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Extropy: Dynamic cumulative past and residual inaccuracy measures with applications

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Abstract. In comparison to PDF-based measures which provided by extropy, the CDF-based measures of cumulative residual and past extropies are generally more stable. In this paper, we extend the concept of cumulative extropy which includes both cumulative inaccuracy and dynamic versions, such as cumulative residual inaccuracy and cumulative past inaccuracy. By using these measures, we characterize specific lifetime distributions and explore some generalized results. Additionally, we find that these inaccuracy measures can uniquely identify the underlying distributions, and we also characterize a few specific lifetime distributions. Two non-parametric estimators for this measure are provided and their performances are compared via some simulation studies. Finally, we illustrate our method in a real data set.

1. Introduction

The concept of entropy was initially introduced by physicist Boltzmann [15] to quantify the disorder in a physical system. However, information theory owes its origins to Shannon's [20] groundbreaking work on communication, where he introduced a mathematical definition of information known as Shannon entropy. Since then, many different entropy and information measures have been developed in both parametric and non-parametric contexts, and are widely used across various fields. A significant advancement in this area is Kerridge's [11] inaccuracy measure, which offers a non-parametric extension of Shannon entropy. Assume that X and Y be two non-negative continuous random variables with cumulative distribution functions (CDFs) $F(x) = P(X \leq x)$ and $G(x) = P(Y \leq x)$, and probability density functions (PDFs) $f(x)$ and $g(x)$, respectively. If $F(x)$ is the actual distribution corresponding to the observations and $G(x)$ is the distribution assigned by the experimenter, then the Kerridge inaccuracy measure of X and Y is given by

$$H(f,g) = -E_f[\log g(X)] = -\int_0^{\infty} f(x)\log g(x)dx, \tag{1}$$

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where $\log(\cdot)$ stands for the natural logarithm such that $0\log 0 = 0$. Specially, if $f(x) = g(x)$, it reduces to Shannon differential entropy $H(f) = -E[\log f(X)]$. Kumar and Taneja [12] proposed an alternative measure, based on survival functions instead of probability density functions. If $\overline{F}(x)$ and $\overline{G}(x)$ are the survival functions of X and Y, the cumulative residual inaccuracy measure is defined as:

$$\mathcal{E}(F,G) = -\int_0^\infty \overline{F}(x)\log\overline{G}(x)dx. \tag{2}$$

In case $\overline{G}(x) = \overline{F}(x)$, we have the cumulative residual entropy defined by Rao *et al.* [19]. The cumulative residual inaccuracy measure defined in (2) reduces to the cumulative residual entropy defined by Rao *et al.* [19] when the survival functions of the two random variables are equal i.e. $\overline{G}(x) = \overline{F}(x)$. Analogous to (2), the cumulative past inaccuracy measure can be defined as (see Di Crescenzo and Longobardi [2])

$$\mathcal{CE}(F,G) = -\frac{1}{2}\int_0^\infty F(x)\log G(x)dx, \tag{3}$$

where $F(x)$ is the baseline distribution function and $G(x)$ can be considered as some reference distribution function. When the two distributions coincide, then measure (3) reduces to the cumulative entropy defined by Di Crescenzo and Longobardi di2009cumulative. In recent years, the extropy of a random variable X was introduced in [14] and is defined as

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)dx. \tag{4}$$

Extropy is a measure of uncertainty for random variables that is defined analogously to entropy. Since its introduction by Lad *et al.* in [14], extropy has been further studied and utilized in areas such as [17], [18] and [21] and the references therein. More recently, Jahanshani *et al.* [9] introduced an alternative measure of uncertainty for non-negative random variables called cumulative residual extropy (CRE). The CRE of a non-negative random variable X with SF $\overline{F}(x) = 1 - F(x)$ is defined as

$$\mathcal{J}(F) = -\frac{1}{2}\int_0^\infty \overline{F}^2(x)dx. \tag{5}$$

Compared to extropy, CRE accounts for the survival function of X rather than just its probability density function. Jahanshani *et al.* [9] showed that CRE satisfies several desirable properties as a measure of uncertainty, and derived bounds relating it to extropy. Overall, CRE provides a new perspective on quantifying uncertainty that complements existing measures like entropy and extropy. In analogy to CRE, Kumar *et al.* [12] proposed a measure called cumulative extropy (CE) that quantifies the idle time of a system. The cumulative extropy serves as a valuable tool for assessing the uncertainty related to the past lifetimes of a system. The cumulative extropy of a non-negative random variable X with CDF $F(x)$ is defined as

$$\bar{\mathcal{J}}(X) = -\frac{1}{2}\int_0^\infty F^2(x)dx. \tag{6}$$

Hashempour and Mohammadi [8] introduced two new extropy-based measures called cumulative past extropy inaccuracy (CPEI) and dynamic cumulative past extropy inaccuracy (DCPEI). These provide an alternative approach to calculating extropy compared to traditional methods. CPEI and DCPEI measure the inaccuracy of predicting the realized value of a random variable X based on its cumulative distribution function up to a given point. Hashempour and Mohammadi [8] conducted a study analyzing properties of DCPEI, with a focus on stochastic ordering and characterization. The DCPEI measure provides a new perspective on quantifying uncertainty and unpredictability that complements existing extropy definitions. Further research is needed to explore applications of CPEI and DCPEI in areas such as statistics, economics, and engineering.

The paper introduces dynamic cumulative residual and past inaccuracy measures and investigates their characterization results. The paper is organized as follows. Section 2 focuses on the cumulative residual inaccuracy measure and establishes a lower bound for it. The dynamic cumulative residual inaccuracy is discussed in Section 3, while Section 4 examines the characterization results for this measure, including characterizing certain specific lifetime distributions. In Section 5, the dynamic cumulative past inaccuracy measure and its characterization result are discussed. Finally, the study concludes with some concluding remarks.

2. Cumulative residual extropy inaccuracy measure

Hereafter, we introduce a new measure of inaccuracy measure between two non-negative random variables X and Y with the SFs $\overline{F}$ and $\overline{G}$, respectively. To this end, we define the cumulative residual extropy inaccuracy (CREI) measure by

$$\mathcal{J}(X,Y) = -\frac{1}{2}\int_0^\infty \overline{F}(x)\overline{G}(x)dx, \tag{7}$$

provided that the integral in the right-hand side of (7) is finite. It is worth mentioning that Eq. (7) measures the difference between the true SFs of two lifetime random variables, X and Y. In fact, it takes into account the total discrepancy between the actual and predicted values of the SFs over time. In our discussion, we will be referring to two random variables, namely X and Y, which have identical support. If X and Y are identically distributed, then (7) becomes equivalent to the cumulative residual extropy shown in (5). Let us assume that the random variables X and X_β satisfy the proportional hazard (PH) rate model given as

$$\overline{F}_\beta(x) = [\overline{F}(x)]^\beta,\ x > 0, \tag{8}$$

for some constant $\beta > 0$. In this case, by substituting (8) in (7), it reduces to

$$\mathcal{J}(X,X_\beta) = -\frac{1}{2}\int_0^\infty \overline{F}^{\beta+1}(x)dx,$$

for some constant $\beta > 0$.

Example 2.1. For a non-negative random variable X uniformly distributed over (c,d). The CDF is $F(x) = \frac{x-c}{d-c}$, $c < x < d$. Assuming that the PH holds for random variables X and Y, we can conclude that the CDF of random variable Y is equal to

$$\overline{G}(x) = \left[\frac{d-x}{d-c}\right]^\beta,\ c < x < d,\ \beta > 0. \tag{9}$$

By plugging in the given values into (7) and subsequently simplifying, we obtain the cumulative inaccuracy measure as

$$\mathcal{J}(F;G) = -\frac{d-c}{2(\beta+2)}. \tag{10}$$

In the following, we will derive both lower and upper bounds for the measure of inaccuracy between X and Y, starting with a lower bound expressed in terms of the extropy for the CREI measure.

Proposition 2.2. *If X and Y are non-negative continuous random variables with finite means, then*

$$\mathcal{J}(F;G) \geq -\frac{1}{2}\left[E(Y) - D(X)\right], \tag{11}$$

where $D(X) = -\int_0^\infty \overline{G}(x)\overline{F}(x)\log\overline{F}(x)dx$.

Proof. To prove (11), using the inequality $x \log \frac{x}{y} \geq x - y$ for all $x, y > 0$, we get

$$\int_0^\infty \overline{F}(x)\overline{G}(x) \log \frac{\overline{F}(x)\overline{G}(x)}{\overline{G}(x)} dx \geq \int_0^\infty \left[\overline{F}(x)\overline{G}(x) - \overline{G}(x)\right] dx,$$

next,

$$\begin{aligned}\int_0^\infty \overline{F}(x)\overline{G}(x) \log \overline{F}(x) dx &\geq \int_0^\infty \overline{F}(x)\overline{G}(x) dx - \int_0^\infty \overline{G}(x) dx \\ &= \int_0^\infty \overline{F}(x)\overline{G}(x) dx - E(Y).\end{aligned}$$

In the following, we given

$$\begin{aligned}-\frac{1}{2}\int_0^\infty \overline{F}(x)\overline{G}(x) dx &\geq -\frac{1}{2}\int_0^\infty \overline{F}(x)\overline{G}(x) \log \overline{F}(x) dx - \frac{1}{2}E(Y) \\ &= -\frac{1}{2}\left[E(Y) - D(X)\right].\end{aligned}$$

This confirms the conclusion. □

Proposition 2.3. *Under the conditions of Proposition 2.2, we have*

$$\mathcal{J}(F;G) \leq \frac{1}{2}\left[\bar{\xi}(F;G) + E(X)\right]. \tag{12}$$

Proof. Using the identity $\log(x) \leq x - 1$ for $0 \leq x \leq 1$, we obtain

$$\overline{F}(x) \log \overline{G}(x) \leq \overline{F}(x)\overline{G}(x) - \overline{F}(x), \ x > 0. \tag{13}$$

Integrating both sides of (13) with respect to x over $(0, \infty)$ yields

$$\begin{aligned}\frac{1}{2}\int_0^\infty \overline{F}(x) \log \overline{G}(x) dx &\leq \frac{1}{2}\int_0^\infty \overline{F}(x)\overline{G}(x) dx - \frac{1}{2}\int_0^\infty \overline{F}(x) dx \\ &= \frac{1}{2}\int_0^\infty \overline{F}(x)\overline{G}(x) dx - \frac{1}{2}E(X).\end{aligned}$$

So, we have

$$\frac{1}{2}\bar{\xi}(F;G) \geq \mathcal{J}(F;G) + \frac{1}{2}E(X),$$

and this confirms the result. □

Proposition 2.4. *Let $\overline{F}(x)$ and $\overline{G}(x)$ be the SFs non-negative continuous random variables X and Y with finite means, respectively. Then*

i) $\mathcal{J}(F;G) \geq \max\{-\frac{E(X)}{2}, -\frac{E(Y)}{2}\}$,

ii) $\mathcal{J}(F;G) \leq \frac{1}{2}\ \bar{\xi}(F;G)$,

iii) $\mathcal{J}(F;G) = \frac{1}{2}\left[W_1(x) - E(X)\right]$, *where* $W_1(x) = \int_0^\infty \overline{F}(x)G(x)dx$,

iv) $\mathcal{J}(F;G) = \frac{1}{2}\left[W_2(x) - E(Y)\right]$, *where* $W_2(x) = \int_0^\infty F(x)\overline{G}(x)dx$.

2.1. Dynamic cumulative residual extropy inaccuracy measure

In experiments that involve testing the lifetime of a system, the experimenter typically knows how old the system is at the time of the test. This means that the method for measuring accuracy introduced in (7) cannot be used as it does not account for the age of the system. Instead, the residual lifetime random variable should be used in such situations. These variables reflect the remaining lifetime of a component that has already survived up to time t. To this aim, let us consider two non-negative random variables X and Y, representing the lifetimes of two items, and having the same support. For $t > 0$, let $X_t = [X - t|X > t]$ and $Y_t = [Y - t|Y > t]$ be their respective residual lifetimes. These notions deserve interest in applied fields such as survival analysis, reliability theory, and actuarial science. For each $t > 0$, we denote the survival functions of X_t and Y_t by

$$\overline{F}_t(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \quad \overline{G}_t(x) = \frac{\overline{G}(x+t)}{\overline{G}(t)}, \ x, t > 0, \tag{14}$$

respectively. Let us define the dynamic cumulative residual extropy inaccuracy (DCREI) measure as follows:

$$\begin{aligned} \mathcal{J}(F, G; t) &= -\frac{1}{2}\int_0^\infty \overline{F}_t(x)\overline{G}_t(x)dx \\ &= -\frac{1}{2}\int_t^\infty \frac{\overline{F}(x)\overline{G}(x)}{\overline{F}(t)\overline{G}(t)}dx, \end{aligned} \tag{15}$$

for all $t > 0$. It is evident that the DCREI is non-positive and it reduces to CREI when $t = 0$. The following example illustrate the DCREI.

Example 2.5. Let X and Y be two non-negative random variables having CDFs respectively

$$\overline{F}(x) = \begin{cases} 1 - \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{2}e^{-(x-1)}, & x > 1 \end{cases}$$

and

$$\overline{G}(x) = \begin{cases} 1 - \frac{x^2}{2}, & 0 \le x \le 1 \\ \frac{1}{2}e^{-(x-1)}, & x > 1 \end{cases}$$

It is not hard to verify that the DCREI measure can be obtained as

$$\mathcal{J}(F, G; t) = \begin{cases} \dfrac{3t^4 - 8t^3 - 12t^2 + 48t - 37}{24(t-2)(t^2-2)}, & 0 < t \le 1 \\ -\dfrac{1}{2} & t \ge 1 \end{cases}$$

By substituting $t = 0$, we get the CREI measure as

$$\mathcal{J}(F, G) = \begin{cases} -0.385, & 0 \le x \le 1 \\ -0.5, & x > 1. \end{cases}$$

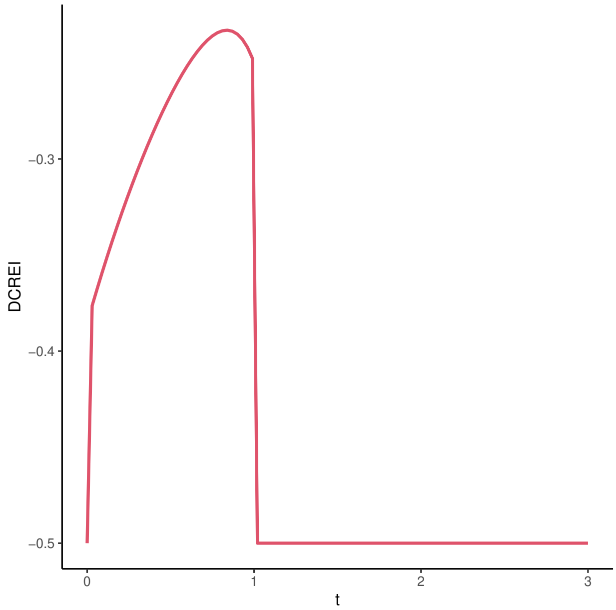

Figure 1: DCREI measure given in Example 2.5.

The behaviour of the DCREI measure $\mathcal{J}(F,G;t)$ for $t \in (0,\infty)$ is shown in Figure 1.

Corollary 2.6. *Suppose that X and Y are non-negative random variables that follow the proportional hazards (PH) model, and let $\overline{F}(x)$ and $\overline{G}(x)$ denote their respective survival functions. If $\mathcal{J}(F,G;t)$ is a decreasing (increasing) function of t for all $t \geq 0$, then we have the inequality*

$$\mathcal{J}(F,G;t) \leq (\geq) \frac{\lambda^{-1}(t)}{2(\beta+1)}, \tag{16}$$

where $\lambda(t)$ is the hazard rate function and β is a constant.

Example 2.7. Suppose X be a non-negative random variable with SF $\overline{F}(x) = \left(1-x^2\right), x \in (0,1)$ and suppose the random variable Y be uniformly distributed over $(0,1)$ with density and survival functions given respectively by $g_Y(x) = 1$ and $\overline{G}_Y(x) = 1-x, x \in (0,1)$. Replacing these values in (15), we obtain the DCREI measure as

$$\mathcal{J}(F,G;t) = \begin{cases} -\frac{(3t+5)(1-t)}{24(t+1)} & ; \quad 0 \leq t < 1 \\ 0 & ; \quad \text{o.w.} \end{cases}$$

Figure 2 depicts the changes in the DCREI measure, denoted by $\mathcal{J}(F,G;t)$, over $(0,1)$ for varying values of t.

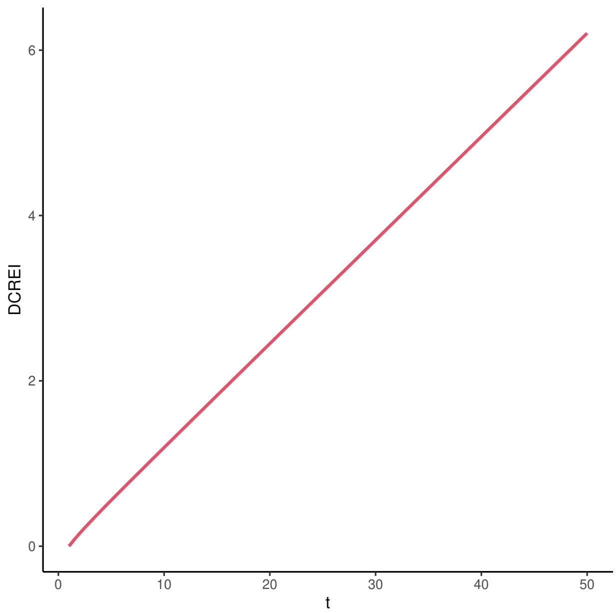

Figure 2: DCREI measure given in Example 2.7.

2.2. Characterization

In the context of the PH equation given by (8), the general characterization problem refers to determining the conditions under which the DCREI measure (15) provides a unique characterization of the distribution function. This problem is of interest because it helps to understand the behavior of the system under certain conditions.

Proposition 2.8. *Under the PH model given by* (8)*, let X and Y be non-negative random variables with respective survival functions denoted by $\overline{F}(x)$ and $\overline{G}(x)$. If for all $t \geq 0$, the function $\mathcal{J}(F; G, t)$ is an increasing function of t, and $\mathcal{J}(F; G, t) > -\infty$, then the survival function $\overline{F}(x)$ of the variable X is uniquely determined by $\mathcal{J}(F; G, t)$.*

Proof. Since X and Y satisfying the PH, the DCRJ (15) can rewrite as

$$\mathcal{J}(F, G; t) \quad = \quad -\frac{1}{2\overline{F}^{\beta+1}(t)} \int_t^\infty \overline{F}^{\delta+1}(x) dx. \tag{17}$$

Differentiating (17) with respect to t on both sides, we obtain

$$\frac{d}{dt}\mathcal{J}(F, G; t) = (\beta + 1)\, \lambda_F(t) \mathcal{J}(F, G; t) + \frac{1}{2}, \tag{18}$$

where $\lambda_F(x) = \frac{f_X(x)}{\bar{F}_X(x)}$ is the HR function.

Assume that F_1, G_1 and F_2, G_2 be two sets of the PDFs satisfying the PH, and let

$$\mathcal{J}(F_1, G_1; t) = \mathcal{J}(F_2, G_2; t), \quad t \geq 0. \tag{19}$$

Differentiating (19) with respect to t on both sides, relation (18) gives

$$\lambda_{F_1}(t)\, (\beta + 1)\, \mathcal{J}(F_1, G_1; t) = \lambda_{F_2}(t)\, (\beta + 1)\, \mathcal{J}(F_2, G_2; t). \tag{20}$$

If for all $t \geq 0$; $\lambda_{F_1}(t) = \lambda_{F2}(t)$, then $\bar{F}_1(x) = \bar{F}_2(x)$ and the proof will be over, otherwise, let $Q = \{t : t \geq 0, \textit{ and } \lambda_{F_1}(t) \neq \lambda_{F_2}(t)\}$, and in following, let the set Q to be non-empty. Hence for some $t_0 \in Q$, $\lambda_{F_1}(t_0) \neq \lambda_{F_2}(t_0)$. Without limiting the scope of the argument, assume that $\lambda_{F_2}(t_0) < \lambda_{F_1}(t_0)$. Using this,

(20) gives $\mathcal{J}(F_1, G_1; t) > \mathcal{J}(F_2, G_2; t)$, for $t = t_0$. Therefore, the set Q is the null set, and this completes the proof. □

One of the important characteristic of a CDF $F(x)$ is the mean residual life (MRL). Let X be a non-negative random variable with CDF $F(x)$, then the MRL of X at time t is given by $\delta_F(t) = E(X - t|X > t)$. Based on the DCREI measure (15), we characterize some specific lifetime distributions in the following result.

Theorem 2.9. *Suppose X and Y are both non-negative continuous random variables that follow the PH* (8)*. Given that X has a MRL of $\delta_F(t)$, then the DCREI measure*

$$\mathcal{J}(F, G; t) = c\delta_F(t),\ c < 0, \tag{21}$$

if, and only if
(a) the random variable X has an exponential distribution for $c = -\frac{1}{2(\beta+1)}$,
(b) the random variable X has an uniform distribution for $-\infty < c_1 = -\frac{1}{\beta+2} < c$,
(c) the random variable X has an finite range distribution for $c < c_2 = -\frac{1}{2(a(\beta+1)+1)} < 0$.

Proof. First we prove the 'if' part. (a) If X has an exponential distribution with $\overline{F}(x) = \exp(-\theta x), \theta > 0$, then the MRL function $\delta_F(t) = \frac{1}{\theta}$. The DCREI measure (15) under PH (2.2) is given as

$$\mathcal{J}(F, G; t) = -\frac{1}{2(\beta + 1)\theta} = c\delta_F(t),$$

for $c = -\frac{1}{2(\beta+1)}$.

(b) If X follows a uniform distribution with $\overline{F}(x) = 1 - x$, and the MRL is $\delta_F(t) = \frac{1-t}{2}$. The DCREI measure (15), under PH (8) is given by

$$\mathcal{J}(F, G; t) = -\frac{1 - t}{2(\beta + 2)} = c\delta_F(t),$$

for $c_1 = -\frac{1}{\beta+2} < c$.

(c) In case X follows a finite range distribution with $f(x) = a(1 - x)^{a-1}, a > 1,\ 0 \le x \le 1$, then the SF is $F(x) = 1 - F(x) = (1 - x)^a$, and the MRL is $\delta_F(t) = \frac{1-t}{a+1}$. The inaccuracy measure (15) under PH (8) is given by

$$\mathcal{J}(F, G; t) = -\frac{1 - t}{2(a(\beta + 1) + 1)} = c\delta_F(t),$$

for $c < c_2 = -\frac{a+1}{2(a(\beta+1)+1)} < 0$. This proves the 'if' part. To prove the ' only if ' part, consider (21) to be valid. Using (17) under PH (8), it gives

$$c\delta_F(t) = -\frac{1}{2\overline{F}^{\beta+1}(t)} \int_t^\infty \overline{F}^{\delta+1}(x)dx. \tag{22}$$

Differentiating it both sides with respect to t, we obtain

$$c\frac{d}{dt}\delta_F(t) = (\beta + 1)\lambda_F(x)(t)\mathcal{J}(F, G; t) + \frac{1}{2}$$

Since $\frac{d}{dt}\delta_F(t) = \lambda_F(t)\delta_F(t) - 1$ and simplify, we obtain

$$\lambda_F(t)\delta_F(t) = \frac{\beta + 1}{c}\lambda_F(t)\mathcal{J}(F, G; t) + \frac{1}{2c} + 1,$$

which implies

$$\frac{d}{dt}\delta_F(t) = \frac{\beta+1}{c}\lambda_F(x)(t)\mathcal{J}(F,G;t) + \frac{1}{2c}.$$

Integrating both sides of this with respect to t over $(0,x)$ yields

$$\delta_F(x) = -\frac{2(\beta+1)c+1}{2\beta c}x + \delta_F(0). \tag{23}$$

A continuous non-negative random variable X has an MRL function $\delta_F(x)$ that is linear in form (23) if, and only if, the distribution of X follows one of three possible distributions: exponential with parameter $c = -\frac{1}{\beta+1}$, uniform with some constant $c_1 < c$, or finite range for some constant $c < c_2$. This information can be found in further detail in Hall and Wellner [7]. The proof of the theorem is complete. □

The following theorem is a new result that extends Theorem 2.9 to allow for c to vary as a function of t. This result is significant because it allows for greater flexibility in the application of Theorem 2.9 in various contexts.

Theorem 2.10. *Assuming that we have two continuous random variables, X and Y, that are both non-negative and follow the PH* (8). *If*

$$\mathcal{J}(F,G;t) = c(t)\delta_F(t), \text{ for } t \geq 0, \tag{24}$$

then

$$\delta(t) = c(t)^{\frac{1}{\beta}}\left[\int_0^t \frac{1+2(\beta+1)c(x)}{2\beta c(x)^{\frac{\beta+1}{\beta}}}dx + C\right],$$

where $C = \delta(0)c(0)^{-\frac{1}{\beta}}$.

Proof. Substituting (24) in (17), we obtain

$$\frac{d}{dt}\mathcal{J}(F,G;t) = (\beta+1)\,\lambda_F(t)\delta(t)c(t) + \frac{1}{2}, \tag{25}$$

differentiating (24) with respect to t and substituting from (25), we obtain

$$\frac{d}{dt}c(t)\delta(t) + c(t)\frac{d}{dt}\delta(t) = (\beta+1)\,\lambda_F(t)\delta(t)c(t) + \frac{1}{2}, \tag{26}$$

substituting $\lambda_F(t)\delta_F(t) = 1 + \frac{d}{dt}\delta(t)$ from (15) and simplifying, we obtain

$$\frac{d}{dt}\delta(t) - \frac{\frac{d}{dt}c(t)}{\beta c(t)}\delta(t) = \frac{1+2(\beta+1)c(t)}{2\beta c(t)}, \tag{27}$$

a linear differential equation in $\delta(t)$. Solving this we obtain (24). □

$$\delta(t) = c(t)^{\frac{1}{\beta}}\left[\frac{(at+b)^{\frac{1-\beta}{\beta}} - b^{\frac{1-\beta}{\beta}}}{a(1-\beta)} + \frac{(at+b)^{\frac{1}{\beta}} - b^{\frac{1}{\beta}}}{a} + \frac{(at+b)^{\frac{1}{\beta}}(\ln(at+b)-(\ln b))}{a} + C\right].$$

This is the final expression for $\delta(t)$ in terms of $c(t) = at+b, a>0, b>0$.

Remark 2.11. (i) For $c(t) = at+b, t>0$ and $a>0$, from (24), we obtain the general model with MRL function

$$\delta(t) = (at+b)^{\frac{1}{\beta}}\left[\frac{(at+b)^{\frac{1-\beta}{\beta}} - b^{\frac{1-\beta}{\beta}}}{a(1-\beta)} + \frac{(at+b)^{\frac{1}{\beta}} - b^{\frac{1}{\beta}}}{a} + \frac{(at+b)^{\frac{1}{\beta}}(\ln(at+b)-(\ln b))}{a} + C\right].$$

(ii) If $a = 0$, we obtain the characterization results given by Theorem 2.9.

(iii) When $\beta = 1$, the expression for $\delta(t)$ simplifies to

$$\delta(t) = c(t)\left[\ln(at+b) - \ln b + C\right].$$

3. Cumulative inaccuracy measure

Hereafter, we introduce a new measure of inaccuracy measure between two non-negative random variables X and Y with the CDFs F and G, respectively. To this end, we define the cumulative inaccuracy (CI) measure by

$$\bar{\mathcal{J}}(X,Y) = -\frac{1}{2}\int_0^\infty F(x)G(x)dx, \tag{28}$$

provided that the integral in the right-hand side of (28) is finite. It is worth mentioning that Eq. (28) measures the difference between the true CDFs of two lifetime random variables, X and Y. In fact, it takes into account the total discrepancy between the actual and predicted values of the SFs over time. In our discussion, we will be referring to two random variables, namely X and Y, which have identical support. If X and Y are identically distributed, then (28) becomes equivalent to the cumulative extropy shown in (5). If two random variables X and $\tilde{X}_\beta$ satisfy the proportional reversed hazard (PRH) model (see e.g. Gupta and Gupta [5]), that is,

$$G(x) = [F(x)]^\beta, \beta > 0. \tag{29}$$

In this case, by substituting (29) in (28), it reduces to

$$\bar{\mathcal{J}}(X, \bar{X}_\beta) = -\frac{1}{2}\int_0^\infty F^{\beta+1}(x)dx,$$

for some constant $\beta > 0$.

Example 3.1. For a non-negative random variable X uniformly distributed over (c, d). The CDF is $F(x) = \frac{x-c}{d-c}$, $c < x < d$. Assuming that the PH holds for random variables X and Y, we can conclude that the CDF of random variable Y is equal to

$$\overline{G}(x) = \left[\frac{d-x}{d-c}\right]^\beta, \; c < x < d, \; \beta > 0. \tag{30}$$

By plugging in the given values into (7) and subsequently simplifying, we obtain the cumulative inaccuracy measure as

$$\mathcal{J}(F;G) = -\frac{d-c}{2(\beta+2)}. \tag{31}$$

3.1. *Dynamic past cumulative inaccuracy measure*

Uncertainty measures in the context of past lifetime distributions have been extensively studied in the literature. For example, Di Crescenzo and Longobardi [3, 4], Nanda and Paul [16], and Kumar *et al.* [13] have all investigated these measures. One specific scenario where uncertainty arises is when a system is observed only at certain preassigned inspection times, and at time t, the system is found to be down. In this case, the uncertainty of the system's remaining life depends on the past, i.e., at which instant in $(0, t)$ the system failed. For $t > 0$, let $X_{(t)} = [X|X \le t]$ and $Y_{(t)} = [Y|Y \le t]$ be their respective past lifetimes. These notions deserve interest in applied fields such as survival analysis, reliability theory, and actuarial science. For each $t > 0$, we denote the distribution functions of $X_{(t)}$ and $Y_{(t)}$ by

$$F_t(x) = \frac{F(x)}{\overline{F}(t)}, \quad G_t(x) = \frac{G(x)}{\overline{G}(t)}, \; 0 \le x \le t, \tag{32}$$

respectively. In analogy with the dynamic cumulative residual extropy inaccuracy measure defined in (15), we now introduce a dynamic version of the cumulative inaccuracy measure referred to the past lifetimes $X_{(t)}$ and $Y_{(t)}$. So, we define the cumulative past inaccuracy (CPI) measure as follows:

$$\begin{aligned}\bar{\mathcal{J}}(F,G;t) &= -\frac{1}{2}\int_0^t F_t(x)G_t(x)dx\\ &= -\frac{1}{2}\int_0^t \frac{F(x)}{F(t)}\frac{G(x)}{G(t)}dx. \end{aligned} \tag{33}$$

When t goes to infinity, the measure (33) reduces to (3). In this situation, the random variable ${}_tX = [X \mid X \leq t]$ is suitable to describe the time elapsed between the failure of a system and the time when it is found to be 'down'. The past lifetime random variable ${}_tX$ is related to two relevant ageing functions, the reversed hazard rate (RHR) defined by $\mu_F(x) = \frac{f(x)}{F(x)}$, and the mean past lifetime (MPT) defined by $\bar{\delta}_F(t) = E(t - X \mid X < t) = \frac{1}{F(t)}\int_0^t F(x)dx$, which are further related as follows:

$$\mu_F(t)\bar{\delta}_F(t) = 1 - \frac{d}{dt}\bar{\delta}_F(t). \tag{34}$$

For further results on PRH function refer to Gupta and Nanda [6].

Example 3.2. Suppose X and Y be two non-negative random variables having distribution functions respectively

$$F(x) = \begin{cases} \exp\{-\frac{1}{2} - \frac{1}{x}\}, & 0 < x \leq 1 \\ \exp\{-2 + \frac{x^2}{2}\}, & 1 < x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

and

$$G(x) = \begin{cases} \frac{x^2}{4}, & 0 < x \leq 2 \\ 1, & x \geq 2 \end{cases}$$

The DCPI measure is given by

$$\bar{\mathcal{J}}(F,G;t) = \begin{cases} -\dfrac{\Gamma\left(-3,\frac{1}{t}\right)\mathrm{e}^{\frac{1}{t}}}{2t^2}, & 0 < t \leq 1 \\ -\dfrac{\Gamma(-3,1)\,\mathrm{e}^{\frac{1}{t}}}{2t^2} - \dfrac{\mathrm{e}^{-\frac{t^2}{2}}\left(\sqrt{\pi}\,\mathrm{i}\left(\mathrm{erf}\left(\frac{\mathrm{i}t}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{\mathrm{i}}{\sqrt{2}}\right)\right) + \sqrt{2}\left(t\mathrm{e}^{\frac{t^2}{2}} - \sqrt{\mathrm{e}}\right)\right)}{2^{\frac{3}{2}}t^2}, & 1 < t \leq 2 \\ -\dfrac{\Gamma(-3,1)\,\mathrm{e}^{\frac{1}{t}}}{2t^2} - \dfrac{\left(\sqrt{\pi}\,\mathrm{i}\left(\mathrm{erf}\left(\sqrt{2}\,\mathrm{i}\right) - \mathrm{erf}\left(\frac{\mathrm{i}}{\sqrt{2}}\right)\right) + 2^{\frac{3}{2}}\mathrm{e}^2 - \sqrt{2}\sqrt{\mathrm{e}}\right)\mathrm{e}^{-\frac{t^2}{2}}}{2^{\frac{3}{2}}t^2} - \dfrac{t-2}{2} & t \geq 2 \end{cases}$$

Graphs of $\bar{\mathcal{J}}(F,G;t))$ for Example (3.2) (left panel) and Example (3.3) (right panel).

Example 3.3. Let X and Y be two non-negative random variables having distribution functions respectively

$$F(x) = \begin{cases} \frac{x^2}{2}, & 0 \leq x < 1 \\ \frac{x^2+2}{6}, & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

and

$$G(x) = \begin{cases} \frac{x^2+x}{4}, & 0 \le x < 1 \\ \frac{x}{2}, & 1 \le x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$$

The DCPI measure is given by

$$\bar{\mathcal{J}}(F,G;t) = \begin{cases} -\dfrac{t\,(4t+5)}{40\,(t+1)}, & 0 < t < 1 \\ -\dfrac{9}{40\,(t^4+t^3)} - \dfrac{t^4+4t^2-5}{8t^3+16t}, & 1 \le t < 2 \\ -\dfrac{9}{40\,(t^4+t^3)} - \dfrac{27}{8\,(t^3+2t)} - \dfrac{t-2}{2}, & t \ge 2 \end{cases}$$

also,the CPI measure is given by

$$\bar{\mathcal{J}}(F,G;t) = \begin{cases} -0.23, & 0 < t \le 1 \\ -0.476, & 1 < t \le 2 \\ -\infty. & t \ge 2 \end{cases}$$

The behaviour of the DCPI measure $\bar{\mathcal{J}}(F,G;t)$ for $t \in (0,\infty)$ is shown in Figure 3.

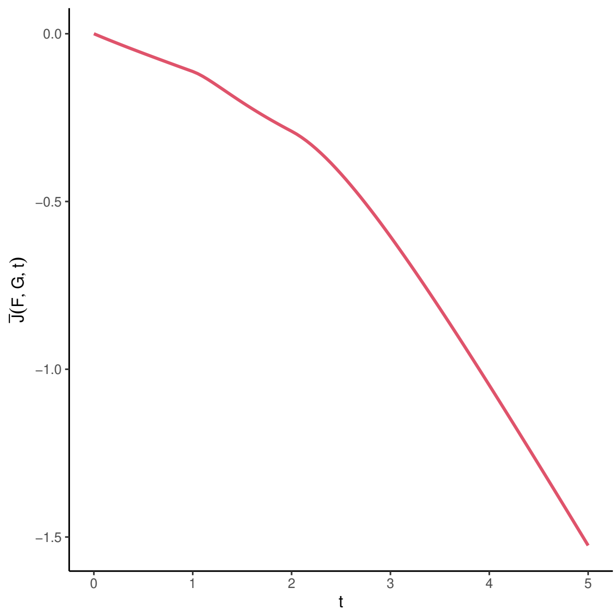

Figure 3: DCREI measure given in Example 3.3.

Analogous to Proposition 2.8, the the next theorem presents the characterization problem for the dynamic measure (33) under the PRHM (29).

Theorem 3.4. *Let X and Y be two non-negative random variables with distribution functions F. and G satisfying the PRHM* (29). *Let* $\xi J(F,G;t) > -\infty,\ \forall t \ge 0$ *be an decreasing function of t, then* $\xi J(F,G;t)$ *uniquely determines the distribution function F of the variable X.*

The proof of the current theorem is similar to the proof presented in Theorem 2.4, and hence omitted.

In the following, we present a characterization of a specific distribution through the utilization of the DCPI measure, which is stated below.

Theorem 3.5. *Assuming that F and G are two CDFs satisfying the PRH model* (29)*, then the DCPI measure*

$$\bar{\mathcal{J}}(F, G; t) = c\eta_F(t), \ 0 > c > -\beta. \tag{35}$$

if, and only if $F(x) = \left(\frac{x}{b}\right)^{-\left(\frac{2c+1}{2c(\beta+1)+1}\right)}, b > 0.$

Proof. Substituting (29) in (33), this gives

$$\xi J(F, G; t) = -\frac{1}{2}\int_0^t \left(\frac{F(x)}{F(t)}\right)^{\beta+1} dx.$$

Differentiating this with respect to t both sides, we obtain

$$\frac{d}{dt}\xi J(F, G; t) = -\frac{1}{2}\left[1 - (\beta+1)\mu_F(t)\int_0^t \left(\frac{F(x)}{F(t)}\right)^{\beta+1} dx\right], \tag{36}$$

Substituting (29) and (33) in Equation (36), we obtain

$$\frac{d}{dt}\xi J(F, G; t) = -(\beta+1)\mu_F(t)\xi J(F, G; t) - \frac{1}{2},$$

Let us take that (35) is valid, then differentiate both side with respect to t, we get

$$\frac{d}{dt}\xi J(F, G; t) = c\frac{d}{dt}\eta_F(t).$$

Put this value into (36), we get

$$c\frac{d}{dt}\eta_F(t) = -(\beta+1)\mu_F(t)\eta_F(t) - \frac{1}{2}.$$

Using (29) and simplify, we obtain

$$\frac{d}{dt}\eta_F(t) = \frac{2c(\beta+1)+1}{2c\beta}. \tag{37}$$

This gives

$$\eta_F(t) = \left(\frac{2c(\beta+1)+1}{2c\beta}\right)t. \tag{38}$$

Divide (37) by (38), we obtain

$$\frac{1 - \frac{d}{dt}\eta_F(t)}{\eta_F(t)} = \mu_F(t) = -\left(\frac{2c+1}{2c(\beta+1)+1}\right)\frac{1}{t}.$$

The RHR and CDF function have a known relationship, which can be expressed as

$$F(x) = e^{\left[\int_0^x \mu_F(t)dt\right]},$$

this gives

$$F(x) = \left(\frac{x}{b}\right)^{-\left(\frac{2c+1}{2c(\beta+1)+1}\right)}, b > 0.$$

Proving the reverse part is a simple and uncomplicated process. □

Example 3.6. Assume X and Y be two non-negative random variables satisfying the PRHM and suppose

$$f_X(x) = \begin{cases} ax^{a-1} & ; \quad \text{if } 0 \leq x < 1,\ a > 0 \\ 0; & \quad \text{o.w.} \end{cases}$$

The distribution function $F(x) = x^a$, and $G(x) = [x]^{a\beta}, \beta > 0$. Substituting these values in (33), after simplification we get

$$\bar{\mathcal{J}}(F, G; t) = -\frac{t}{2(a\beta + a + 1)}, \ a(\beta + 1) \neq -1$$

where $c = \frac{a+1}{2(a\beta+a+1)}$ and mean past lifetime is $\bar{\delta}_F(t) = \frac{t}{a+1}$.

To generalize the result (33) for a specific value of c, we consider the case where c is a function of time t. The following statement presents our result in this more general scenario.

Theorem 3.7. *Let X and Y be two non-negative continuous random variables satisfying the PRHM and*

$$\bar{\mathcal{J}}(F, G; t) = c(t)\eta_F(t), \text{ for } t \geq 0,$$

then

$$\bar{\delta}_F(t) = c(t)^{\frac{1}{\beta}} \left[-\int_0^t \frac{1 + 2(\beta + 1)c(x)}{2\beta c(x)^{\frac{\beta+1}{\beta}}} dx + C \right], \tag{39}$$

where $C = \bar{\delta}_F(0)c(0)^{-\frac{1}{\beta}}$.

Proof. The proof has been omitted since it follows a comparable approach to that of Theorem 2.10. □

Corollary 3.8. *(I) For $c(t) = at + b, t > 0$ and $a > 0$, from* (39)*, we obtain the general model with MRL function*

$$\bar{\delta}_F(t) = (at + b)^{\frac{1}{\beta}} \left[\frac{(at + b)^{\frac{1-\beta}{\beta}} - b^{\frac{1-\beta}{\beta}}}{a(1 - \beta)} - \frac{(at + b)^{\frac{1}{\beta}} - b^{\frac{1}{\beta}}}{a} - \frac{(at + b)^{\frac{1}{\beta}}(\ln(at + b) - (\ln b))}{a} + C \right].$$

(II) If $a = 0$, we obtain the characterization results given by Theorem 3.5.

(III) When $\beta = 1$, the expression for $\delta(t)$ simplifies to

$$\delta(t) = c(t) \left[\ln(at + b) - \ln b + C \right].$$

4. Non-parametric estimator and simulation study

In this section, two estimators are proposed. By using a simulated data set, we illustrate the usefulness of the proposed estimators. Our estimators are based on the estimation of $F(x)$ in the formula of DCREI. One of the traditionally estimator of $F(x)$ is the empirical cumulative distribution function (ECDF) given by

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x),$$

where $I(.)$ stands for an indicator function. Therefore our first estimator is obtained by substituting CDF $F(x)$ by ECDF as

$$\mathcal{J}_1(\hat{F}_n, G; t) = -\frac{1}{2} \int_t^{\infty} \frac{\overline{F}_n(x)\overline{G}(x)}{\overline{F}_n(t)\overline{G}(t)} dx = -\frac{1}{2} \int_t^{\infty} \frac{\sum_{i=1}^{n} I(X_i > x)\overline{G}(x)}{\sum_{i=1}^{n} I(X_i > t)\overline{G}(t)} dx. \tag{40}$$

In recent years, smoothed versions of ECDF have been taken into account by researchers. It can be obtained by integrating $\hat{f}_h$ as

$$\hat{F}_h(x) = \int_{-\infty}^{x} \hat{f}_h(t)dt = \frac{1}{n}\sum_{i=1}^{n} W(\frac{x - X_i}{h_n}), \tag{41}$$

where $\hat{f}_h$ is the kernel density estimation of PDF $f(x)$, and h_n is a bandwidth parameter and $W(x)$ is defined as

$$W(x) = \int_{-\infty}^{x} K(t)dt,$$

where $K(x)$ is a kernel density function. So, our second estimator is obtained by substituting CDF $F(x)$ by $\hat{F}_h(x)$ as

$$\mathcal{J}_2(\hat{F}_h, G; t) = -\frac{1}{2}\int_t^{\infty} \frac{\overline{F}_h(x)\overline{G}(x)}{\overline{F}_h(t)\overline{G}(t)}dx = -\frac{1}{2}\int_t^{\infty} \frac{\sum_{i=1}^{n}(1 - W(\frac{x-X_i}{h_n}))\overline{G}(x)}{\sum_{i=1}^{n}(1 - W(\frac{t-X_i}{h_n}))\overline{G}(t)}dx. \tag{42}$$

In this article, we consider the normal kernel function as $K(x) = 1/\sqrt{2\pi}\exp(-x^2/2)$ and so, $W(x)$ is its corresponding CDF. The issue of selecting bandwidth h_n is crucial due to smoothness. Here, we observed that $h_n = n^{-1/2}$ has the plausible performance in terms of the bias and mean square error (MSE) contrasting other smoothing methods based on h_n.

4.1. Simulation study

For comparing the proposed estimators $\mathcal{J}_1(\hat{F}_n, G; t)$ and $\mathcal{J}_2(\hat{F}_h, G; t)$ for estimating $\mathcal{J}(F, G; t)$, here, we consider some situations and we obtain the bias MSE of the proposed estimators using simulated data.In each case, we run the simulation $B = 5000$ times and we let sample size n vary in $\{25, 50, 75\}$. For the first case, an exponential distribution is selected. We let the parameter $\lambda = 1$ for the actual distribution of data and $\lambda = 2, 5, 7$ for PDF g and $t = 0.5, 1, 1.5$. For the second case, we consider the beta distribution. We let the parameter $(\alpha, \beta) = (1, 1)$ for the actual distribution of data and $(\alpha, \beta) = \{(2, 3), (6, 3), (7, 7)\}$ for PDF g and $t = 0.1, 0.6, 0.9$. The results are provided in Tables 1-2. From the results of these tables, it can be found that both of $\mathcal{J}_1(\hat{F}_n, G; t)$ and $\mathcal{J}_2(\hat{F}_h, G; t)$ perform well in terms of the bias and MSE. Also, the method based on $\mathcal{J}_2(\hat{F}_h, G; t)$ outperform for estimating the DCREI measure in terms of the MSE for most configurations.

exponential(1)			$\lambda = 2$		$\lambda = 5$		$\lambda = 7$	
t	n		$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_1$	$\mathcal{J}_2$
0.5	30	bias	0.0108	0.0110	0.0409	0.0409	0.0320	0.0321
		MSE	0.0004	0.0004	0.0018	0.0017	0.0011	0.0011
	50	bias	0.0111	0.0113	0.0402	0.0403	0.0316	0.0316
		MSE	0.0004	0.0003	0.0017	0.0017	0.0010	0.0010
	75	bias	0.0112	0.0112	0.0406	0.0406	0.0320	0.0320
		MSE	0.0003	0.0003	0.0017	0.0015	0.0010	0.0010
1	30	bias	0.0114	0.0122	0.0404	0.0406	0.0321	0.0321
		MSE	0.0011	0.0007	0.0018	0.0018	0.0011	0.0011
	50	bias	0.0111	0.0114	0.0404	0.0406	0.0318	0.0319
		MSE	0.0007	0.0005	0.0017	0.0017	0.0011	0.0011
	75	bias	0.0116	0.0117	0.0407	0.0407	0.0319	0.0319
		MSE	0.0005	0.0003	0.0017	0.0017	0.0011	0.0010
1.5	30	bias	0.0170	0.0164	0.0399	0.0406	0.0321	0.0324
		MSE	0.0029	0.0018	0.0020	0.0019	0.0012	0.0012
	50	bias	0.0119	0.0121	0.0404	0.0405	0.0315	0.0315
		MSE	0.0012	0.0008	0.0018	0.0018	0.0011	0.0011
	75	bias	0.0117	0.0119	0.0406	0.0406	0.0321	0.0322
		MSE	0.0008	0.0006	0.0018	0.0018	0.0011	0.0010

Table 1: Bias and MSE for $\mathcal{J}_1(\hat{F}_n, G; t)$ and $\mathcal{J}_2(\hat{F}_h, G; t)$ for exponential distribution.

beta(1,1)			$(\alpha,\beta) = (2,3)$		$(\alpha,\beta) = (6,3)$		$(\alpha,\beta) = (7,7)$	
t	n		$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_1$	$\mathcal{J}_2$
0.1	30	bias	0.0415	0.0369	0.0464	0.0381	0.0505	0.0449
		MSE	0.0019	0.0017	0.0026	0.0017	0.0027	0.0023
	50	bias	0.0409	0.0379	0.0451	0.0397	0.0504	0.0465
		MSE	0.0017	0.0015	0.0023	0.0017	0.0027	0.0024
	75	bias	0.0405	0.0385	0.0445	0.0408	0.0495	0.0469
		MSE	0.0017	0.0015	0.0022	0.0016	0.0025	0.0022
0.6	30	bias	0.0158	0.0126	0.0185	0.0141	0.0128	0.0107
		MSE	0.0003	0.0002	0.0004	0.0002	0.0002	0.0001
	50	bias	0.0154	0.0133	0.0180	0.0152	0.0127	0.0112
		MSE	0.0003	0.0002	0.0004	0.0002	0.0002	0.0001
	75	bias	0.0154	0.0139	0.0177	0.0156	0.0126	0.0116
		MSE	0.0003	0.0002	0.0004	0.0001	0.0002	0.0001
0.9	30	bias	0.0019	0.0003	0.0029	0.0003	0.0027	0.0009
		MSE	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	50	bias	0.0039	0.0001	0.0025	0.0013	0.0026	0.0011
		MSE	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	75	bias	0.0035	0.0013	0.0039	0.0014	0.0025	0.0013
		MSE	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2: Bias and MSE for $\mathcal{J}_1(\hat{F}_n, G; t)$ and $\mathcal{J}_2(\hat{F}_h, G; t)$ for beta distribution.

5. Real data

Here, we consider the real data set to show the behavior of the estimators in real world. The following data set has been provided by [1]. They are showing the lifetime of 50 devices.

0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55,60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86

this dataset was also studied by [10]. They fitted some distributions to this data set. Candidate distributions for fitting to this data set are generalized Gompertz (GG), extension of the generalized exponential (EGE) and extension of the generalized Gompertz distribution (EGG). The CDF of EGG is

$$F(x;\alpha,\beta,\tau,c) = (1 - (1 - \frac{\beta\tau}{c}(e^{cx} - 1))^{1/\beta})^{\alpha}. \tag{43}$$

If $\beta = 0$, then (43) reduces to GG which has the following CDF as

$$F(x;\alpha,a,b) = (1 - e^{-\frac{\tau}{c}(e^{cx}-1)})^{\alpha}. \tag{44}$$

Also, if $c \to 0$, then (43) reduces to EGE. The p-value of Kolmogorov-Smirnov (K-S) statistics for EGG, GG and EGE are 0.3041, 0.5273 and 0.1763, respectively. We consider two cases for estimating $\mathcal{J}(F,G;t)$. In both cases, we consider the GG as the actual distribution of the data and EGG or EGE as a one assigned by the experimenter. The values of $\mathcal{J}(F,G;t)$ as well as two estimators $\mathcal{J}_1(\hat{F}_n,G;t)$ and $\mathcal{J}_2(\hat{F}_h,G;t)$ as a function of t are depicted in Figure 4. From Figure 4, it can be seen that the performance of the two estimators is well and in this case we can see that $F_n(x) \approx \hat{F}_h(x)$.

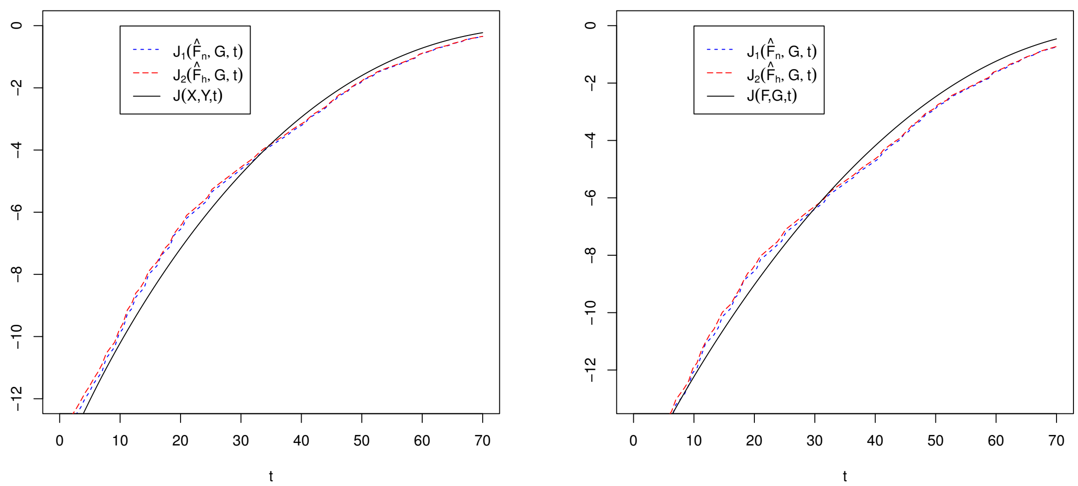

Figure 4: The plot of $\mathcal{J}_1(\hat{F}_n,G;t)$, $\mathcal{J}_2(\hat{F}_h,G;t)$ and $\mathcal{J}(F_n,G;t)$: left plot is GG as actual and EGE as a distribution assigned by experimenter and right plot is GG as actual and EGG as a distribution assigned by the experimenter.

6. Conclusions

This research paper introduced measures for dynamic cumulative residual and past inaccuracy, and explored their characterization results under different hazard models. Specifically, the proportional hazard model was used to analyze dynamic cumulative residual extropy inaccuracy, while the proportional reversed hazard model was used for dynamic cumulative past inaccuracy. We found that these inaccuracy measures can uniquely determine the underlying distributions. Also, we provided characterization results for some specific lifetime models. Finally, we gave two non-parametric estimations for the proposed measure and the efficiency of these estimators was compared via some simulation studies.

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