot\sigma) = 0$, $Q(g, \widetilde{\nabla}\cdot\sigma) = 0$, $Q(S,\sigma) = 0$, $Q(S,\nabla\sigma) = 0$, $Q(S, R\cdot\sigma) = 0$, $Q(g, C\cdot\sigma) = 0$ and $Q(S, \widetilde{C}\cdot\sigma) = 0$ for an invariant submanifold of a lorentzian trans-Sasakian manifold. Finally,, we have classified the properties reduced by both the ambient manifold and the necessary submanifold.

In this connection, we need the following proposition for later used.

Proposition 1.1. *Let $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an lorentzian trans-Sasakain manifold, we denote the Riemannian curvature and Ricci tensors by $\widetilde{R}$ and $\widetilde{S}$, respectively, then we have*

$$\begin{aligned}\widetilde{R}(X,Y)\xi &= (\alpha^2+\beta^2)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y]\\ &+ Y(\alpha)\phi X - X(\alpha)\phi Y + Y(\beta)\phi^2 X - X(\beta)\phi^2 Y,\end{aligned} \tag{6}$$

$$\eta(\widetilde{R}(X,Y)Z) = (\alpha^2+\beta^2)g(\eta(Y)X-\eta(X)Y,Z) \tag{7}$$

$$\widetilde{R}(\xi,X)\xi = (\alpha^2+\beta^2-\xi(\beta))\phi^2X + (2\alpha\beta-\xi(\alpha))\phi X, \tag{8}$$

$$\begin{aligned}&\widetilde{S}(X,\xi) = [2n(\alpha^2+\beta^2)-\xi(\beta)]\eta(X) + (2n-1)X(\beta)\\ &- (\phi X)(\alpha)\end{aligned} \tag{9}$$

for all vector fields X, Y on $\widetilde{M}^{2n+1}$[7].

Now, let M be an immersed submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of M in $\widetilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{10}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{11}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the induced connections on M and $\Gamma(T^{\perp}M)$ and σ and A are called the second fundamental form and shape operator of M, respectively. Also, $\Gamma(TM)$ denotes the set differentiable vector fields on M. They are related by

$$g(A_V X, Y) = g(\sigma(X,Y), V). \tag{12}$$

The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{13}$$

for all $X, Y, Z \in \Gamma(TM)$[2]. If $\widetilde{\nabla}\sigma = 0$, then submanifold is said to be its second fundamental form is parallel. On the other hand, the submanifold M is said to be Chaki pseudo-parallel if there exists a 1-form γ such that

$$\begin{aligned}(\widetilde{\nabla}_X\sigma)(Y,Z) &= 2\gamma(X)\sigma(Y,Z) + \gamma(Y)\sigma(X,Z)\\ &+ \gamma(Z)\sigma(X,Y),\end{aligned} \tag{14}$$

for all $X, Y \in (TM)$.

By R, we denote the Riemannian curvature tensor of the submanifold M, we have the following Gauss equation

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X\sigma)(Y,Z) \\ &- (\widetilde{\nabla}_Y\sigma)(X,Z), \end{aligned} \tag{15}$$

for all $X, Y, Z \in \Gamma(TM)$.

On the other hand, the concircular curvature tensor C on a pseudo-Riemannian manifold (M^{2n+1}, g) is defined as follows

$$C(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{16}$$

for all $X, Y, Z \in \Gamma(TM)$[6], where τ denotes the scalar curvature of M^{2n+1}.

For a $(0,k)$-type tensor field T, $k \geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M, g), $Q(A,T)$-Tachibana operator is defined by

$$\begin{aligned} Q(A,T)(X_1, X_2, ..., X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, ..., X_k)... \\ &- T(X_1, X_2, ...X_{k-1}, (X \wedge_A Y)X_k),, \end{aligned} \tag{17}$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$[6], where the endormorphism $\wedge_A$ is defined by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{18}$$

2. Invariant Submanifolds of Lorentzian trans-Sasakain manifolds.

The geometry of submanifolds of a contact(paracontact, contact and product structures) metric manifold is depend on the behaviour of contact metric structure ϕ. Namely, a submanifold M of a lorentzian trans-Sasakian manifold is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for any vector feld X tangent to M at every point of M. In other words, $\phi(TM) \subset (TM)$ at each point of M.

In the submanifolds theory, we note that the geometry of invariant submanifolds inherits almost all properties of the ambient manifolds. Therefore, invariant submanifolds have an active and fruitful research field played a significant role in the development of modern differential geometry. In this connection, the papers related to invariant submanifolds has been studied and studies continue on different structures..

In the rest of this paper, we will assume that M is an invariant submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}$ unless otherwise stated.

So we need the following Theorem for later used.

Theorem 2.1. *Let M be an immersed submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}(\phi, \xi, \eta, g)$. By R and σ, we denote the Riemannian curvature tensor and second fundamental form of submanifold M, respectively. Then the following relations hold;*

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi, \tag{19}$$

$$\sigma(X,\phi Y) = \sigma(\phi X, Y) = \phi\sigma(X,Y), \tag{20}$$

$$\sigma(X,\xi) = A_V\xi = 0, \tag{21}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. Since the proofs are a result of simple calculations, we think that to be unnecessary to give. □

Theorem 2.2. *Let M be an invariant submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}$. If $Q(S,\sigma) = 0$, then M is either totally geodesic submanifold or $\xi(\beta) = \alpha^2 + \beta^2$.*

Proof. $Q(S,\sigma) = 0$ means that

$$\begin{aligned} Q(S,\sigma)(U,V;X,Y) &= \sigma((X \wedge_S Y)U, V) + \sigma(V, (X \wedge_S Y)U) \\ &= \sigma(S(Y,U)X - S(X,U)Y, V) \\ &+ \sigma(U, S(Y,V)X - S(X,V)Y) = 0, \end{aligned} \tag{22}$$

for all $X, Y, U, V \in \Gamma(TM)$. Taking $X = V = \xi$ in (22) and by using (9), we have

$$-2n(\alpha^2 + \beta^2 - \xi(\beta))\sigma(U,Y) = 0,$$

which proves our assertion. □

Theorem 2.3. *Let M be a invariant submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}$. If $Q(g,\sigma) = 0$ if and only if M is totally geodesic submanifold.*

Proof. $Q(g,\sigma) = 0$ implies that

$$Q(g,\sigma)(U,V;X,Y) = \sigma((X \wedge_g Y)U, V) + \sigma(U, (X \wedge_g Y)V) = 0,$$

for all $X, Y, U, V \in \Gamma(TM)$. Substituting $Y = U = \xi$ in the last equality, we can conclude $\sigma(X,V) = 0$. The converse is obvious. □

Theorem 2.4. *Let M be an invariant submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}$. If $Q(S,\widetilde{\nabla}\cdot\sigma) = 0$, then at least one of the following holds;*
1.) M is a totally geodesic,
2.) $\xi(\beta) = \alpha^2 + \beta^2$,
3.) $\alpha^2 - \beta^2 = 0$.

Proof. If M is an invariant submanifold and $Q(S,\nabla\cdot\sigma) = 0$, then we have

$$\begin{aligned} Q(S,\widetilde{\nabla}\cdot\sigma)(U,V,Z;X,Y) &= (\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z) + (\widetilde{\nabla}_U\sigma)((X \wedge_S Y)V, Z) \\ &+ (\widetilde{\nabla}_U\sigma)(V, (X \wedge_S Y)Z) = 0, \end{aligned}$$

for all $X, Y, U, V, Z \in \Gamma(TM)$. For $Y = Z = \xi$, this yields to

$$\begin{aligned} (\widetilde{\nabla}_{(X\wedge_S\xi)U}\sigma)(V,\xi) &+ (\widetilde{\nabla}_U\sigma)((X \wedge_S \xi)V, \xi) \\ &+ (\widetilde{\nabla}_U\sigma)(V, (X \wedge_S \xi)\xi) = 0. \end{aligned} \tag{23}$$

Here, non-zero components of the first term have

$$\begin{aligned} (\widetilde{\nabla}_{(X\wedge_S\xi)U}\sigma)(V,\xi) &= -\sigma(\nabla_{(X\wedge_\xi)U}\xi, V) = -\sigma(\nabla_{S(\xi,U)X-S(X,U)\xi}\xi, V) \\ &= -S(\xi,U)\sigma(\nabla_X\xi, V) = S(\xi,U)\sigma(\alpha\phi X + \beta^2\phi X, V) \\ &= S(\xi,U)[\alpha\phi\sigma(X,V) + \beta\sigma(X,V)]. \end{aligned} \tag{24}$$

For the non-zero components of the second term, we have

$$\begin{aligned} (\widetilde{\nabla}_{(X\wedge_S\xi)U}\sigma)(V,\xi) &= -\sigma(\nabla_U, S(\xi,V)X - S(X,V)\xi) = -S(\xi,V)\sigma(\nabla_U\xi, X) \\ &= S(\xi,V)\sigma(\alpha\phi U + \beta\phi^2 U, X) \\ &= S(\xi,V)[\alpha\phi\sigma(U,X) + \beta\sigma(U,X)]. \end{aligned} \tag{25}$$

Finally,

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(V,(X\wedge_S\xi)\xi) &= (\widetilde{\nabla}_U\sigma)(V,S(\xi,\xi)X-S(X,\xi)\xi)=(\widetilde{\nabla}_U\sigma)(S(\xi,\xi)X,V)\\ &- (\widetilde{\nabla}_U\sigma)(S(X,\xi)\xi,V)\\ &= (\widetilde{\nabla}_U\sigma)(S(\xi,\xi)X,V)-S(\xi,X)\sigma(\nabla_U\xi,V)\\ &= (\widetilde{\nabla}_U\sigma)(S(\xi,\xi)X,V)+S(X,\xi)[\alpha\phi\sigma(U,V)+\beta\sigma(U,V)]. \end{aligned}\tag{26}$$

For $V=\xi$, (24), (25) and (26) are put in (23), we have

$$\begin{aligned}S(\xi,\xi)[\alpha\phi\sigma(X,U) &+ \beta\sigma(X,U)]+(\widetilde{\nabla}_U\sigma)(S(\xi,\xi)X,\xi)\\ &= S(\xi,\xi)[\alpha\phi\sigma(X,U)+\beta\sigma(X,U)]-\sigma(\nabla_U\xi,S(\xi,\xi)X)\\ &= 0.\end{aligned}$$

Also taking into account (9) and (19), we obtain

$$-2n(\alpha^2+\beta^2-\xi(\beta))[\alpha\phi\sigma(X,U)+\beta\sigma(X,U)]=0. \tag{27}$$

Now applying ϕ to (27) and using (19), we have

$$-2n(\alpha^2+\beta^2-\xi(\beta))[\alpha\sigma(X,U)+\beta\phi\sigma(X,U)]=0. \tag{28}$$

From (27) and (28), we can derive

$$-2n[\alpha^2+\beta^2-\xi(\beta)](\alpha^2-\beta^2)\sigma(U,V)=0.$$

This proves our assertions. □

Theorem 2.5. *Let M be an invariant submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}$. If $Q(g,\widetilde{\nabla}\cdot\sigma)=0$, then M is either totally geodesic submanifold or $\alpha^2-\beta^2=0$.*

Proof.

$$Q(g,\widetilde{\nabla}\cdot\sigma)(U,V,Z;X,Y)=(\widetilde{\nabla}_U\sigma)((X\wedge_g Y)V,Z)+(\widetilde{\nabla}_U\sigma)(V,(X\wedge_g Y)Z)=0,$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. Taking $Y=V=\xi$ in last equality and by means of (13) and (18), we obtain

$$-(\widetilde{\nabla}_U\sigma)(X,Z)+\eta(X)\sigma(\nabla_U\eta(X)\xi,Z)-\sigma(\nabla_U\xi,\eta(Z)X)=0,$$

or

$$-(\widetilde{\nabla}_U\sigma)(X,Z)-\eta(X)\sigma(\alpha\phi U+\beta\phi^2U,Z)+\eta(Z)\sigma(\alpha\phi U+\beta\phi^2U,X)=0,$$

which implies for $Z=\xi$

$$\begin{aligned}\sigma(\nabla_U\xi,X)-\sigma(\alpha\phi U+\beta\phi^2U,X) &= -2(\alpha\phi\sigma(U,X)+\beta\sigma(U,X))\\ &= 0.\end{aligned}\tag{29}$$

Applying ϕ to (29) and we consider (19), we have

$$\alpha\sigma(U,X)+\beta\phi\sigma(U,X)=0. \tag{30}$$

From (29) and (30) we conclude

$$(\alpha^2-\beta^2)\sigma(U,X)=0.$$

This completes the proof. □

Theorem 2.6. *let M be an invariant submanifold of lorentzian trans-Sasakian manifold* $\widetilde{M}(\phi, \xi, \eta, g)$. *If* $Q(g, \widetilde{R} \cdot \sigma) = 0$, *then at least one of the following holds:*
1. M is a totally geodesic submanifold,
2. $\eta(\nabla(\beta - \alpha)) = (\beta - \alpha)^2$
3. $\eta(\nabla(\beta + \alpha)) = (\beta + \alpha)^2$

Proof. $Q(g, \widetilde{R} \cdot \sigma) = 0$ leads to

$$(\widetilde{R}(X, Y) \cdot \sigma)((U \wedge_g V)Z, W) + (\widetilde{R}(X, Y) \cdot \sigma)(Z, (U \wedge_g V)Z) = 0,$$

for all $X, Y, Z, U, V, W \in \Gamma(T\widetilde{M})$. For $Z = U = W = \xi$, this equality implies

$$\begin{aligned} & (\widetilde{R}(X, Y) \cdot \sigma)(\eta(V)\xi + V, \xi) = R^{\perp}(X, Y)\sigma(\eta(V)\xi, \xi) - \sigma(R(X, Y)\eta(V)\xi, \xi) \\ - \; & \sigma(\eta(V)\xi, R(X, Y)\xi) + R^{\perp}(X, Y)\sigma(V, \xi) - \sigma(R(X, Y)V, \xi) - \sigma(V, R(X, Y)\xi) \\ = \; & 0. \end{aligned}$$

Taking into account of (6), (19) and (19), we can infer

$$\begin{aligned} & \sigma(V, (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y]) \\ + \; & \sigma(V, Y(\alpha)\phi X - X(\alpha)\phi Y + Y(\beta)\phi^2 X - X(\beta)\phi^2 Y) = 0. \end{aligned} \tag{31}$$

Also, if ξ is taken instead of X in (31) and by means of (1) and (21), we reach at

$$[\alpha^2 + \beta^2 - \xi(\beta)]\sigma(V, Y) + 2\alpha\beta\phi\sigma(V, Y) = 0. \tag{32}$$

Applying ϕ to (32) and taking into accont that (19), we have

$$[\alpha^2 + \beta^2 - \xi(\beta)]\phi\sigma(V, Y) + 2\alpha\beta\sigma(V, Y) = 0. \tag{33}$$

by combining (32) and (33), we observe

$$\{[\alpha^2 + \beta^2 - \xi(\beta)]^2 - [2\alpha\beta - \xi(\alpha)]^2\}\sigma(Y, V) = 0.$$

This completes the proof. □

Theorem 2.7. *Let M be an invariant submanifold of a lorentzian trans-Sasakian manifold* $\widetilde{M}$. *If* $Q(S, \widetilde{R} \cdot \sigma) = 0$, *then at least one of the following holds;*
1.) M is a totally geodesic,
2.) $\xi(\beta) = \alpha^2 + \beta^2$
3.) $\xi(\beta - \alpha) = (\alpha - \beta)^2$.
4.) $\xi(\beta + \alpha) = (\alpha + \beta)^2$.

Proof. $Q(S, \widetilde{R} \cdot \sigma) = 0$ has the form

$$\begin{aligned} Q(S, \widetilde{R} \cdot \sigma)(U, V, Z, W; X, Y) \; = \; & (\widetilde{R}(X, Y) \cdot \sigma)((U \wedge_S V)Z, W) \\ + \; & (\widetilde{R}(X, Y) \cdot \sigma)(Z, (U \wedge_S V)W) = 0. \end{aligned}$$

For $X = U = W = \xi$, this yields to

$$\begin{aligned} (\widetilde{R}(\xi, Y) \cdot \sigma)(S(V, Z)\xi - S(\xi, Z)V, \xi) \; + \; & (\widetilde{R}(\xi, Y) \cdot \sigma)(Z, S(V, \xi)\xi - S(\xi, \xi)V) \\ = \; & 0. \end{aligned}$$

Non-zero components of this expansion give us

$$\begin{aligned} S(\xi, Z)\sigma(V, R(\xi, Y)\xi) \; - \; & S(V, \xi)\sigma(Z, R(\xi, Y)\xi) - R^{\perp}(\xi, Y)S(\xi, \xi)\sigma(V, Z) \\ & + S(\xi, \xi)\sigma(R(\xi, Y)V, Z) + S(\xi, \xi)\sigma(R(\xi, Y)Z, V) \\ = \; & 0. \end{aligned} \tag{34}$$

Taking $Z = \xi$ in (34) and by using (8) and (21), we verify

$$\begin{aligned} S(\xi,\xi)\sigma(R(\xi,Y)\xi,V) &= \Big\{-[2n(\alpha^2+\beta^2)-\xi(\beta)]+(2n-1)\xi(\beta)\Big\} \\ &\otimes\sigma(V,(\alpha^2+\beta^2-\xi(\beta))\phi^2 Y+(2\alpha\beta-\xi(\alpha))\phi Y)=0. \end{aligned}$$

This is equivalent to

$$\begin{aligned} 2n(\alpha^2+\beta^2-\xi(\beta))\Big[(\alpha^2+\beta^2-\xi(\beta))\sigma(V,Y)+(2\alpha\beta-\xi(\alpha))\phi\sigma(V,Y)\Big] \\ = 0. \end{aligned} \quad (35)$$

If Y is taken instead of ϕY in (35) and using (19), we obtain

$$2n(\alpha^2+\beta^2-\xi(\beta))\Big[(\alpha^2+\beta^2-\xi(\beta))\phi\sigma(V,Y)+(2\alpha\beta-\xi(\alpha))\sigma(V,Y)\Big]=0.$$

From the last two-equalities, we conclude that

$$2n(\alpha^2+\beta^2-\xi(\beta))[(\alpha^2+\beta^2-\xi(\beta))^2-(2\alpha\beta-\xi(\alpha))^2]\sigma(Y,V)=0,$$

which proves our assertion. □

Theorem 2.8. *Let M be an invariant submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}$. If $Q(g,C\cdot\sigma)=0$, then at least one of the following holds;*
1. M is totally geodesic submanifold,
2. The scalar curvture τ of $\widetilde{M}$ satisfies $\tau = 2n(2n+1)[(\alpha\pm\beta)^2-\xi(\alpha\pm\beta)]$

Proof. $Q(g,C\cdot\sigma)=0$ is of

$$(C(X,Y)\cdot\sigma)((U\wedge_g V)Z,W)+(C(X,Y)\cdot\sigma)(Z,(U\wedge_g V)Z)=0,$$

for all $X,Y,Z,U,V,W\in\Gamma(T\widetilde{M})$.
Here, taking $Z=U=W=\xi$, we have

$$\begin{aligned} &(C(\xi,Y)\cdot\sigma)(g(V,Z)\xi-\eta(Z)V,\xi)+(C(\xi,Y)\cdot\sigma)(Z,\eta(V)\xi+V) \\ =\ &(C(\xi,Y)\cdot\sigma)(g(V,Z)\xi,\xi)-(C(\xi,Y)\cdot\sigma)(\eta(Z)V,\xi) \\ +\ &(C(\xi,Y)\cdot\sigma)(Z,\eta(V)\xi)+(C(\xi,Y)\cdot\sigma)(g(V,Z)\xi,\xi)(Z,V)=0. \end{aligned}$$

As non-zero components in these expansions, one can easily to see

$$\begin{aligned} \eta(Z)\sigma(V,C(\xi,Y)\xi) &+ R^{\perp}(\xi,Y)\sigma(Z,V)-\sigma(C(\xi,Y)Z,V) \\ &- \sigma(Z,C(\xi,Y)V)=0. \end{aligned} \quad (36)$$

In (36), setting $Z=\xi$ and consider (6), (16), (21) we reach at

$$\begin{aligned} \sigma(V,C(\xi,Y)\xi) &= \left(\alpha^2+\beta^2-\xi(\beta)-\frac{\tau}{2n(2n+1)}\right)\sigma(V,Y) \\ &+ [2\alpha\beta-\xi(\alpha)]\phi\sigma(V,Y)=0. \end{aligned} \quad (37)$$

Replacing ϕY by Y in the last inequality and making the necessary revisions, we get

$$\begin{aligned} \left(\alpha^2+\beta^2-\xi(\beta)-\frac{\tau}{2n(2n+1)}\right)\phi\sigma(V,Y) &+ [2\alpha\beta-\xi(\alpha)]\sigma(V,Y) \\ &= 0. \end{aligned} \quad (38)$$

Thus (37) and (38) give us

$$\left(\left[\alpha^2+\beta^2-\xi(\beta)-\frac{\tau}{2n(2n+1)}\right]^2-[2\alpha\beta-\xi(\alpha)]^2\right)\sigma(V,Y)=0,$$

which proves our assertions. □

Theorem 2.9. *Let M be an invariant submanifold of a lorentzian trans-Sasakian manifold $\widetilde{M}$. If $Q(S, C\cdot\sigma) = 0$, then at least one of the following holds;*
1. M is totally geodesic submanifold,
2.The scalar curvture τ of $\widetilde{M}$ satisfies $\tau = \pm 2n(2n+1)(2\alpha\beta - \xi(\alpha))$ or $\tau = 2n(2n+1)[(\alpha\pm\beta)^2 - \xi(\alpha\pm\beta)]$.

Proof. $Q(S, C\cdot\sigma) = 0$ is of the form

$$(C(X,Y)\cdot\sigma)((U\wedge_S V)Z, W) + (C(X,Y)\cdot\sigma)(Z, (U\wedge_S V)Z) = 0,$$

for all $X, Y, U, V, Z \in \Gamma(T\widetilde{M})$.
Also these decompositions give us for $Z = U = W = \xi$

$$\begin{aligned} & (C(\xi,Y)\cdot\sigma)(S(V,Z)\xi - S(Z,\xi)V, \xi) + (C(\xi,Y)\cdot\sigma)(Z, S(V,\xi) - S(\xi,\xi)V) \\ = \ & (C(\xi,Y)\cdot\sigma)(S(V,Z)\xi, \xi) - (C(\xi,Y)\cdot\sigma)(S(Z,\xi)V, \xi) \\ + \ & (C(\xi,Y)\cdot\sigma)(Z, S(V,\xi)\xi) - (C(\xi,Y)\cdot\sigma)(S(\xi,\xi)V, Z) = 0. \end{aligned}$$

If these statements are written clearly and the necessary revisions are made, we have non-zero components

$$\begin{aligned} & S(\xi,Z)\sigma(V, C(\xi,Y)\xi) - S(\xi,V)\sigma(Z, C(\xi,Y)\xi) - R^{\perp}(\xi,Y)S(\xi,\xi)\sigma(V,Z) \\ & + S(\xi,\xi)\sigma(C(\xi,Y)Z, V) + S(\xi,\xi)\sigma(Z, C(\xi,Y)V) = 0. \end{aligned} \tag{39}$$

Next, replacing ξ by Z in (39), by using (6) and (16), we can see

$$\begin{aligned} & S(\xi,\xi)\sigma(C(\xi,Y)\xi, V) = S(\xi,\xi)(\alpha^2 + \beta^2 - \xi(\beta) - \frac{\tau}{2n(2n+1)})\sigma(V,Y) + \\ + \ & S(\xi,\xi)(2\alpha\beta - \xi(\alpha))\phi\sigma(V,Y) = 0, \end{aligned}$$

that is,

$$\begin{aligned} & 2n(\alpha^2 + \beta^2 - \xi(\beta))\left\{[\alpha^2 + \beta^2 - \xi(\beta - \frac{\tau}{2n(2n+1)})]^2 - [2\alpha\beta - \xi(\alpha)]^2\right\}\sigma(V,Y) \\ & = 0. \end{aligned}$$

This completes the proof. □

Nex, we will build an example to illustrate our topic.

Example 2.10. *Let $\widetilde{M}^5 = \{(x_1, x_2, x_3, x_4, t) \in \mathbf{E}^5 : t \neq 0\}$be a 5-dimensional differentiable manifold with the standart coordinate system (x_1, x_2, x_3, x_4, t). Then vector fields*

$$\begin{aligned} E_1 &= t(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial t}), \ E_2 = t\frac{\partial}{\partial x_2}, \ E_3 = t(\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial t}), \\ E_4 &= t\frac{\partial}{\partial x_4}, \ E_5 = \xi = \frac{\partial}{\partial t} \end{aligned}$$

are the linear independent at each points of $\widetilde{M}$, that is, these vector fields are basis of tangent space of $\widetilde{M}$. Now, we define,respectively, the contact structure and metric tensor ϕ and g by

$$\phi E_1 = -E_2, \ \phi E_2 = E_1, \ \phi E_3 = -E_4, \ \phi E_4 = E_3, \ \phi E_5 = 0,$$

and

$$\begin{aligned} g(E_i, E_j) &= \delta_{ij}, \ 1 \leq i, j \leq 4 \\ g(E_5, E_5) &= -1 \end{aligned}$$

then we can easily verify that

$$\phi^2 X = X + \eta(X)\xi, \ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \ \eta(X) = g(X,\xi).$$

So, $\widetilde{M}^5(\phi,\xi,\eta,g)$ is a 5-dimensional contact metric manifold. By direct calculations, we can get the non-zero components of Lie-bracket as

$$[E_i, E_5] = -\frac{1}{t}E_i, \ \ 1 \le i \le 4,$$

$$[E_1, E_2] = x_2E_2 - t^2E_5, \ \ [E_1, E_3] = -x_4E_1 + x_2E_3, \ \ [E_1, E_4] = x_2E_4.$$

In view of Kozsul formulae, we can find the following non-zero components of connections as

$$\widetilde{\nabla}_{E_1}E_5 = -\frac{1}{t}E_1 + \frac{1}{2}t^2E_2, \ \widetilde{\nabla}_{E_2}E_5 = -\frac{1}{2}t^2E_1 - \frac{1}{t}E_2$$
$$\widetilde{\nabla}_{E_3}E_5 = -\frac{1}{t}E_3 + \frac{1}{2}t^2E_4, \ \ \widetilde{\nabla}_{E_4}E_5 = -\frac{1}{2}t^2E_3 - \frac{1}{t}E_4.$$

By the straightforward calculations, by using (5),we can observe $\alpha = \frac{1}{2}t^2$ and $\beta = -\frac{1}{t}$. This tell us that $\widetilde{M}^5(\phi,\xi,\eta,g)$ is a 5-dimensional trans-Sasakain manifold with $\alpha = \frac{1}{2}t^2$ and $\beta = -\frac{1}{t}$.
Now, we consider vector fields

$$e_1 = t(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + (x_2 + x_4)\frac{\partial}{\partial t}), \ \ e_2 = t(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}), \ \ e_3 = \frac{\partial}{\partial t}.$$

These vector fields are the linear dependent. By **D**, *let's denote the distribution spanned by these vectors. One can observe* **D** *is integrable and involutive. By M, we denote the its integral manifold, then M manifold is a submanifold of $\widetilde{M}^5(\phi,\xi,\eta,g)$. On can easily to see that*

$$\phi e_1 = -e_2, \ \ \phi e_2 = e_1, \ \ \text{and} \ \ \phi e_3 = 0.$$

This tell us that M is a 3-dimensional invariant submanifold of a trans-Sasakaim manifold $\widetilde{M}^5(\phi,\xi,\eta,g)$. On the other hand, by direct calculations, we verify that $\sigma(e_1,e_2) = 0$, that is, M is totally geodesic submanifold and $\xi(\alpha) = t$, $\xi(\beta) = \frac{1}{t^2}$.

References

[1] P. Uygun, M. Atçeken and T. Mert, *Certain results for invariant submanifolds of an almost α-cosymplectic (κ,ν,μ)-space,* Mathematical Sciences and Applications E-Notes. **12(2)** (2024), 93-100.

[2] M. Atçeken, P. Uygun, *Characterizations for totally geodesic submanifolds of (κ,μ)-paracontact metric manifolds,* Korean J. Math. **28(3)** (2020), 555-571.

[3] I. Mihai, A. Oiaga, R. Rosca, *Lorentzian Kenmotsu manifolds having two skew-symmetric conformal vector fields,* Bull. Math. Soc. Sci. Math. Roumania. **42** (1999), 237-251.

[4] A. Mondal, *Some curves of tree dimensional lorentzian trans-Sasakain manifolds,* Palestine Journal of Mathematics. **9(2)** (2020), 841-847.

[5] S. K. Chaubey, *Trans-Sasakian manifold satisfying certain conditions,* TWMS J. App. Eng. Math. **9(2)** (2019), 305-314.

[6] T. Mert, M. Atçeken, *Characterizations for pseudoparalel submanifolds of Lorentz-Sasakian space forms,* Malaya J. Mat.**12(01)** (2024), 31-42.

[7] U.C. De, K. De, *On lorentzian trans-Sasakian manifolds,*. Commun. Fac. Sci.Univ.Ank.Series A1. **62(2)** (2013), 37-51.

[8] T. Mert, S. Pandey, M. Atçeken, U.C. De, *Invariant and Anti-Invariant Submanifolds of Generalized Sasakian Space Forms,* Lobachevskii Journal of Mathematics. **45** (2024), 2895-2909.

[9] S. Deshmukh, F. Al-Solamy, *A Note on Compact Trans-Sasakian Manifolds,* Mediterr. J. Math. **13(4)** (2016), 2099-2104.

[10] S. Deshmukh, U.C. De, F. Al-Solamy, *Tran-Sasakian Manifolds homothetic to Sasakian Manifolds*, Publ. Math. Debrecen. **88(3-4)** (2016), 439-448.

[11] V. F. Kirichenko, *On the geometry of nearly trans-Sasakian manifolds,* (Russian) Dokl.Akad. Nauk. **397(6)** (2004), 733-736.