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An upper bound estimate for a class of split variational-hemivariational-like inequalities on Hadamard manifolds

Tran Hong Mo[a], **Vo Minh Tam**[b,*]

[a]*Ho Chi Minh City Open University, Vietnam*
[b]*Department of Mathematics, Dong Thap University, Cao Lanh City 870000, Dong Thap Province, Vietnam*

Abstract. In this paper, we introduce a class of split variational-hemivariational-like inequalities (for short, SVHLI) in the setting of Hadamard manifolds. Then a new regularized gap function (for short, RG-function) for the problem SVHLI is established under suitable assumptions. Furthermore, using the properties of strongly monotone, strong nonexpanding and skew-symmetric mappings and the generalized subdifferentials in the sense of Clarke, we provide an upper bound for the problem SVHLI in terms of RG-functions. Finally, we give some examples to illustrate our main results.

1. Introduction

The notion of gap functions was introduced by Auslender [2] which is known as a valuable tool for solving variational inequalities in the form of associated optimization problems. In general, the Auslender gap function is nondifferentiable. To get over this disadvantage, Fukushima [11] originally proposed a new gap function for variational inequalities which is called the regularized gap function (for short, RG-function). In [11, 34], the authors provided upper bound estimates (error bound) for variational inequalities by using RG-functions. This interesting results have extended to the different RG-functions and upper bounds for various kinds of problems for instance optimization problems, equilibrium problems, variational inequalities and hemivariational inequalities, see e.g. [6, 8, 15, 29] and references therein. Besides, variational inclusions and related problems have been studied in many different directions, see [9, 10, 18, 20, 23, 27].

On the other hand, many significant concepts and methods of nonlinear analysis and optimization have been extended from Euclidean spaces to Riemannian manifolds, see [26, 32]. This development has obtained some advantages that some nonconvex and nonsmooth problems in the setting of linear spaces can be seen as convex and smooth ones in the aspect of Riemannian geometry. Many various problems have investigated in Riemannian manifolds or Hadamard manifolds, see e.g., [5, 22, 24, 33] and the references therein. Recently, Hung et al. [16, 17] established some results on RG-functions and global error bounds

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* Corresponding author: Vo Minh Tam
Email addresses: mo.th@ou.edu.vn (Tran Hong Mo), vmtam@dthu.edu.vn (Vo Minh Tam)
ORCID iDs: https://orcid.org/0009-0009-6921-3518 (Tran Hong Mo), https://orcid.org/0000-0002-3959-5449 (Vo Minh Tam)

for mixed quasi-hemivariational inequalities and mixed vector equilibrium-like problems on Hadamard manifolds.

The theory of split variational inequalities was introduced by Censor et al. [7]. Split variational inequalities have many applications in practical problems arising from signal recovery (inverse problems), image processing and intensity-modulated radiation therapy planning. Many authors have studied various kinds of split variational inequalities, see e.g., [3, 12, 13, 25] and the references therein. Recently, Hung et al. [14] developed the results of RG-functions and upper bound estimates for a class of split mixed vector quasivariational inequalities in real Hilbert spaces. Tam et al. [30] introduced a class of split hemivariational inequalities on Hadamard manifolds and studied the Levitin-Polyak well-posedness to such problems. However, up to now, there has not been any work devoted to the investigate of RG-functions and upper bound estimates for split hemivariational (variational-hemivariational) inequalities on Hadamard manifolds.

Motivated and inspired by the above, the purpose of this paper is to investigate a class of split variational-hemivariational-like inequalities (for short, SVHLI) in the setting of Hadamard manifolds. Then we introduce an RG-function of the problem SVHLI under suitable assumptions on Hadamard manifolds. Furthermore, we establish an upper bound estimate for the problem SVHLI by using the properties of strongly monotone, strong nonexpanding and skew-symmetric mappings and the Clarke's generalized subdifferentials. Finally, we give some examples to illustrate our main results.

2. Preliminaries

In this section, we introduce some definitions and known results on Riemannian manifolds which will be used throughout the paper, see e.g., [28, 32].

Given a connected m-dimensional Riemannian manifold $\mathcal{M}$, we denote by $T_w\mathcal{M}$ the tangent space of $\mathcal{M}$ at w and $\langle\cdot,\cdot\rangle_w$ the scalar product on $T_w\mathcal{M}$ with the associated norm $\|\cdot\|_w$, where the subscript w is sometimes omitted. The tangent space $T_w\mathcal{M}$ is a usual finite-dimensional space for each $w \in \mathcal{M}$. Then $T\mathcal{M} = \bigcup_{w\in\mathcal{M}} T_w\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$ and it is naturally a manifold. Let $\gamma : [a,b] \to \mathcal{M}$ be a piecewise smooth curve joining w to z, that is, $\gamma(a) = w$ and $\gamma(b) = z$, we can define the length of γ by $L(\gamma) := \int_a^b \|\gamma'\|_\gamma dt$. Then for any $w, z \in \mathcal{M}$, the Riemannian distance $d_{\mathcal{R}}(w,z)$, which induces the original topology on $\mathcal{M}$, is defined by minimizing this length over the set of all such curves joining w to z.

A Riemannian manifold is *complete* if for any $w \in \mathcal{M}$ all geodesics emanating from w are defined all over $\mathbb{R}$. We know that if $\mathcal{M}$ is complete, then any two points in $\mathcal{M}$ can be joined by a minimal geodesic. Moreover, $(\mathcal{M}, d)$ is a complete metric space and bounded closed subsets are compact (Hopf-Rinow Theorem).

A *Hadamard manifold* $\mathcal{M}$ is a complete simply connected Riemannian manifold of nonpositive sectional curvature. Then, the exponential map $\exp_w : T_w\mathcal{M} \to \mathcal{M}$ at w is defined by $\exp_w z = \gamma_z(1, w)$ for each $z \in T_w\mathcal{M}$, where $\gamma(\cdot) = \gamma_z(\cdot, w)$ is the geodesic starting at w with the velocity z, that is $\gamma(0) = w$ and $\gamma'(0) = z$. It is easy to see that $\exp_w(tz) = \gamma_z(t)$ for each real number t. Take $w \in \mathcal{M}$. Let $\exp_w^{-1} : \mathcal{M} \to T_w\mathcal{M}$ be the inverse of the exponential map. Note that the map $\exp_w$ is diffeomorphism on $T_w\mathcal{M}$ for any $w \in \mathcal{M}$ and $d_{\mathcal{R}}(w,z) = \|\exp_z^{-1} w\|$. For any two points $w, z \in \mathcal{M}$, there exists a unique normalized geodesic γ joining w to z such that $\gamma(t) = \exp_w\left(t \exp_w^{-1} z\right)$ for all $t \in [0,1]$.

Lemma 2.1. (see [1], p.3) *Let $\mathcal{M}$ be a Hadamard manifold and $u, z \in \mathcal{M}$. For $t \in (0,1)$ and a point $x_t = \gamma(t) = \exp_u(t \exp_u^{-1} z)$ on the geodesic $\gamma : [0,1] \to \mathcal{M}$ joining u to z, we have $\exp_u^{-1} x_t = t \exp_u^{-1} z$.*

Definition 2.2. (see [32]) A set $Q \subset \mathcal{M}$ is said to be *geodesic convex* if for any two distinct points u and z in Q, the geodesic joining u to z is contained in Q, that is, if $\gamma : [0,1] \to \mathcal{M}$ is a geodesic such that $u = \gamma(0)$ and $z = \gamma(1)$, then $\gamma(s) = \exp_u\left(s \exp_u^{-1} z\right) \in Q$, for all $s \in [0,1]$.

Definition 2.3. (see [32]) Let $\mathcal{M}$ be a Hadamard manifold. A real-valued function $\kappa\colon \mathcal{M} \to \mathbb{R}$ is said to be *geodesic convex* if, for any $u, z \in \mathcal{M}$ and $t \in [0, 1]$,

$$\kappa\left(\exp_z\left(t\exp_u^{-1} z\right)\right) \leq (1-t)\kappa(u) + t\kappa(z).$$

Definition 2.4. (see [19]) Let $\mathcal{M}$ be a Riemannian manifold. A real-valued function $\kappa : \mathcal{M} \to \mathbb{R}$ said to be:

(a) *Lipschitz of rank L* on a given subset Q of $\mathcal{M}$ if

$$|\kappa(u) - \kappa(z)| \leq Ld_{\mathcal{R}}(u, z), \quad \forall u, z \in Q.$$

(b) *Lipschitz near* $z \in \mathcal{M}$ if it is Lipschitz of some rank on an open neighborhood of z.

(c) *locally Lipschitz* on $\mathcal{M}$ if it is Lipschitz near z, for every $z \in \mathcal{M}$.

Definition 2.5. (see [19]) Let $\kappa : \mathcal{M} \to \mathbb{R}$ be a locally Lipschitz function on a Riemannian manifold $\mathcal{M}$. The *Clarke's generalized directional derivative* of κ at $y \in \mathcal{M}$ in direction $z \in T_y\mathcal{M}$, denoted by $\kappa^0(y; z)$, is defined as

$$\kappa^0(y; z) := \limsup_{v \to y, t \downarrow 0} \frac{\kappa \circ \varphi^{-1}\left(\varphi(v) + t\mathrm{d}\varphi(y)(z)\right) - \kappa \circ \varphi^{-1}(\varphi(v))}{t}, \tag{1}$$

where (φ, U) is a chart at y.

Indeed, $\kappa^0(y; z) = \left(\kappa \circ \varphi^{-1}\right)^0 (\varphi(y); \mathrm{d}\kappa(y)(z))$, where the direction $\mathrm{d}\kappa(y)(z)$ is the image of the tangent vector z when modelling $\mathcal{M}$ in $\mathbb{R}^m$. Note that this definition is not dependent on charts. Taking into account $0_y \in T_y\mathcal{M}$, one has

$$\kappa^0(y; z) = \left(\kappa \circ \exp_y\right)^0 (0_y; z).$$

Next, some basic properties of the Clarke's generalized directional derivative are provided by the following lemma.

Lemma 2.6. (see [19], Theorem 2.4) *Let $\mathcal{M}$ be a Riemannian manifold, $y \in \mathcal{M}$ and $\kappa\colon \mathcal{M} \to \mathbb{R}$ be Lipschitz of rank L on an open neighbourhood O_y of y. Then the following assertions hold:*

(i) *For each $u \in O_y$, the function $z \mapsto \kappa^0(u; z)$ is finite, positively homogeneous and subadditive on $T_u\mathcal{M}$, and satisfies*

$$\|\kappa^0(u; z)\| \leq L\|z\|, \text{ for all } z \in T_u\mathcal{M};$$

(ii) *$\kappa^0(\cdot; \cdot)$ is upper semicontinuous on $O_y \times T_u\mathcal{M}$ as a function of (u, z) and, as a function of z alone, is Lipschitz of rank L on $T_u\mathcal{M}$, for each $z \in O_y$.*

Definition 2.7. (see [16], Definition 4.1) Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_{\mathcal{R}})$ be a Hadamard manifold, $\emptyset \neq Q \subset \mathcal{M}$ and $F : Q \to T\mathcal{M}$. Then

(a) $h\colon T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$ is said to be *(F, σ)-strongly monotone* on Q if there exists $\sigma > 0$ such that for any $x, y \in Q$ with $x \neq y$,

$$h\left(F(x), \exp_x^{-1} y\right) + h\left(F(y), \exp_y^{-1} x\right) \leq -\sigma d_{\mathcal{R}}^2(x, y).$$

(b) $\varphi\colon Q \times Q \to \mathbb{R}$ is said to be *skew-symmetric* if for each $(x, y) \in Q \times Q$,

$$\varphi(x, x) - \varphi(x, y) - \varphi(y, x) + \varphi(y, y) \geq 0.$$

3. Main results

3.1. Problem formulation and RG-functions

Throughout the paper, unless otherwise specified, $(\mathcal{M}_1, \langle\cdot,\cdot\rangle_{\mathcal{R}_1})$ and $(\mathcal{M}_2, \langle\cdot,\cdot\rangle_{\mathcal{R}_2})$ are Hadamard manifolds and P and Q are nonempty, compact and geodesic convex subsets of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Let $h\colon T\mathcal{M}_1 \times T\mathcal{M}_1 \to \mathbb{R}$ and $k\colon T\mathcal{M}_2 \times T\mathcal{M}_2 \to \mathbb{R}$ be two bifunctions such that $h(x,\mathbf{0}) = k(u,\mathbf{0}) = 0$, for all $x \in T\mathcal{M}_1$ and $u \in T\mathcal{M}_2$, $\Psi\colon \mathcal{M}_1 \to \mathbb{R}$ and $\Phi\colon \mathcal{M}_2 \to \mathbb{R}$ be two locally Lipschitz functions, $\psi\colon P \times P \to \mathbb{R}$ and $\varphi\colon Q \times Q \to \mathbb{R}$ be two bifunctions such that $\psi(x,x) = 0$ and $\varphi(u,u) = 0$ for all $x \in P$, $u \in Q$, $A\colon P \to Q$ be a bounded linear operator, and $F\colon P \to T\mathcal{M}_1$ and $G\colon Q \to T\mathcal{M}_2$ be two vector fields.

We now consider the following split variational-hemivariational-like inequalities (for short, SVHLI) in the setting of Hadamard manifolds, which consists of finding $(x^*, u^*) \in \mathcal{M}_1 \times \mathcal{M}_2$ such that

$$\begin{cases} x^* \in P,\ u^* \in Q,\ u^* = A(x^*), \\ h\left(F(x^*), \exp_{x^*}^{-1} y\right) + \psi(x^*, y) + \Psi^0\left(x^*; \exp_{x^*}^{-1} y\right) \geq 0,\ \forall y \in P, \\ k\left(G(u^*), \exp_{u^*}^{-1} v\right) + \varphi(u^*, v) + \Phi^0\left(u^*; \exp_{u^*}^{-1} v\right) \geq 0,\ \forall v \in Q, \end{cases} \tag{2}$$

where $\Psi^0(x;v)$ (resp., $\Phi^0(u;w)$) denotes the Clarke's generalized directional derivative at the point $x \in P$ (resp., $u \in Q$) in the direction $v \in T_x\mathcal{M}_1$ (resp., $w \in T_u\mathcal{M}_2$).

If $h(\cdot,\cdot) = \langle\cdot,\cdot\rangle_{\mathcal{R}_1}$ and $k(\cdot,\cdot) = \langle\cdot,\cdot\rangle_{\mathcal{R}_2}$, then the problem SVHLI (2) reduces to the split hemivariational inequality problem considered in Tam et al. [30].

We denote by

$$\mathbf{S}(h, F, \psi, \Psi) = \left\{x \in P : h\left(F(x), \exp_x^{-1} y\right) + \psi(x,y) + \Psi^0\left(x; \exp_x^{-1} y\right) \geq 0,\ \forall y \in P\right\},$$

and

$$\mathbf{S}(k, G, \varphi, \Phi) = \left\{u \in Q : k\left(G(u), \exp_u^{-1} v\right) + \varphi(u,v) + \Phi^0\left(u; \exp_u^{-1} v\right) \geq 0,\ \forall v \in Q\right\}.$$

Note that if $(x^*, u^*) \in \mathcal{M}_1 \times \mathcal{M}_2$ is a solution of SVHLI (2), then $u^* = A(x^*)$. Thus, we can define the solution set of SVHLI (2) as follows:

$$\mathbf{S} = \{x \in \mathcal{M}_1 : x \in \mathbf{S}(h, F, \psi, \Psi), u = A(x), u \in \mathbf{S}(k, G, \varphi, \Phi)\}.$$

In this paper, we always assume that $S \neq \emptyset$.

Let us introduce the exact definition of gap functions for SVHLI (2) as follows.

Definition 3.1. A real-valued function $\mathbf{d}\colon \mathcal{M}_1 \to \mathbb{R}$ is said to be a *gap function* for the problem SVHLI if the following properties are satisfied:

(a) $\mathbf{d}(x) \geq 0$ for all $x \in P$;
(b) for any $x^* \in P$, $\mathbf{d}(x^*) = 0$ if and only if x^* is a solution of the problem SVHLI.

Now, we consider the following function $\Delta_{\alpha,\beta}^{A,\varepsilon}\colon \mathcal{M}_1 \to \mathbb{R}$ defined by

$$\Delta_{\alpha,\beta}^{A,\varepsilon}(x) = \Omega_\alpha(x) + \varepsilon\Theta_\beta(A(x)) \tag{3}$$

where $\varepsilon, \alpha, \beta > 0$ and functions $\Omega_\alpha\colon \mathcal{M}_1 \to \mathbb{R}$ and $\Theta_\beta\colon \mathcal{M}_2 \to \mathbb{R}$ are defined by

$$\Omega_\alpha(x) = \sup_{y \in P}\left\{-h\left(F(x), \exp_x^{-1} y\right) - \psi(x,y) - \Psi^0\left(x; \exp_x^{-1} y\right) - \frac{1}{2\alpha} d_{\mathcal{R}_1}^2(x,y)\right\}, \tag{4}$$

and

$$\Theta_\beta(u) = \sup_{v\in Q}\left\{-k\left(G(u), \exp_u^{-1} v\right) - \varphi(u,v) - \Phi^0\left(u; \exp_u^{-1} v\right) - \frac{1}{2\beta} d^2_{\mathcal{R}_2}(u,v)\right\}, \tag{5}$$

for all $(x,u) \in \mathcal{M}_1 \times \mathcal{M}_2$.

Because of the presentation of regularized terms $\frac{1}{2\alpha} d^2_{\mathcal{R}_1}(x,y)$ and $\frac{1}{2\beta} d^2_{\mathcal{R}_2}(u,v)$, this is the reason why we call the function $\Delta^{A,\varepsilon}_{\alpha,\beta}$ by the RG-function for problem SVHLI.

Now, we prove that $\Delta^{A,\varepsilon}_{\alpha,\beta}$ is a gap function for the problem SVHLI.

Theorem 3.2. *Assume that the following conditions hold for the problem SVHLI:*

(i) *P, Q are geodesic convex;*

(ii) *h, k are positively homogeneous in the second component;*

(iii) *Ψ, Φ are locally Lipschitz functions;*

(iv) *ψ, φ are geodesic convex in the second component;*

(v) *For any $x \in P$, $A(x) \in Q$.*

Then, the function $\Delta^{A,\varepsilon}_{\alpha,\beta}$ for any $\varepsilon, \alpha, \beta > 0$, defined by (3) *is a gap function for the problem SVHLI.*

Proof. (a) For any $\varepsilon, \alpha, \beta > 0$ and $x \in P$, we have

$$\begin{aligned}\Omega_\alpha(x) &= \sup_{y\in P}\left\{-h\left(F(x), \exp_x^{-1} y\right) - \psi(x,y) - \Psi^0\left(x; \exp_x^{-1} y\right) - \frac{1}{2\alpha} d^2_{\mathcal{R}_1}(x,y)\right\} \\ &\geq -h\left(F(x), \exp_x^{-1} x\right) - \psi(x,x) - \Psi^0\left(x; \exp_x^{-1} x\right) - \frac{1}{2\alpha} d^2_{\mathcal{R}_1}(x,x) \\ &= -h\left(F(x), \mathbf{0}\right) - \Psi^0\left(x; \mathbf{0}\right) = 0.\end{aligned}$$

By the condition (v), we obtain $A(x) \in Q$ and so we get that $\Theta_\beta(A(x)) \geq 0$. Thus,

$$\Delta^{A,\varepsilon}_{\alpha,\beta}(x) = \Omega_\alpha(x) + \varepsilon\Theta_\beta(A(x)) \geq 0$$

for all $x \in P$.

(b) Suppose that there exists $x_0 \in P$, $\Delta^{A,\varepsilon}_{\alpha,\beta}(x_0) = 0$. Since $\Omega_\alpha(x) \geq 0$, $\Theta_\beta(A(x)) \geq 0$ for all $x \in P$ and $\varepsilon > 0$, $\Omega_\alpha(x_0) = 0$, $\Theta_\beta(A(x_0)) = 0$. It follows that, for any $v \in Q$,

$$0 = \sup_{v\in Q}\left\{-k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right) - \varphi(A(x_0), v) - \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right) - \frac{1}{2\beta} d^2_{\mathcal{R}_2}(A(x_0), v)\right\}.$$

Equivalently, for any $v \in Q$,

$$\frac{1}{2\beta} d^2_{\mathcal{R}_2}(A(x_0), v) \geq -k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right) - \varphi(A(x_0), v) - \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right).$$

For each $v \in Q$ and $\rho \in (0,1)$, we set $v_\rho := \exp_{A(x_0)}\left(\rho \exp^{-1}_{A(x_0)} v\right)$. As Q is a geodesic convex set, $v_\rho \in Q$, and hence

$$\frac{1}{2\beta} d^2_{\mathcal{R}_2}(A(x_0), v_\rho) \geq -k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v_\rho\right) - \varphi(A(x_0), v_\rho) - \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v_\rho\right). \tag{6}$$

By Lemma 2.1 and condition (ii), we have

$$-k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v_\rho\right) = -k\left(G(A(x_0)), \rho \exp^{-1}_{A(x_0)} v\right) = -\rho k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right). \tag{7}$$

It follows from Lemma 2.6 (i) that the function $w \mapsto \Phi^0(A(x_0); w)$ is positively homogeneous on $T_{A(x_0)}\mathcal{M}_2$. Thanks to Lemma 2.1, we have

$$-\Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v_\rho\right) = -\Phi^0\left(A(x_0); \rho \exp^{-1}_{A(x_0)} v\right) = -\rho\Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right). \tag{8}$$

As φ is geodesic convex in the second component and $\varphi(A(x_0), A(x_0)) = 0$, we have

$$-\varphi(A(x_0), v_\rho) \geq -(1-\rho)\varphi(A(x_0), A(x_0)) - \rho\varphi(A(x_0), v) = -\rho\varphi(A(x_0), v). \tag{9}$$

Moreover,

$$d_{\mathcal{R}_2}(A(x_0), v_\rho) = \left\|\exp^{-1}_{A(x_0)}\left(\exp_{A(x_0)}\left(\rho \exp^{-1}_{A(x_0)} v\right)\right)\right\|_{\mathcal{R}_2} = \left\|\rho \exp^{-1}_{A(x_0)} v\right\|_{\mathcal{R}_2} = \rho d_{\mathcal{R}_2}(A(x_0), v). \tag{10}$$

From (6)-(10), we have

$$\frac{\rho^2}{2\beta} d^2_{\mathcal{R}_2}(A(x_0), v) \geq \rho\left[-k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right) - \varphi(A(x_0), v) - \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right)\right].$$

which implies that

$$\frac{\rho}{2\beta} d^2_{\mathcal{R}_2}(A(x_0), v) \geq -k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right) - \varphi(A(x_0), v) - \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right). \tag{11}$$

In (11), letting $\rho \to 0^+$, we obtain

$$k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right) + \varphi(A(x_0), v) + \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right) \geq 0. \tag{12}$$

that is, $A(x_0) \in \mathbf{S}(k, G, \varphi, \Phi)$. By the similar way, it follows that $\Omega_\alpha(x_0) = 0$ implies $x_0 \in \mathbf{S}(h, F, \psi, \Psi)$. Thus $x_0 \in \mathbf{S}$.

Conversely, if $x_0 \in \mathbf{S}$, then $x_0 \in \mathbf{S}(h, F, \psi, \Psi)$ and $A(x_0) \in \mathbf{S}(k, G, \varphi, \Phi)$. It follows from $A(x_0) \in \mathbf{S}(k, G, \varphi, \Phi)$ that for any $v \in Q$,

$$k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right) + \varphi(A(x_0), v) + \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right) \geq 0.$$

This implies that

$$-k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right) - \varphi(A(x_0), v) - \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right) \leq 0.$$

for all $v \in Q$. Thus, we have

$$\begin{aligned}\Theta_\beta(A(x_0)) &= \sup_{v \in Q}\left\{-k\left(G(A(x_0)), \exp^{-1}_{A(x_0)} v\right) - \varphi(A(x_0), v) - \Phi^0\left(A(x_0); \exp^{-1}_{A(x_0)} v\right) - \frac{1}{2\beta} d^2_{\mathcal{R}_2}(A(x_0), v)\right\} \\ &\leq 0.\end{aligned}$$

Similarly, since $x_0 \in \mathbf{S}(h, F, \psi, \Psi)$, $\Omega_\alpha(x_0) \leq 0$. Therefore, for $\varepsilon > 0$, we have

$$\Delta^{A,\varepsilon}_{\alpha,\beta}(x_0) = \Omega_\alpha(x_0) + \varepsilon\Theta_\beta(A(x_0)) \leq 0.$$

Combining with $\Delta^{A,\varepsilon}_{\alpha,\beta}(x_0) \geq 0$, we have $\Delta^{A,\varepsilon}_{\alpha,\beta}(x_0) = 0$. Thus, $\Delta^{A,\varepsilon}_{\alpha,\beta}$ is a gap function for the problem SVHLI. This completes the proof. □

Remark 3.3. It follows from Theorem 3.2 that

$$\mathbf{S} = \{x \in P : \Delta^{A,\varepsilon}_{\alpha,\beta}(x) = 0\},$$

that is, to find the solutions of the problem SVHLI, we only solve the equation

$$\Delta^{A,\varepsilon}_{\alpha,\beta}(x) = 0$$

with noting that $\Delta^{A,\varepsilon}_{\alpha,\beta}(x) \geq 0$ for all $x \in P$.

Remark 3.4. Since the RG-function for split hemivariational inequality problems in the setting of Hadamard manifolds has not been considered in previous references, it would be interesting to continue the study of error bounds the problem SVHLI based on RG-functions (see Subsection 3.2).

Now we give an example to illustrate computing the RG-function of the problem SVHLI under such conditions.

Example 3.5. Let $\mathcal{M}_1 = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\langle \cdot, \cdot \rangle_{\mathcal{R}_1}$ be a Riemannian metric defined by

$$\langle u, w \rangle_{\mathcal{R}_1} = \frac{1}{x^2} uw, \quad x \in \mathcal{M}_1, \quad \forall u, w \in T_x\mathcal{M}_1,$$

where the tangent plane at $x \in \mathcal{M}_1$, denoted by $T_x\mathcal{M}_1$, is precisely $\mathbb{R}$, Moreover, for any $x, y \in \mathcal{M}_1$ the Riemannian distance is given by $d_{\mathcal{R}_1} : \mathcal{M}_1 \times \mathcal{M}_1 \to \mathbb{R}_+$,

$$d_{\mathcal{R}_1}(x, y) = \left| \ln \frac{x}{y} \right|.$$

Then, $(\mathcal{M}_1, \langle \cdot, \cdot \rangle_{\mathcal{R}_1})$ is a Hadamard manifold with sectional curvature 0, see e.g. [4].

For $x \in \mathcal{M}_1$, the exponential map $\exp_x : T_x\mathcal{M}_1 \to \mathcal{M}_1$ and the inverse exponential map $\exp_x^{-1} : \mathcal{M}_1 \to T_x\mathcal{M}_1$ are defined by

$$\exp_x(sv) = xe^{(\frac{v}{x})s} \text{ and } \exp_x^{-1} y = x \ln \frac{y}{x}.$$

Now we consider another Hadamard manifold. Let $\mathbb{P}^n$ be the set of the symmetric matrices, $\mathbb{P}^n_+$ be the cone of the symmetric positive semi-definite matrices and $\mathbb{P}^n_{++}$ be the cone of the symmetric positive-definite matrices both $n \times n$.

For $U, W \in \mathbb{P}^n_+$, $W \geq U$ (or $U \leq W$) means that $W - U \in \mathbb{P}^n_+$ and $W > U$ (or $U < W$) means that $W - U \in \mathbb{P}^n_{++}$.

Let $(\mathcal{M}_2, \langle \cdot, \cdot \rangle_{\mathcal{R}_2})$ be a Riemannian manifold, where $\mathcal{M}_2 = \mathbb{P}^n_{++}$ and $\langle \cdot, \cdot \rangle_{\mathcal{R}_2}$ is a Riemannian metric defined by

$$\langle U, W \rangle_{\mathcal{R}_2} = \operatorname{tr}(X^{-1}UX^{-1}W), \quad X \in \mathcal{M}_2, \quad \forall U, W \in T_X\mathcal{M}_2,$$

where $\operatorname{tr}(A)$ denotes the trace of matrix $A \in \mathbb{P}^n$ and $T_X\mathcal{M}_2 \simeq \mathbb{P}^n$, with the corresponding norm denoted by $\| \cdot \|_{\mathcal{R}_2}$. Then, $(\mathcal{M}_2, \langle \cdot, \cdot \rangle_{\mathcal{R}_2})$ is a Hadamard manifold, see e.g., [21, Theorem 1.2, p. 325].

For each $U \in \mathcal{M}_2$, the exponential map $\exp_U : T_U\mathcal{M}_2 \to \mathcal{M}_2$ and the inverse exponential map $\exp_U^{-1} : \mathcal{M}_2 \to T_U\mathcal{M}_2$ are defined by

$$\exp_U(W) = U^{1/2} e^{(U^{-1/2}WU^{-1/2})} U^{1/2} \text{ and } \exp_U^{-1} W = U \ln(U^{-1}W).$$

Moreover, the Riemannian distance between $X, Y \in \mathcal{M}_2$ is given by

$$d_{\mathcal{R}_2}(U, W) = \left[\operatorname{tr}\left(\ln^2(U^{-1/2}WU^{-1/2}) \right) \right]^{1/2} = \left[\sum_{i=1}^{n} \ln^2 \left(\lambda_i(U^{-1/2}WU^{-1/2}) \right) \right]^{1/2}.$$

where $\lambda_i(U^{-1/2}WU^{-1/2}),\ i = 1, ..., n,$ denotes the ith eigenvalue of the matrix $U^{-1/2}WU^{-1/2}$.

We now consider the problem SVHLI with its data on the different Hadamard manifolds $(\mathcal{M}_1, \langle\cdot,\cdot\rangle_{\mathcal{R}_1})$ and $(\mathcal{M}_2, \langle\cdot,\cdot\rangle_{\mathcal{R}_2})$. Let the sets P and Q be defined by

$$P = \left\{x \in \mathbb{R} : x = e^{\frac{1}{2}+s}, s \in [0, 1]\right\} \subset \mathcal{M}_1, \text{ and}$$
$$Q = \left\{U \in \mathbb{P}^n_{++} : \left(\sqrt{e}\right)^n \le \det U \le 5^n\right\} \subset \mathcal{M}_2,$$

the functions $\Psi\colon \mathcal{M}_1 \to \mathbb{R}$, $\Phi\colon \mathcal{M}_2 \to \mathbb{R}$, $\psi\colon P \times P \to \mathbb{R}$ and $\varphi\colon Q \times Q \to \mathbb{R}$ given by

$$\Psi(x) = \ln^2 x \quad \text{and} \quad \Phi(U) = 0, \quad \forall x \in \mathcal{M}_1, U \in \mathcal{M}_2;$$
$$\psi(x, y) = \ln y - \ln x \quad \text{and} \quad \varphi(U, W) = \ln\det W - \ln\det U, \quad \forall x, y \in P, \forall U, W \in Q,$$

and operator $A\colon P \to Q$ and the vector fields $F\colon P \to T\mathcal{M}_1$ and $G\colon Q \to T\mathcal{M}_2$ defined by

$$A(x) = \operatorname{diag}(x, x, \dots, x), \quad \forall x \in P;$$
$$F(x) = 2\sqrt{\frac{\ln x}{x}} \quad \text{and} \quad G(U) = U \quad \forall x \in P, \forall U \in Q.$$

We also consider the functions $h\colon T\mathcal{M}_1 \times T\mathcal{M}_1 \to \mathbb{R}$ and $k\colon T\mathcal{M}_2 \times T\mathcal{M}_2 \to \mathbb{R}$ given by

$$h(u, v) = u^2 v \quad \text{and} \quad k(W, Z) = (\ln\det W)\operatorname{tr}(U^{-1}WU^{-1}Z),$$

for all $u, v \in T\mathcal{M}_1$ and all $W, Z \in T_U\mathcal{M}_2$, $U \in Q$.

Furthermore, the Clarke's generalized directional derivative of Ψ at $x \in P$ in the direction $w \in T_x\mathcal{M}_1$ is given by $\Psi^0(x; w) = \dfrac{2w\ln x}{x}$, and so

$$\Psi^0(x; \exp_x^{-1} y) = 2\ln x\,(\ln y - \ln x), \quad \forall x, y \in P.$$

Similarly, $\Phi^0(U; W) = 0$ for all $U \in Q$ and $W \in T_U\mathcal{M}_2$.

For any $x, y \in P$ and $U, V \in Q$, we have

$$h\left(F(x), \exp_x^{-1} y\right) = (F(x))^2 \exp_x^{-1} y = 4\ln x(\ln y - \ln x)$$

and

$$\begin{aligned} k\left(G(U), \exp_U^{-1} V\right) &= (\ln\det U)\operatorname{tr}(U^{-1}UU^{-1}U\ln(U^{-1}V)) \\ &= (\ln\det U)\operatorname{tr}\ln(U^{-1}V) = (\ln\det U)\left(\ln\det(U^{-1}V)\right) \\ &= (\ln\det V - \ln\det U)\ln\det U. \end{aligned}$$

Then,

$$\begin{aligned} \mathbf{S}(h, F, \psi, \Psi) &= \left\{x \in P : h\left(F(x), \exp_x^{-1} y\right) + \psi(x, y) + \Psi^0\left(x; \exp_x^{-1} y\right) \ge 0,\ \forall y \in P\right\} \\ &= \{x \in P : (1 + 6\ln x)(\ln y - \ln x) \ge 0,\ \forall y \in P\} = \left\{\sqrt{e}\right\} \end{aligned}$$

and

$$\begin{aligned} \mathbf{S}(k, G, \varphi, \Phi) &= \left\{U \in Q : k\left(G(U), \exp_U^{-1} V\right) + \varphi(U, V) + \Phi^0\left(U; \exp_U^{-1} V\right) \ge 0,\ \forall V \in Q\right\} \\ &= \{U \in Q : (1 + \ln\det U)(\ln\det V - \ln\det U) \ge 0,\ \forall V \in Q\} \\ &= \{U \in \mathbb{P}^n_{++} : \det U = \left(\sqrt{e}\right)^n\}. \end{aligned}$$

Since $A\left(\sqrt{e}\right) = \text{diag}\left(\sqrt{e}, \sqrt{e}, \ldots, \sqrt{e}\right)$, $\det A\left(\sqrt{e}\right) = \left(\sqrt{e}\right)^n$. Therefore, the solution set of SVHLI (2) is

$$\mathbf{S} = \{x \in \mathcal{M}_1 : x \in \mathbf{S}(h, F, \psi, \Psi), U \in \mathbf{S}(k, G, \varphi, \Phi), U = A(x)\} = \left\{\sqrt{e}\right\}.$$

We now check all assumptions in Theorem 3.2. Indeed, it is easy to verify that P and Q are nonempty, compact and geodesic convex subsets. The positive homogeneity in the second component of h and k follows from the definition. It is easily seen that the functions Ψ and Φ are locally Lipschitz, $\psi(x, x) = 0$ for all $x \in P$ and $\varphi(U, U) = 0$ for all $U \in Q$. Moreover, an easy computation shows that $A(x) \in Q$ for all $x \in P$. The geodesic convexity in the second component of ψ comes from Example 2.5 in [16], while the one of φ follows from the well-known properties that $\det(UV) = \det U \det V$, $\text{tr}\,(UV) = \text{tr}\,(VU)$, $\text{tr}\,(\alpha U) = \alpha \text{tr}\,(U)$ for any U, V $\alpha \in \mathbb{R}$ (see [36]), and $\ln\det U = \text{tr}\,(\ln U)$ for any $U \in \mathbb{P}^n_{++}$ (see [31]). We thus get all assumptions in Theorem 3.2 are satisfied. Thus, $\Delta^{A,\varepsilon}_{\alpha,\beta}$ is a gap function for the problem SVHLI for any $\varepsilon, \alpha, \beta > 0$.

For example, let $\alpha = \dfrac{1}{2}$, we get from (4) that for any $x \in P$,

$$\begin{aligned}
\Omega_\alpha(x) &= \sup_{y\in P}\left\{-h\left(F(x), \exp_x^{-1} y\right) - \psi(x, y) - \Psi^0\left(x; \exp_x^{-1} y\right) - \frac{1}{2\alpha} d^2_{\mathcal{R}_1}(x, y)\right\}\\
&= \sup_{y\in P}\left\{(1 + 6\ln x)(\ln x - \ln y) - (\ln x - \ln y)^2\right\}\\
&= \sup_{y\in P}\left\{(1 + 5\ln x + \ln y)(\ln x - \ln y)\right\}\\
&= \left(\frac{3}{2} + 5\ln x\right)\left(\ln x - \frac{1}{2}\right).
\end{aligned}$$

Similarly, take $\beta = 1$. From the property $\ln\det U = \text{tr}\,(\ln U) = \sum_{i=1}^{n} \ln(\lambda_i(U))$ for any $U \in \mathbb{P}^n_{++}$, where $\lambda_i(U), i = 1, ..., n$, denotes the ith eigenvalue of U ([31]), for any $x \in P$, we obtain

$$\begin{aligned}
\Theta_\beta(A(x)) &= \sup_{V\in Q}\left\{-k\left(G(A(x)), \exp^{-1}_{A(x)} V\right) - \varphi(A(x), V) - \Phi^0\left(A(x); \exp^{-1}_{A(x)} V\right) - \frac{1}{2\beta} d^2_{\mathcal{R}_2}(A(x), V)\right\}\\
&= \sup_{V\in Q}\left\{-(\ln\det A(x) + 1)\left(\ln\det A(x)^{-1}V\right) - \frac{1}{2}\sum_{i=1}^{n} \ln^2\left(\lambda_i(A(x)^{-1/2}VA(x)^{-1/2})\right)\right\}\\
&= \sup_{V\in Q}\left\{-(n\ln x + 1)\sum_{i=1}^{n}\ln\left(\lambda_i(x^{-1}V)\right) - \frac{1}{2}\sum_{i=1}^{n}\ln^2\left(\lambda_i(x^{-1}V)\right)\right\}.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, together with calculating the maximum of the function $f(t) = -(n\ln x + 1)\,t - \frac{1}{2}t^2$ on $\left[n\left(\frac{1}{2} - \ln x\right),\ n(\ln 5 - \ln x)\right]$, one gets

$$\Theta_\beta(A(x)) = n\left(\ln x - \frac{1}{2}\right)\left[\frac{5}{4} + \left(n - \frac{1}{2}\right)\ln x\right], \forall x \in P.$$

Hence, for $\alpha = \dfrac{1}{2}$, $\beta = 1$ and any $\varepsilon > 0$, one has

$$\Delta^{A,\varepsilon}_{\alpha,\beta}(x) = \begin{cases} 0 & \text{if } x = \sqrt{e} \in \mathbf{S};\\ \left(\ln x - \frac{1}{2}\right)\left[\frac{3}{2} + \frac{5}{4}\varepsilon n + \left(5 + \left(n - \frac{1}{2}\right)\varepsilon n\right)\ln x\right] \geq 0 & \text{if } x \in P. \end{cases}$$

Thus, $\Delta^{A,\varepsilon}_{\alpha,\beta}$ is a gap function for the problem SVHLI with $\alpha = \dfrac{1}{2}$ and $\beta = 1$ and any $\varepsilon > 0$.

3.2. Global error bounds for the problem SVHLI

Based on the RG-function $\Delta^{A,\varepsilon}_{\alpha,\beta}$, we now establish a global error bound for the problem SVHLI under suitable assumptions.

Theorem 3.6. *Let x^* be the solution to the problem* SVHLI*. Suppose that all assumptions of Theorem 3.2 are satisfied and the following conditions hold:*

(i) *h is (F, σ_1)-strongly monotone on P and k is (G, σ_2)-strongly monotone on Q;*

(ii) *for any $x, y \in P$, there exists $\delta_1 > 0$ such that*

$$\Psi^0\left(x; \exp_x^{-1} y\right) + \Psi^0\left(y; \exp_y^{-1} x\right) \leq \delta_1 d^2_{\mathcal{R}_1}(x, y);$$

(iii) *for any $u, v \in Q$, there exists $\delta_2 > 0$ such that*

$$\Phi^0\left(u; \exp_u^{-1} v\right) + \Phi^0\left(v; \exp_v^{-1} u\right) \leq \delta_2 d^2_{\mathcal{R}_2}(u, v);$$

(iv) *ψ and φ are skew-symmetric;*

(v) *A is L-strongly nonexpanding on P, i.e., there exists $L > 0$ such that*

$$d_{\mathcal{R}_2}(A(u), A(v)) \geq L d_{\mathcal{R}_1}(u, v), \quad \forall u, v \in P.$$

Then, for any $\varepsilon > 0$, $\alpha > 0$ and $\beta > 0$ satisfying $\sigma_2 - \delta_2 - \frac{1}{2\beta} > 0$ and

$$\sum_{i=1}^{2} (\varepsilon L^2)^{i-1}(\sigma_i - \delta_i) - \frac{\alpha\varepsilon L^2 + \beta}{2\alpha\beta} > 0 \tag{13}$$

one has for each $x \in P$,

$$d_{\mathcal{R}_1}(x, x^*) \leq \sqrt{\frac{\Delta^{A,\varepsilon}_{\alpha,\beta}(x)}{\sum_{i=1}^{2} (\varepsilon L^2)^{i-1}(\sigma_i - \delta_i) - \dfrac{\alpha\varepsilon L^2 + \beta}{2\alpha\beta}}}. \tag{14}$$

Proof. For any $\varepsilon, \alpha, \beta > 0$ and any $x \in \mathcal{M}_1$, by the definition of the function $\Delta^{A,\varepsilon}_{\alpha,\beta}$ we have

$$\begin{aligned}
\Delta^{A,\varepsilon}_{\alpha,\beta}(x) \geq & - h\left(F(x), \exp_x^{-1} x^*\right) - \psi(x, x^*) - \Psi^0\left(x; \exp_x^{-1} x^*\right) \\
& - \varepsilon k\left(G(A(x)), \exp^{-1}_{A(x)} A(x^*)\right) - \varepsilon\varphi(A(x), A(x^*)) \\
& - \varepsilon\Phi^0\left(A(x); \exp^{-1}_{A(x)} A(x^*)\right) - \frac{1}{2\alpha} d^2_{\mathcal{R}_1}(x, x^*) - \varepsilon\frac{1}{2\beta} d^2_{\mathcal{R}_2}(A(x), A(x^*)).
\end{aligned} \tag{15}$$

Since x^* is the solution of the problem SVHLI, we have

$$h\left(F(x^*), \exp_{x^*}^{-1} x\right) + \psi(x^*, x) + \Psi^0\left(x^*; \exp_{x^*}^{-1} x\right) \geq 0 \tag{16}$$

and

$$k\left(G(A(x^*)), \exp^{-1}_{A(x^*)} A(x)\right) + \varphi(A(x^*), A(x)) + \Phi^0\left(A(x^*); \exp^{-1}_{A(x^*)} A(x)\right) \geq 0. \tag{17}$$

Using the conditions (i)–(iv) and (16), we have

$$
\begin{aligned}
& - h\left(F(x), \exp_x^{-1} x^*\right) - \psi(x, x^*) - \Psi^0\left(x; \exp_x^{-1} x^*\right) \\
& \geq h\left(F(x^*), \exp_{x^*}^{-1} x\right) + \psi(x^*, x) + \Psi^0\left(x^*; \exp_{x^*}^{-1} x\right) + (\sigma_1 - \delta_1)\, d_{\mathcal{R}_1}^2(x, x^*) \\
& \geq (\sigma_1 - \delta_1)\, d_{\mathcal{R}_1}^2(x, x^*). \qquad (18)
\end{aligned}
$$

By the argument to get (18), one has

$$
\begin{aligned}
& - k\left(G(A(x)), \exp_{A(x)}^{-1} A(x^*)\right) - \varphi(A(x), A(x^*)) - \Phi^0\left(A(x); \exp_{A(x)}^{-1} A(x^*)\right) \\
& \geq k\left(G(A(x^*)), \exp_{A(x^*)}^{-1} A(x)\right) + \varphi(A(x^*), A(x)) + \Phi^0\left(A(x^*); \exp_{A(x^*)}^{-1} A(x)\right) + (\sigma_2 - \delta_2)\, d_{\mathcal{R}_2}^2(A(x), A(x^*)) \\
& \geq (\sigma_2 - \delta_2)\, d_{\mathcal{R}_2}^2(A(x), A(x^*)). \qquad (19)
\end{aligned}
$$

Combining (15), (18) and (19), using the condition (v) and the assumption $\sigma_2 - \delta_2 - \frac{1}{2\beta} > 0$, we obtain

$$
\begin{aligned}
\Delta_{\alpha,\beta}^{A,\varepsilon}(x) & \geq \left(\sigma_1 - \delta_1 - \frac{1}{2\alpha}\right) d_{\mathcal{R}_1}^2(x, x^*) + \varepsilon\left(\sigma_2 - \delta_2 - \frac{1}{2\beta}\right) d_{\mathcal{R}_2}^2(A(x), A(x^*)) \\
& \geq \left(\sigma_1 - \delta_1 - \frac{1}{2\alpha} + \varepsilon L^2\sigma_2 - \varepsilon L^2\delta_2 - \varepsilon\frac{L^2}{2\beta}\right) d_{\mathcal{R}_1}^2(x, x^*) \\
& = \left(\sum_{i=1}^{2}(\varepsilon L^2)^{i-1}(\sigma_i - \delta_i) - \frac{\alpha\varepsilon L^2 + \beta}{2\alpha\beta}\right) d_{\mathcal{R}_1}^2(x, x^*). \qquad (20)
\end{aligned}
$$

By the condition (13), it follows from (20) that

$$
d_{\mathcal{R}_1}(x, x^*) \leq \sqrt{\frac{\Delta_{\alpha,\beta}^{A,\varepsilon}(x)}{\sum_{i=1}^{2}(\varepsilon L^2)^{i-1}(\sigma_i - \delta_i) - \dfrac{\alpha\varepsilon L^2 + \beta}{2\alpha\beta}}},
$$

that is the inequality (14) holds. The proof is complete. □

We provide an example for computing global error bounds for the problem via the RG-Function $\Delta_{\alpha,\beta}^{A,\varepsilon}$.

Example 3.7. We maintain the assumptions on the manifolds, sets, and bifunctions, operator, vector fields as in Example 3.5. We showed that the assumptions in Theorem 3.2 are satisfied. We begin by checking the assumptions in Theorem 3.6. Let us first show that h is (F, σ_1)-strongly monotone on P with $\sigma_1 = 4$ and k is (G, σ_2)-strongly monotone on Q. Indeed, for any $x, y \in P$ and $x \neq y$, one has

$$
\begin{aligned}
h\left(F(x), \exp_x^{-1} y\right) + h\left(F(y), \exp_y^{-1} x\right) & = 4\ln x(\ln y - \ln x) + 4\ln y(\ln x - \ln y) \\
& = -4(\ln x - \ln y)^2 \\
& = -4d_{\mathcal{R}_1}^2(x, y).
\end{aligned}
$$

Moreover, according to the well-known property that for any a positive semidefinite matrix A, $0 \leq \operatorname{tr}\left(A^2\right) \leq (\operatorname{tr} A)^2$ (see [35]), it follows that k is (G, σ_2)-strongly monotone on Q with $\sigma_2 = 1$. Infact, for any $U, V \in Q$ and

$U \neq V$,

$$\begin{aligned} k\left(G(U), \exp_U^{-1} V\right) + k\left(G(V), \exp_V^{-1} U\right) &= \ln\det U\left(\ln\det(U^{-1}V)\right) + \ln\det V\left(\ln\det(V^{-1}U)\right) \\ &= -\left(\ln\det(U^{-1}V)\right)^2 \\ &= -\left(\ln\det(U^{-1/2}VU^{-1/2})\right)^2 \\ &= -\left[\operatorname{tr}\left(\ln(U^{-1/2}VU^{-1/2})\right)\right]^2 \\ &\leq -\left[\operatorname{tr}\left(\ln^2(U^{-1/2}VU^{-1/2})\right)\right] = -d^2_{\mathcal{R}_2}(U, V). \end{aligned}$$

We proceed to show that the condition (ii) holds and (iii) does so. Indeed, for any $x, y \in P$, one has

$$\begin{aligned} \Psi^0\left(x; \exp_x^{-1} y\right) + \Psi^0\left(y; \exp_y^{-1} x\right) &= 2\ln x(\ln y - \ln x) + 2\ln y(\ln x - \ln y) \\ &= -2(\ln x - \ln y)^2 \\ &= -2d^2_{\mathcal{R}_1}(x, y), \end{aligned}$$

and $\Phi^0\left(U; \exp_U^{-1} V\right) + \Phi^0\left(V; \exp_V^{-1} U\right) = 0$ for all $U, V \in Q$. From what has already been showed, it may be concluded that (ii) holds with any $\delta_1 > 0$ and (iii) does so with any $\delta_2 > 0$, hence, we can take $\delta_1 = 1$ and $\delta_2 = \dfrac{1}{4}$.

It is clear that ψ and φ are skew-symmetric. It remains to check that the condition (v) is satisfied. In fact, since A is a diagonal matrix, it follows that for any $x, y \in P$, $A(x)^{-1/2}A(y)A(x)^{-1/2} = \operatorname{diag}\left(\frac{y}{x}, \frac{y}{x}, \ldots, \frac{y}{x}\right)$. Hence, for any $x, y \in P$, we have

$$\begin{aligned} d_{\mathcal{R}_2}(A(x), A(y)) &= \left[\sum_{i=1}^n \ln^2\left(\lambda_i(A(x)^{-1/2}A(y)A(x)^{-1/2})\right)\right]^{1/2} \\ &= \left[\sum_{i=1}^n \ln^2\left(\lambda_i\left(\operatorname{diag}\left(\frac{y}{x}, \frac{y}{x}, \ldots, \frac{y}{x}\right)\right)\right)\right]^{1/2} \\ &= \left[\sum_{i=1}^n \ln^2\left(\frac{y}{x}\right)\right]^{1/2} = \left[n\ln^2\left(\frac{y}{x}\right)\right]^{1/2} \\ &= \sqrt{n}\left|\ln\frac{y}{x}\right| = \sqrt{n}\, d_{\mathcal{R}_1}(x, y), \end{aligned}$$

This means that A is L-strongly nonexpanding on P with $L = \sqrt{n}$. We thus get all assumptions in Theorem 3.6 are also fulfilled.

Moreover, according to Example 3.5, we have the solution set of SVHLI is

$$\mathbf{S} = \{x \in \mathcal{M}_1 : x \in \mathbf{S}(h, F, \psi, \Psi), U \in \mathbf{S}(k, G, \varphi, \Phi), U = A(x)\} = \{\sqrt{e}\},$$

which implies that $d_{\mathcal{R}_1}(x, x^*) = d_{\mathcal{R}_1}(x, \sqrt{e}) = |\ln x - \ln\sqrt{e}| = \ln x - \dfrac{1}{2}$ for all $x \in P$.

From Example 3.5 we obtain $\Delta^{A,\varepsilon}_{\alpha,\beta}$ is a gap function for the problem SVHLI with $\alpha = \dfrac{1}{2}$, $\beta = 1$, $n = 2$ and $\varepsilon = \frac{1}{2}$,

$$\Delta^{A,\varepsilon}_{\alpha,\beta}(x) = \left(\ln x - \frac{1}{2}\right)\left(\frac{11}{4} + \frac{13}{2}\ln x\right), \ \forall x \in P.$$

For the constants shown above, one gets

$$\begin{cases} \sigma_2 - \delta_2 - \frac{1}{2\beta} = \dfrac{1}{4} > 0 \\ \sum_{i=1}^{2}(\varepsilon L^2)^{i-1}(\sigma_i - \delta_i) - \dfrac{\alpha\varepsilon L^2 + \beta}{2\alpha\beta} = \dfrac{9}{4} > 0 \end{cases}$$

Then, for any $x \in P$,

$$\begin{aligned} \sqrt{\frac{\Delta^{A,\varepsilon}_{\alpha,\beta}(x)}{\sum_{i=1}^{2}(\varepsilon L^2)^{i-1}(\sigma_i - \delta_i) - \dfrac{\alpha\varepsilon L^2 + \beta}{2\alpha\beta}}} &= \frac{2}{3}\sqrt{\Delta^{A,\varepsilon}_{\alpha,\beta}(x)} \\ &= \frac{2}{3}\sqrt{\left(\ln x - \frac{1}{2}\right)\left(\frac{11}{4} + \frac{13}{2}\ln x\right)} \\ &\geq \frac{5\sqrt{2}}{3}\sqrt{\left(\ln x - \frac{1}{2}\right)^2} \\ &\geq \left(\ln x - \frac{1}{2}\right) = d_{\mathcal{R}_1}(x, x^*). \end{aligned}$$

4. Conclusions

In this paper, we investigated a class of split variational-hemivariational-like inequalities (SVHLI) on Hadamard manifolds. A new regularized gap function (RG-function) of the problem SVHLI was established in Theorem 3.2. Then we developed an global upper bound for the problem SVHLI based on the RG-function under suitable assumptions, see Theorem 3.6. Besides, we provided two examples to compute the RG-function and the global upper bound to illustrate our main results on the problem SVHLI, see Example 3.5 and Example 3.7.

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