Filomat 39:8 (2025), 2789–2804
https://doi.org/10.2298/FIL2508789H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Complete convergence and complete f-moment convergence for m-negatively associated random variables

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Abstract. In this paper, the complete convergence and complete f-moment convergence for arrays of rowwise m-negatively associated (m-NA, for short) random variables are studied, which generalize and improve the corresponding ones of Hu et al. (2009) and Wang et al. (2023). The complete moment convergence for arrays of rowwise m-NA random variables is also obtained as an auxiliary conclusion.

1. Introduction

It is known that the complete convergence plays a significant role in probability theory and mathematical statistics. The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows.
Definition 1.1. *A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to the constant c if for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P(|X_n - c| > \varepsilon) < \infty.$$

By the Borel-Cantelli lemma, this implies that $X_n \to c$ almost surely. Hence, the complete convergence is stronger than almost sure convergence.

Chow (1988) introduced the following concept of complete moment convergence, which is much stronger than complete convergence.
Definition 1.2. *For a sequence $\{X_n, n \geq 1\}$ of random variables, if*

$$\sum_{n=1}^{\infty} a_n E(b_n^{-1}|X_n| - \varepsilon)_+^r < \infty$$

2020 *Mathematics Subject Classification*. Primary 60F15.
Keywords. Complete convergence; complete f-moment convergence; complete moment convergence; m-negatively associated random variables.
Received: 05 November 2023; Accepted: 13 Janurary 2025
Communicated by Biljana Popović
Supported by the Key Project of Natural Science Research of Anhui Colleges (2022AH051723), the Outstanding Young and Backbone Teachers' Domestic Visiting and Training Program in Higher Education Institutions (gxgnfx2022061), and the Key Construction Discipline of Chaohu University (kj22zdjsxk01).
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for some $r > 0$ and any $\varepsilon > 0$, where $x_+ = \max\{x, 0\}$, $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are two sequences of positive numbers, then $\{X_n, n \geq 1\}$ is said to exhibit complete r-th moment convergence.

Wu et al. (2019) put forward a more general concept of convergence, i.e., complete f-moment convergence as follows.

Definition 1.3. *Let $\{S_n, n \geq 1\}$ be a sequence of random variables, $\{c_n, n \geq 1\}$ be a sequence of positive constants and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function with $f(0) = 0$. We say that $\{S_n, n \geq 1\}$ converges f-moment completely, if*

$$\sum_{n=1}^{\infty} c_n Ef(\{|S_n| - \varepsilon\}_+) < \infty \quad \textit{for all} \quad \varepsilon > 0,$$

where herein and after, $a_+ = \max\{0, a\}$.

Taking a special case $f(t) = t^r, t \geq 0$, the complete f-moment convergence degenerates to complete r-th moment convergence. Wu et al. (2019) also proved that

$$\sum_{n=1}^{\infty} c_n Ef(\{|S_n| - \varepsilon/2\}_+) \geq \delta \sum_{n=1}^{\infty} c_n P(|S_n| > \varepsilon),$$

where $\delta = f(\varepsilon/2) > 0$. That is to say, the complete f-moment convergence is stronger than complete convergence. Therefore, the study of complete f-moment convergence is of general interest in limit theory.

Let $\{k_n, n \geq 1\}$ be a sequence of positive integers. Hu et al. (2009) established the following result on complete convergence for m-negatively associated (m-NA, for short) random variables. The concept of m-NA random variables will be given in Section 2.

Theorem 1.1. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise m-NA random variables and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that for any $\varepsilon > 0$ and some $\delta > 0$, $\eta \geq 2$,*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) < \infty$$

and

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}|^2 I(|X_{nk}| \leq \delta) \right)^{\eta} < \infty.$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1 \leq j \leq k_n} \left| \sum_{k=1}^{j} (X_{nk} - EX_{nk} I(|X_{nk}| \leq \delta)) \right| > \varepsilon \right) < \infty.$$

Theorem 1.1 was later on generalized to several dependence structures. For examples, Wu et al. (2014) extended it for extended negatively dependent random variables and improved $\eta \geq 2$ to $\eta \geq 1$; Shen et al. (2016) extended it to complete moment convergence for negatively supper-additive dependent random variables; Hu et al. (2015) obtained the desired result for partial sums of extended negatively dependent random variables, which also improves the condition

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}|^2 I(|X_{nk}| \leq \delta) \right)^{\eta} < \infty$$

of Theorem 1.1 to

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\left|X_{nk} I(|X_{nk}| \leq \delta) - EX_{nk} I(|X_{nk}| \leq \delta)\right|^p \right)^{\eta} < \infty$$

for some $\eta > 0$ and $0 < p \le 2$; Wang et al. (2017) extended the results of Hu et al. (2015) for widely orthant dependent random variables under the assumptions

$$\sum_{n=1}^{\infty} c_n g(k_n) \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) < \infty$$

and

$$\sum_{n=1}^{\infty} c_n g(k_n) \left(\sum_{k=1}^{k_n} E\left|X_{nk} I(|X_{nk}| \le \delta) - EX_{nk} I(|X_{nk}| \le \delta)\right|^p \right)^{\eta} < \infty,$$

where $g(k_n)$ is the dominating coefficient of widely orthant dependent random variables. For the definitions of extended negatively dependent random variables, negatively supper-additive dependent random variables and widely orthant dependent random variables, one can see in Liu (2009), Hu (2000), and Wang et al. (2013), respectively.

Recently, Wang et al. (2023) investigated the complete f-moment convergence for m-NA random variables based on Theorem 1.1, as follows.

Theorem 1.2. *Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise m-NA random variables, $\{c_n, n \ge 1\}$ be a sequence of positive constants, $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function with $f(0) = 0$ and $\eta \ge 1$ be a constant. Suppose that the following conditions hold:*

(1) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef(192m|X_{nk}|I(|X_{nk}| > \varepsilon)) < \infty$ for any $\varepsilon > 0$;

(2) there exist constants $0 < p \le 2$ and $\delta > 0$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \le \delta) \right)^2 < \infty;$$

(3) $\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \frac{\delta}{384m}) \to 0$, as $n \to \infty$.

(4) Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be the inverse function for $f(t)$, that is, $g(f(t)) = t, t \ge 0$ and $s(t) = \max_{\delta \le x \le g(t)} \frac{x}{f(x)}$. Assume that the constants η and δ and the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the condition

$$\int_{f(\delta)}^{\infty} g^{-2}(t) s(t)\, dt < \infty.$$

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n Ef\left(\left\{ \max_{1 \le j \le k_n} \left| \sum_{k=1}^{j} (X_{nk} - EX_{nk} I(|X_{nk}| \le \delta)) \right| - \varepsilon \right\}_+ \right) < \infty.$$

We point out that the conditions of Theorem 1.2 are limited. Assumptions (2) and (4) can be improved to much more general case. More details are given in Section 3. For this purpose, the current study will further investigate the complete convergence and the complete f-moment convergence for arrays of rowwise m-NA random variables, which extend and improve Theorem 1.1 and Theorem 1.2 under some weaker conditions.

Throughout this paper, the symbol C represents a positive constant which may vary in different places. Let $I(A)$ be the indicator function of the set A. This work is organized as follows: Some preliminary concepts and lemmas are provided in Section 2. Main results are stated in Section 3. The proofs of the main results are presented in Section 4.

2. Preliminaries

In this section, we recall some concepts of dependent random variables and give some lemmas which will be used in proving our main results.

The concept of negatively associated (NA, for short) random variables was introduced by Joag-Dev and Proschan (1983) as follows.

Definition 2.1. *A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be NA if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$ and any real coordinatewise nondecreasing functions f_1 on $\mathbb{R}^A$ and f_2 on $\mathbb{R}^B$,*

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \le 0,$$

whenever the covariance above exists. An infinite family of random variables is NA if every finite subfamily is NA.

As pointed out and proved by Joag-Dev and Proschan (1983), a number of well known multivariate distributions, such as multinomial, convolution of unlike multinomial, multivariate hypergeometric, Dirichlet, permutation distribution, negatively correlated normal distribution, random sampling without replacement, and joint distribution of ranks all possess the NA property.

As a general extension of NA random variables, the following concept of m-NA random variables was raised by Hu et al. (2007).

Definition 2.2. *Let $m \ge 1$ be a fixed integer. A sequence $\{X_n, n \ge 1\}$ of random variables is said to be m-NA if for any $n \ge 2$ and any $i_1, i_2, \cdots, i_n$ such that $|i_j - i_k| \ge m$ for all $1 \le j \ne k \le n$, we have that $X_{i1}, X_{i2}, \cdots, X_{in}$ are NA.*

In many real-world scenarios, the dependencies between random variables may not conform to strict negative association. The m-NA framework provides a way to model such scenarios by allowing for limited positive associations. On the other hand, verifying the m-NA condition may be easier than verifying the NA condition, especially when dealing with large sets of random variables. Moreover, if we take $m = 1$, then the m-NA structure will degenerate to NA. In summary, the m-NA framework offers a more flexible and general approach to modeling negative dependence structures among random variables, making it advantageous in certain situations compared to the traditional NA framework.

The following lemma is a basic property for m-NA random variables, which can be found in Shen et al. (2015a).

Lemma 2.1. *If $\{X_n, n \ge 1\}$ is a sequence of m-NA random variables and $f_n(\cdot), n \ge 1$ are all nondecreasing (or nonincreasing) functions, then $\{f_n(X_n), n \ge 1\}$ is still m-NA.*

The following exponential inequality for m-NA random variables plays a significant role in the proofs of the main results, which was proved in Remark 2.1 of Wu et al. (2015).

Lemma 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of m-NA random variables with zero means and finite second moments. Denote $B_n =: \sum_{k=1}^{n} EX_k^2$. Then for all $x > 0$, $y > 0$ and $n \ge m$,*

$$P\left(\max_{1\le j\le n}\left|\sum_{k=1}^{j} X_k\right| \ge x\right) \le 2m\sum_{k=1}^{n} P(|X_k| > y) + 8m\left(1 + \frac{3xy}{2mB_n}\right)^{-\frac{x}{12my}}.$$

3. Main results

We now present our main results as follows.

Theorem 3.1. *Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise m-NA random variables and $\{c_n, n \ge 1\}$ be a sequence of positive constants. Suppose that the following two conditions hold:*

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) < \infty$ for any $\varepsilon > 0$;

(ii) there exist $\eta > 0$, $\delta > 0$ and $0 < p \le 2$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\left|X_{nk}I(|X_{nk}| \le \delta) - EX_{nk}I(|X_{nk}| \le \delta)\right|^p\right)^{\eta} < \infty.$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(X_{nk}-EX_{nk}I(|X_{nk}|\le\delta))\right|>\varepsilon\right)<\infty. \tag{1}$$

Remark 3.1. It is obvious that condition (*ii*) in Theorem 3.1 improves the corresponding assumption in Theorem 1.1. Even when we take $p = 2$, we still have

$$\begin{aligned}&\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}|\le\delta)-EX_{nk}I(|X_{nk}|\le\delta)|^2\right)^{\eta}\\ \le\ & C\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}|\le\delta)\right)^{\eta}<\infty.\end{aligned}$$

Moreover, $\eta \ge 2$ in Theorem 1.1 is improved to $\eta > 0$. Hence, the result of Theorem 3.1 generalizes and improves the corresponding one of Theorem 1.1.

Corollary 3.1. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise m-NA random variables with zero means and* $\{c_n, n \ge 1\}$ *be a sequence of positive constants. Suppose that conditions (i) and (ii) of Theorem 3.1 hold. If there exists a constant* $\delta_1 > 0$ *such that* $\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta_1) \to 0$ *as* $n \to \infty$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j} X_{nk}\right|>\varepsilon\right)<\infty. \tag{2}$$

Under some stronger conditions, we can obtain the complete f-moment convergence for arrays of rowwise m-NA random variables as follows.

Theorem 3.2. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise m-NA random variables,* $\{c_n, n \ge 1\}$ *be a sequence of positive constants,* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function with* $f(0) = 0$ *and* $\eta \ge 1$ *be a constant. Suppose that the following conditions hold:*

(a) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef(96m\eta|X_{nk}|I(|X_{nk}| > \varepsilon)) < \infty$ *for any* $\varepsilon > 0$;

(b) there exist constants $0 < p \le 2$ *and* $\delta > 0$ *such that*

$$\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}|\le\delta)-EX_{nk}I(|X_{nk}|\le\delta)|^p\right)^{\eta}<\infty;$$

(c) $\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \frac{\delta}{192m\eta}) \to 0$, *as* $n \to \infty$.

(d) Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be the inverse function for* $f(t)$, *that is,* $g(f(t)) = t, t \ge 0$ *and* $s(t) = \max_{\delta\le x\le g(t)} \frac{x}{f(x)}$. *Assume that the constants* η *and* δ *and the function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfy the condition*

$$\int_{f(\delta)}^{\infty} g^{-\eta}(t)s(t)\,dt<\infty.$$

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n Ef\left(\left\{\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(X_{nk}-EX_{nk}I(|X_{nk}|\le\delta))\right|-\varepsilon\right\}_+\right)<\infty. \tag{3}$$

Furthermore, if $EX_{nk} = 0$ *for each* $1 \le k \le k_n$ *and* $n \ge 1$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n Ef\left(\left\{\max_{1\le j\le k_n}\left|\sum_{k=1}^{j} X_{nk}\right|-\varepsilon\right\}_+\right)<\infty. \tag{4}$$

Remark 3.2. Taking $\eta = 2$, conditions (a), (c) and (d) in Theorem 3.2 equal to corresponding ones in Theorem 1.2. However, condition (b) is still weaker than (2) in Theorem 1.2. We give a simple example to show the superiority of our result to that of Wang et al. (2023).

Example 3.1. Let $p = 2$, $c_n = 1$, $k_n = n$, $X_k \sim U[-\delta, \delta]$ and $X_{nk} = (n^2k)^{-1/4}X_k$ for each $1 \le k \le n$ and $n \ge 1$. Then it is easy to check that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \le \delta) \right)^2 = \frac{\delta^2}{3} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} n^{-1}k^{-1/2} \right)^2 = \infty.$$

That is to say, Theorem 1.2 is unavailable. However, the condition (b) in Theorem 3.2 hold true from

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \le \delta) - EX_{nk}I(|X_{nk}| \le \delta)|^2 \right)^{\eta} = \frac{\delta^2}{3} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} n^{-1}k^{-1/2} \right)^{\eta} < \infty$$

for all $\eta > 2$. Moreover, as pointed out in Wu et al. (2019), if the function $f(x)/x$ is increasing, then the assumption (d) can be written as:

$$\int_{\delta}^{\infty} \frac{f(t)}{t^{1+\eta}}\, dt < \infty.$$

Under this situation, Theorem 1.2 does not work if $f(t) = t^q$ for some $q \ge 2$. However, Theorem 3.2 is also valid. The aforementioned statements reveal that our result improves the corresponding one of Wang et al. (2023).

Taking $f(t) = t^q, t \ge 0, q > 0$ in Theorem 3.2, we can obtain the following complete q-th moment convergence for m-NA random variables.

Theorem 3.3. *Let $q > 0$, $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise m-NA random variables and $\{c_n, n \ge 1\}$ be a sequence of positive constants. Suppose that the following conditions hold:*

(1) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|^q I(|X_{nk}| > \varepsilon) < \infty$ *for any* $\varepsilon > 0$;

(2) there exist constants $\eta > \max(1, q), 0 < p \le 2,$ *and* $\delta > 0$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \le \delta) - EX_{nk}I(|X_{nk}| \le \delta)|^p \right)^{\eta} < \infty;$$

(3) $\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \frac{\delta}{192m\eta}) \to 0$, *as* $n \to \infty$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E \left\{ \max_{1 \le j \le k_n} \left| \sum_{k=1}^{j} (X_{nk} - EX_{nk}I(|X_{nk}| \le \delta)) \right| - \varepsilon \right\}_+^q < \infty.$$

Furthermore, if $EX_{nk} = 0$ *for each* $1 \le k \le k_n$ *and* $n \ge 1$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n E \left\{ \max_{1 \le j \le k_n} \left| \sum_{k=1}^{j} X_{nk} \right| - \varepsilon \right\}_+^q < \infty.$$

Remark 3.3. We point out that (3) and (4) in Theorem 3.2 also hold true under the conditions similar to those in Theorem 3.3 if we take $f(t) = t^q l(t), t \ge 0, q > 0$, where $l(t)$ is any slowly varying function.

4. Proofs of the main results

Proof of Theorem 3.1. Since $\varepsilon > 0$ is arbitrary, we may assume without loss of generality that $\frac{\varepsilon}{48m\eta} < \delta$. The proof will be conducted under the following three cases.

Case 1: $1 \le p \le 2,\ \eta \ge 1$.

Under this case, we denote $N_1 = \left\{n : \sum_{k=1}^{k_n} P(|X_{nk}| > \frac{\varepsilon}{4\eta}) < 1\right\}$. Noting that for $n \in \mathbb{N} - N_1$, $\sum_{k=1}^{k_n} P(|X_{nk}| > \frac{\varepsilon}{4\eta}) \ge 1$, so we can obtain by condition (i) that

$$\begin{aligned}
&\sum_{n\in\mathbb{N}-N_1} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta))\right| > \varepsilon\right)\\
\le\ & \sum_{n\in\mathbb{N}-N_1} c_n \le \sum_{n\in\mathbb{N}-N_1} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \frac{\varepsilon}{4\eta})\\
\le\ & \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \frac{\varepsilon}{4\eta}) < \infty. \qquad (1)
\end{aligned}$$

Therefore, it suffices to consider the case $n \in N_1$. Define for $1 \le k \le k_n$,

$$\begin{aligned}
Y_{nk} &= -\frac{\varepsilon}{48m\eta} I(X_{nk} < -\frac{\varepsilon}{48m\eta}) + X_{nk} I(|X_{nk}| \le \frac{\varepsilon}{48m\eta}) + \frac{\varepsilon}{48m\eta} I(X_{nk} > \frac{\varepsilon}{48m\eta}),\\
Z_{nk} &= \frac{\varepsilon}{48m\eta} I(X_{nk} < -\frac{\varepsilon}{48m\eta}) + X_{nk} I(\frac{\varepsilon}{48m\eta} < |X_{nk}| \le \delta) - \frac{\varepsilon}{48m\eta} I(X_{nk} > \frac{\varepsilon}{48m\eta}),\\
U_{nk} &= X_{nk} I(|X_{nk}| > \delta).
\end{aligned}$$

It is easy to obtain that

$$\begin{aligned}
&\sum_{n\in N_1} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta))\right| > \varepsilon\right)\\
=\ & \sum_{n\in N_1} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(Y_{nk} - EY_{nk} + Z_{nk} - EZ_{nk} + U_{nk})\right| > \varepsilon\right)\\
\le\ & \sum_{n\in N_1} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(Y_{nk} - EY_{nk})\right| > \frac{\varepsilon}{2}\right) + \sum_{n\in N_1} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(Z_{nk} - EZ_{nk})\right| > \frac{\varepsilon}{2}\right)\\
& + \sum_{n\in N_1} c_n P\left(\max_{1\le k\le k_n} |X_{nk}| > \delta\right)\\
=:\ & I_1 + I_2 + I_3. \qquad (2)
\end{aligned}$$

On one hand, by C_r-inequality and Jensen's inequality we have that

$$\begin{aligned}
& E|X_{nk}I(|X_{nk}| \le \frac{\varepsilon}{48m\eta}) - EX_{nk}I(|X_{nk}| \le \frac{\varepsilon}{48m\eta})|^p\\
=\ & E|X_{nk}I(|X_{nk}| \le \delta) - EX_{nk}I(|X_{nk}| \le \delta)\\
& -X_{nk}I(\frac{\varepsilon}{48m\eta} < |X_{nk}| \le \delta) + EX_{nk}I(\frac{\varepsilon}{48m\eta} < |X_{nk}| \le \delta)|^p\\
\le\ & 2^{p-1}E|X_{nk}I(|X_{nk}| \le \delta) - EX_{nk}I(|X_{nk}| \le \delta)|^p\\
& +2^{p-1}E|X_{nk}I(\frac{\varepsilon}{48m\eta} < |X_{nk}| \le \delta) - EX_{nk}I(\frac{\varepsilon}{48m\eta} < |X_{nk}| \le \delta)|^p\\
\le\ & 2^{p-1}E|X_{nk}I(|X_{nk}| \le \delta) - EX_{nk}I(|X_{nk}| \le \delta)|^p
\end{aligned}$$

$$
\begin{aligned}
&+2^{2p-1}E|X_{nk}|^pI(\frac{\varepsilon}{48m\eta}<|X_{nk}|\le\delta)\\
\le\ & 2^{p-1}E|X_{nk}I(|X_{nk}|\le\delta)-EX_{nk}I(|X_{nk}|\le\delta)|^p\\
&+2^{2p-1}\delta^pP(|X_{nk}|>\frac{\varepsilon}{48m\eta}).
\end{aligned}
$$

On the other hand, it follows from Lemma 2.1 that $\{Y_{nk}, 1\le k\le k_n, n\ge 1\}$ is still an array of rowwise m-NA random variables with $|Y_{nk}-EY_{nk}|\le\frac{\varepsilon}{24m\eta}$ for all $1\le k\le k_n$ and $n\ge 1$. Hence, applying Lemma 2.2 with $x=\frac{\varepsilon}{2}$, $y=\frac{\varepsilon}{24m\eta}$ and $B_n=\sum_{k=1}^{k_n}E(Y_{nk}-EY_{nk})^2$, we get by the C_r-inequality and conditions (i) and (ii) that

$$
\begin{aligned}
I_1 \le\ & C\sum_{n\in N_1}c_n\sum_{k=1}^{k_n}P(|Y_{nk}-EY_{nk}|>\frac{\varepsilon}{24m\eta})\\
&+C\sum_{n\in N_1}c_n\left(\sum_{k=1}^{k_n}E(Y_{nk}-EY_{nk})^2\right)^{\eta}\\
\le\ & C\sum_{n\in N_1}c_n\left(\sum_{k=1}^{k_n}E|Y_{nk}-EY_{nk}|^p\right)^{\eta}\\
\le\ & C\sum_{n\in N_1}c_n\left(\sum_{k=1}^{k_n}E|X_{nk}I(|X_{nk}|\le\frac{\varepsilon}{48m\eta})-EX_{nk}I(|X_{nk}|\le\frac{\varepsilon}{48m\eta})|^p\right)^{\eta}\\
&+C\sum_{n\in N_1}c_n\left(\sum_{k=1}^{k_n}P(|X_{nk}|>\frac{\varepsilon}{48m\eta})\right)^{\eta}\\
\le\ & C\sum_{n\in N_1}c_n\left(\sum_{k=1}^{k_n}E|X_{nk}I(|X_{nk}|\le\delta)-EX_{nk}I(|X_{nk}|\le\delta)|^p\right)^{\eta}\\
&+C\sum_{n\in N_1}c_n\left(\sum_{k=1}^{k_n}P(|X_{nk}|>\frac{\varepsilon}{48m\eta})\right)^{\eta}\\
\le\ & C\sum_{n=1}^{\infty}c_n\left(\sum_{k=1}^{k_n}E|X_{nk}I(|X_{nk}|\le\delta)-EX_{nk}I(|X_{nk}|\le\delta)|^p\right)^{\eta}\\
&+C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}P(|X_{nk}|>\frac{\varepsilon}{48m\eta})<\infty. \qquad (3)
\end{aligned}
$$

For I_2, we derive by C_r-inequality and condition (i) that

$$
\begin{aligned}
I_2 \le\ & C\sum_{n\in N_1}c_nE\left|\sum_{k=1}^{k_n}(Z_{nk}-EZ_{nk})\right| \le C\sum_{n\in N_1}c_n\sum_{k=1}^{k_n}E|Z_{nk}|\\
\le\ & C\sum_{n\in N_1}c_n\sum_{k=1}^{k_n}E|X_{nk}|I(\frac{\varepsilon}{48m\eta}<|X_{nk}|\le\delta)+C\sum_{n\in N_1}c_n\sum_{k=1}^{k_n}P(|X_{nk}|>\frac{\varepsilon}{48m\eta})\\
\le\ & C\sum_{n\in N_1}c_n\sum_{k=1}^{k_n}P(|X_{nk}|>\frac{\varepsilon}{48m\eta})\\
\le\ & C\sum_{n=1}^{\infty}c_ng(k_n)\sum_{k=1}^{k_n}P(|X_{nk}|>\frac{\varepsilon}{48m\eta})<\infty. \qquad (4)
\end{aligned}
$$

For I_3, we can obtain by condition (*i*) that

$$I_3 \leq \sum_{n\in N_1} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) < \infty. \tag{5}$$

Hence, the proof of Case 1 is completed by (1)-(5).

Case 2: $1 \leq p \leq 2,\ 0 < \eta < 1$**.**

In this case we define $N_2 = \left\{ n : \sum_{k=1}^{k_n} E\,|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p < 1 \right\}$. Noting that if $n \in \mathbb{N} - N_2$, $\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p \geq 1$, so we have by condition (*ii*) that for any $\eta > 0$,

$$\begin{aligned}
&\sum_{n\in\mathbb{N}-N_2} c_n P\left(\max_{1\leq j\leq k_n} \left| \sum_{k=1}^{j} (X_{nk} - EX_{nk}I(|X_{nk}| \leq \delta)) \right| > \varepsilon \right) \\
\leq\ & \sum_{n\in\mathbb{N}-N_2} c_n \leq \sum_{n\in\mathbb{N}-N_2} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p \right)^{\eta} \\
\leq\ & \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p \right)^{\eta} < \infty.
\end{aligned}$$

It retains to consider the case $n \in N_2$. Note that

$$\begin{aligned}
&\sum_{n\in N_2} c_n \sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p \\
\leq\ & \sum_{n\in N_2} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p \right)^{\eta} \\
\leq\ & \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p \right)^{\eta} < \infty.
\end{aligned}$$

Hence, by applying the result obtained in Case 1 with $\eta = 1$ and $1 \leq p \leq 2$, we obtain the desired result under Case 2.

Up to now, we have proved that (1) holds when $1 \leq p \leq 2$ and $\eta > 0$. We now prove that (1) also holds when $0 < p < 1$ and $\eta > 0$.

Case 3: $0 < p < 1,\ \eta > 0$**.**

Noting that $E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)| \leq 2\delta$, we can obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\,|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)| \right)^{\eta} \\
\leq\ & (2\delta)^{(1-p)\eta} \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p \right)^{\eta} < \infty.
\end{aligned}$$

That is to say, the two conditions of Theorem 3.1 are satisfied with $p = 1$ and $\eta > 0$, which we have proved in the former two cases. In other words, condition (*ii*) under Case 3 is a stronger assumption than that under Cases 1 and 2, which of course can guarantee the validity of the result. Therefore, the proof of Theorem 3.1 is completed. □

Proof of Corollary 3.1. It follows from $\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta_1) \to 0$ and $EX_{nk} = 0$ for each $1 \le k \le k_n$, $n \ge 1$ that there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\max_{1\le j\le k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| \le \delta_1)\right| = \max_{1\le j\le k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| > \delta_1)\right| \le \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta_1) < \frac{\varepsilon}{4}. \quad (6)$$

The rest of the proof will be considered under the following two cases.

Case 1: $\delta_1 \ge \delta$.

Let $N_3 = \left\{n : \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) < \frac{\varepsilon}{4\delta_1}\right\}$. For all $n \in \mathbb{N} - N_3$ we have $\sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \ge \frac{\varepsilon}{4\delta_1}$ and thus by condition (i),

$$\sum_{n\in\mathbb{N}-N_3} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j} X_{nk}\right| > \varepsilon\right) \le \sum_{n\in\mathbb{N}-N_3} c_n \le \frac{4\delta_1}{\varepsilon}\sum_{n\in\mathbb{N}-N_3} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \le \frac{4\delta_1}{\varepsilon}\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) < \infty. \quad (7)$$

It follows by conditions (a) and (b) that the conclusion (1) of Theorem 3.1 holds. Now to obtain (2), it suffices to show that for all n large enough with $n \in N_3$,

$$\max_{1\le j\le k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| \le \delta)\right| < \frac{\varepsilon}{2}.$$

For $n > n_0$ and $n \in N_3$, we obtain by (6) that

$$\begin{aligned}
&\max_{1\le j\le k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| \le \delta)\right| \\
\le\ & \max_{1\le j\le k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| \le \delta_1)\right| + \sum_{k=1}^{k_n} E|X_{nk}|I(\delta < |X_{nk}| \le \delta_1) \\
\le\ & \frac{\varepsilon}{4} + \delta_1 \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \\
\le\ & \frac{\varepsilon}{4} + \delta_1 \cdot \frac{\varepsilon}{4\delta_1} \\
=\ & \frac{\varepsilon}{2},
\end{aligned}$$

which together with (1) obtains

$$\begin{aligned}
&\sum_{n\in N_3} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j} X_{nk}\right| > \varepsilon\right) \\
\le\ & \sum_{n\in N_3, n\le n_0} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j} X_{nk}\right| > \varepsilon\right) \\
& + \sum_{n\in N_3, n>n_0} c_n P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta))\right| + \max_{1\le j\le k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| \le \delta)\right| > \varepsilon\right)
\end{aligned}$$

$$\leq \quad C + \sum_{n\in N_3, n>n_0} c_n P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j}(X_{nk} - EX_{nk}I(|X_{nk}| \leq \delta))\right| > \frac{\varepsilon}{2}\right) < \infty. \tag{8}$$

Hence we obtain the desired result by (7) and (8) for Case 1.

Case 2: $\delta_1 < \delta$.

Let $N_4 = \left\{n : \sum_{k=1}^{k_n} P(|X_{nk}| > \delta_1) < \frac{\varepsilon}{4\delta}\right\}$. Note that if $n \in \mathbb{N} - N_4$, we have $\sum_{k=1}^{k_n} P(|X_{nk}| > \delta_1) \geq \frac{\varepsilon}{4\delta}$. Hence, by condition (i) we have

$$\begin{aligned}\sum_{n\in\mathbb{N}-N_4} c_n P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j} X_{nk}\right| > \varepsilon\right) &\leq \sum_{n\in\mathbb{N}-N_4} c_n \leq \frac{4\delta}{\varepsilon}\sum_{n\in\mathbb{N}-N_4} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \delta_1) \\ &\leq \frac{4\delta}{\varepsilon}\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \delta_1) < \infty. \end{aligned} \tag{9}$$

Similar to the proof of Case 1, we only need to show that for all sufficiently large $n \in N_4$,

$$\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| \leq \delta)\right| < \frac{\varepsilon}{2}.$$

For $n > n_0$ and $n \in N_4$, we obtain

$$\begin{aligned}&\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| \leq \delta)\right| \\ \leq\ & \max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j} EX_{nk}I(|X_{nk}| \leq \delta_1)\right| + \sum_{k=1}^{k_n} E|X_{nk}|I(\delta_1 < |X_{nk}| \leq \delta) \\ \leq\ & \frac{\varepsilon}{4} + \delta \sum_{k=1}^{k_n} P(|X_{nk}| > \delta_1) \\ \leq\ & \frac{\varepsilon}{4} + \delta \cdot \frac{\varepsilon}{4\delta} \\ =\ & \frac{\varepsilon}{2}.\end{aligned}$$

Similar to the proof of (8), we also obtain that

$$\sum_{n\in N_4} c_n P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j} X_{nk}\right| > \varepsilon\right) < \infty. \tag{10}$$

A combination of (9) and (10) gives (2) under Case 2. This completes the proof of the corollary. □

Proof of Theorem 3.2. Since function f is increasing, $m \geq 1$ and $\eta \geq 1$, it follows from condition (a) that

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef(|X_{nk}|I(|X_{nk}| > \varepsilon)) < \infty \text{ for any } \varepsilon > 0.$$

Now we state that the conditions (i) and (ii) of Theorem 3.1 hold. For all $\varepsilon > 0$, it follows from Markov's inequality that

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) \leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef(|X_{nk}|I(|X_{nk}| > \varepsilon)) < \infty,$$

which implies that condition (*i*) of Theorem 3.1 holds. Condition (*ii*) of Theorem 3.1 holds trivially by (*b*). Thus all the conditions of Theorem 3.1 are satisfied.

Denote $S_n = \max_{1\le j\le k_n} |\sum_{k=1}^{j}(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta))|$. It can be checked that

$$\begin{aligned}\sum_{n=1}^{\infty} c_n Ef(\{S_n-\varepsilon\}_+) &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P(S_n > \varepsilon + g(t))dt \\ &= \sum_{n=1}^{\infty} c_n \int_0^{f(\delta)} P(S_n > \varepsilon + g(t))dt + \sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} P(S_n > \varepsilon + g(t))dt \\ &=: J_1 + J_2. \end{aligned} \tag{11}$$

By Theorem 3.1, it follows that

$$J_1 \le f(\delta)\sum_{n=1}^{\infty} c_n P(S_n > \varepsilon) < \infty. \tag{12}$$

Thus, to prove (3), we need to show $J_2 < \infty$. Note that

$$\begin{aligned} J_2 \le\ & \sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} P\left(S_n > g(t), \bigcup_{k=1}^{k_n}\{|X_{nk}| > g(t)\}\right)dt \\ & + \sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} P\left(S_n > g(t), \bigcap_{k=1}^{k_n}\{|X_{nk}| \le g(t)\}\right)dt \\ \le\ & \sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} P\left(\bigcup_{k=1}^{k_n}\{|X_{nk}| > g(t)\}\right)dt \\ & + \sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(X_{nk}I(|X_{nk}| \le g(t)) - EX_{nk}I(|X_{nk}| \le \delta))\right| > g(t)\right)dt \\ =:\ & J_3 + J_4. \end{aligned} \tag{13}$$

It follows from condition (*a*) and Markov's inequality that

$$J_3 \le \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{f(\delta)}^{\infty} P(f(|X_{nk}|) > t)dt \le C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef(|X_{nk}|I(|X_{nk}| > \delta)) < \infty. \tag{14}$$

Now we turn to estimate J_4. For fixed $n \ge 1, 1 \le k \le k_n$ and $t \ge f(\delta)$, denote

$$\begin{aligned}\xi_{nk} &= -g(t)I(X_{nk} < -g(t)) + X_{nk}I(|X_{nk}| \le g(t)) + g(t)I(X_{nk} > g(t)), \\ \eta_{nk} &= -g(t)I(X_{nk} < -g(t)) + g(t)I(X_{nk} > g(t)).\end{aligned}$$

It follows from Lemma 2.1 that $\{\xi_{nk} - E\xi_{nk}, 1 \le k \le k_n, n \ge 1\}$ is still an array of m-NA random variables. Note that

$$\begin{aligned} & P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(X_{nk}I(|X_{nk}| \le g(t)) - EX_{nk}I(|X_{nk}| \le \delta))\right| > g(t)\right) \\ \le\ & P\left(\max_{1\le j\le k_n}\left|\sum_{k=1}^{j}(\xi_{nk} - E\xi_{nk} - \eta_{nk} + E\eta_{nk})\right| + \sum_{k=1}^{k_n} E|X_{nk}|I(\delta < |X_{nk}| \le g(t)) > g(t)\right). \end{aligned} \tag{15}$$

By assumption (c), we can obtain that

$$\max_{t\geq f(\delta)}\frac{1}{g(t)}\sum_{k=1}^{k_n}E|X_{nk}|I(\delta<|X_{nk}|\leq g(t))\leq\delta^{-1}\sum_{k=1}^{k_n}E|X_{nk}|I(|X_{nk}|>\delta)\to 0,\ \text{as}\ n\to\infty.$$

Hence, for all n large enough,

$$\sum_{k=1}^{k_n}E|X_{nk}|I(\delta<|X_{nk}|\leq g(t))<g(t)/2, t\geq f(\delta),$$

which together with (15) yields that for all n large enough,

$$\begin{aligned}
& P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j}(X_{nk}I(|X_{nk}|\leq g(t))-EX_{nk}I(|X_{nk}|\leq\delta))\right|>g(t)\right)\\
\leq\ & P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j}(\xi_{nk}-E\xi_{nk}-\eta_{nk}+E\eta_{nk})\right|>g(t)/2\right)\\
\leq\ & P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j}(\eta_{nk}-E\eta_{nk})\right|>g(t)/4\right)+P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j}(\xi_{nk}-E\xi_{nk})\right|>g(t)/4\right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
J_4\ \leq\ & C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j}(\eta_{nk}-E\eta_{nk})\right|>g(t)/4\right)dt\\
& +C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}P\left(\max_{1\leq j\leq k_n}\left|\sum_{k=1}^{j}(\xi_{nk}-E\xi_{nk})\right|>g(t)/4\right)dt\\
=:\ & J_5+J_6. \qquad (16)
\end{aligned}$$

Noting that $|\eta_{nk}|=g(t)I(|X_{nk}|>g(t))$, we obtain by Markov's inequality and condition (a) that

$$\begin{aligned}
J_5\ \leq\ & C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}\frac{1}{g(t)}\sum_{k=1}^{k_n}E|\eta_{nk}|dt\\
\leq\ & C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}\int_{f(\delta)}^{\infty}P(|X_{nk}|>g(t))dt\\
\leq\ & C\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}Ef(|X_{nk}|I(|X_{nk}|>\delta))<\infty. \qquad (17)
\end{aligned}$$

Hence it suffices to deal with J_6. Noting that $\{\xi_{nk}-E\xi_{nk}, 1\leq k\leq k_n, n\geq 1\}$ is still an array of m-NA random variables, we apply Lemma 2.2 with $x=g(t)/4$ and $y=g(t)/(48m\eta)$, where η satisfies the assumption (d), to obtain

$$\begin{aligned}
J_6\ \leq\ & C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}P\left(\max_{1\leq k\leq k_n}|\xi_{nk}-E\xi_{nk}|>\frac{g(t)}{48m\eta}\right)dt\\
& +C\sum_{n=1}^{\infty}c_n\int_{f(\delta)}^{\infty}\left(\frac{B_n}{g^2(t)}\right)^{\eta}dt\\
=:\ & J_7+J_8, \qquad (18)
\end{aligned}$$

where $B_n = \sum_{k=1}^{k_n} E(\xi_{nk} - E\xi_{nk})^2$.

It follows from assumption (c) and Markov's inequality that $\sum_{k=1}^{k_n} P(|X_{nk}| > \frac{\delta}{192m\eta}) \to 0$ as $n \to \infty$. Consequently, for all n large enough, $\sum_{k=1}^{k_n} P(|X_{nk}| > \frac{\delta}{192m\eta}) \leq \frac{1}{384m\eta}$. Thus,

$$
\begin{aligned}
& \max_{t\geq f(\delta)} \max_{1\leq k\leq k_n} \frac{1}{g(t)}|E\xi_{nk}| \leq \max_{t\geq f(\delta)} \max_{1\leq k\leq k_n} \frac{1}{g(t)}E|\xi_{nk}| \\
\leq\ & \max_{t\geq f(\delta)} \max_{1\leq k\leq k_n} [\frac{1}{g(t)}E|X_{nk}|I(|X_{nk}| \leq \frac{\delta}{192m\eta}) + \frac{1}{g(t)}E|X_{nk}|I(\frac{\delta}{192m\eta} < |X_{nk}| \leq g(t)) \\
& +P(|X_{nk}| > g(t))] \\
\leq\ & \delta^{-1} \cdot \frac{\delta}{192m\eta} + 2\sum_{k=1}^{k_n} P(|X_{nk}| > \frac{\delta}{192m\eta}) \\
\leq\ & \frac{1}{192m\eta} + 2 \cdot \frac{1}{384m\eta} = \frac{1}{96m\eta}.
\end{aligned}
$$

Hence by condition (a) we have

$$
\begin{aligned}
J_7 &\leq C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} P\left(\max_{1\leq k\leq k_n} |\xi_{nk}| > \frac{g(t)}{96m\eta}\right) dt \\
&\leq C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} P\left(\max_{1\leq k\leq k_n} |X_{nk}| > \frac{g(t)}{96m\eta}\right) dt \quad (\text{since } |\xi_{nk}| \leq |X_{nk}|) \\
&\leq C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{f(\delta)}^{\infty} P\left(|X_{nk}| > \frac{g(t)}{96m\eta}\right) dt \\
&\leq C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef(96m\eta|X_{nk}|I(|X_{nk}| > \delta)) < \infty. \qquad (19)
\end{aligned}
$$

By C_r-inequality and Markov's inequality, we have

$$
\begin{aligned}
J_8 &= C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} g^{-2\eta}(t) B_n^{\eta}\, dt \\
&= C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} g^{-2\eta}(t) \left(\sum_{k=1}^{k_n} E(\xi_{nk} - E\xi_{nk})^2\right)^{\eta} dt \\
&\leq C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} g^{-2\eta}(t) \left(\sum_{k=1}^{k_n} E(X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta))^2\right)^{\eta} dt \\
&\quad +C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} g^{-2\eta}(t) \left(\sum_{k=1}^{k_n} EX_{nk}^2 I(\delta < |X_{nk}| \leq g(t))\right)^{\eta} dt \\
&\quad +C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} \left(\sum_{k=1}^{k_n} P(|X_{nk}| > g(t))\right)^{\eta} dt \\
&=: J_9 + J_{10} + J_{11}. \qquad (20)
\end{aligned}
$$

Noting that $|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)| \leq 2\delta$, we obtain by assumptions (b) and (d) that

$$
J_9 \leq C(2\delta)^{(2-p)\eta} \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}I(|X_{nk}| \leq \delta) - EX_{nk}I(|X_{nk}| \leq \delta)|^p\right)^{\eta} \int_{f(\delta)}^{\infty} g^{-2\eta}(t)\, dt
$$

$$< \quad \infty. \tag{21}$$

For J_{10}, it follows from condition (c) that for all n large enough,

$$\sum_{k=1}^{k_n} E|X_{nk}|I(\delta < |X_{nk}| \le g(t)) \le \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) \le \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \frac{\delta}{192m\eta}) < 1,$$

which together with $\eta \ge 1$ yields that

$$\left(\sum_{k=1}^{k_n} E|X_{nk}|I(\delta < |X_{nk}| \le g(t))\right)^{\eta} \le \sum_{k=1}^{k_n} E|X_{nk}|I(\delta < |X_{nk}| \le g(t)).$$

Therefore, by conditions (a) and (d) we have

$$\begin{aligned}
J_{10} &\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} g^{-2\eta}(t)\left(g(t)\sum_{k=1}^{k_n} E|X_{nk}|I(\delta < |X_{nk}| \le g(t))\right)^{\eta} dt \\
&\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} g^{-\eta}(t)\left(\sum_{k=1}^{k_n} E\frac{|X_{nk}|I(\delta < |X_{nk}| \le g(t))}{f(|X_{nk}|I(\delta < |X_{nk}| \le g(t)))} f(|X_{nk}|I(\delta < |X_{nk}| \le g(t)))\right) dt \\
&\le C\sum_{n=1}^{\infty} c_n \int_{f(\delta)}^{\infty} g^{-\eta}(t)s(t)\left(\sum_{k=1}^{k_n} Ef(|X_{nk}|I(\delta < |X_{nk}| \le g(t)))\right) dt \\
&\le C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef(|X_{nk}|I(|X_{nk}| > \delta)) \int_{f(\delta)}^{\infty} g^{-\eta}(t)s(t)\, dt < \infty.
\end{aligned} \tag{22}$$

Finally, we will show that $J_{11} < \infty$. Noting that $t \ge f(\delta)$, it follows from Markov's inequality and condition (c) that

$$\sum_{k=1}^{k_n} P(|X_{nk}| > g(t)) \le \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \le \delta^{-1} \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) \to 0, \text{ as } n \to \infty,$$

which implies that for all n large enough,

$$\sum_{k=1}^{k_n} P(|X_{nk}| > g(t)) < 1$$

and thus for all $t \ge f(\delta)$,

$$\left(\sum_{k=1}^{k_n} P(|X_{nk}| > g(t))\right)^{\eta} \le \sum_{k=1}^{k_n} P(|X_{nk}| > g(t)).$$

Hence, we have by condition (a) that

$$\begin{aligned}
I_{11} &\le C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{f(\delta)}^{\infty} P(|X_{nk}| > g(t))\, dt \\
&\le C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} Ef(|X_{nk}|I(|X_{nk}| > \delta)) < \infty.
\end{aligned} \tag{23}$$

Therefore, by (11)-(23), (3) follows immediately. The proof of (4) is completely analogous to that of Corollary 3.1 and thus is omitted here. The proof of the theorem is completed. □

Acknowledgements

The authors are very grateful to the anonymous referees and the editor for their many valuable comments and suggestions.

Availability of data and materials

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that there are no competing interests.

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