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Consecutive optimization of the weighted quadrature formulas with derivative

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Abstract. This article addresses the derivation and analysis of a weighted optimal quadrature formula in the Hilbert space $W_2^{(2,1)}(0,1)$, where functions φ with prescribed properties reside. The quadrature formula is expressed as a linear combination of function values and its first-order derivative at equidistant nodes in the interval $[0,1]$. The coefficients are determined by minimizing the norm of the error functional in the dual space $W_2^{(2,1)*}(0,1)$. The error functional is defined as the difference between the integral of a function over the interval and the quadrature approximation. The key results include explicit expressions for the coefficients and the norm of the error functional.

The optimization problem is formulated and solved, leading to a system of linear equations for the coefficients. Analytical solutions of the system are obtained via the Sobolev method, which provides an explicit expression for the optimal coefficients. The convergence with the exact values of the integrals is analyzed via numerical experiments.

1. Introduction

Quadrature formulas are used to estimate definite integrals and are essential for solving differential and integral equations numerically. These formulas involve nodes and coefficients, and there are various methods available to construct them. One such method is the variational approach, which is based on the principles and functionals of variational calculus. The primary objective of this approach is to identify the nodes and weights that minimize or maximize a specific functional related to the estimation of the integral. Several studies have been conducted on this approach [4, 5, 7, 9, 12, 16, 21, 23, 24, 27, 37].

In general, the following quadrature rules can be used to compute integrals with a weight function

$$\int_a^b p(x)f(x)\mathrm{d}x \approx \sum_{\beta=1}^{N} C_\beta f(x_\beta),$$

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where $p(x)$ is a integrable over the interval (a, b) weight function. By including the effect of $p(x)$ in the quadrature weights and points, one can obtain an accurate approximation of the integral even for integrands that behave poorly, such as singular, discontinuous, or highly oscillatory functions.

The main advantage of using weighted quadrature rules is that they can handle the "bad" behaviour of $p(x)$, which is known and can be factored out into the weight function. By designing a quadrature rule with $p(x)$ taken into account, one can achieve fast convergence, provided that the remaining factor $f(x)$ is smooth. Moreover, the same rule can be used for different $f(x)$ as long as $p(x)$ remains the same.

The challenges inherent in devising weighted quadrature formulas involving derivatives have garnered significant attention from researchers, yielding various findings across different domains.

For instance, in the study by Shadimetov and Nuraliev [31], the quadrature summation comprises values of the integrand at internal nodes alongside the first, third, and fifth derivatives of the integrand at the endpoints of the integration interval. The optimal coefficients for quadrature formulas were determined, and the norm of the optimal error functional was computed for any natural number N and any $m \geq 6$ utilizing the Sobolev method. This approach relies on a discrete analogue of the differential operator d^{2m}/dx^{2m}. Notably, for $m = 6, 7$, the optimality of the classical Euler-Maclaurin quadrature formula was established. Beyond $m = 8$, novel optimal quadrature formulas were derived. Similarly, in [30], for $m = 4$ and $m = 5$, the classical Euler-Maclaurin quadrature formula was found to be optimal. Starting from $m = 6$, new optimal quadrature formulas were obtained. In [19], such results were also obtained.

A large number of scientific studies have been devoted to the construction of optimal quadrature formulas with weight $p(x) = e^{2\pi i \omega x}$ (highly oscillatory) in the sense of Sard and their application. Previous studies [8, 10, 14, 15, 20, 22, 33, 38] have focused on the construction of optimal quadrature formulas for the numerical integration of Fourier integrals in the Hilbert and Sobolev spaces and applied these results to computed tomography problems.

Even in cases where $p(x)$ is weakly singular [3, 6, 11, 13, 32] and singular [1, 2, 28], optimal quadrature formulas were constructed in different Hilbert and Sobolev spaces.

The paper is structured as follows: Section 2 presents the auxiliary results; Section 3 presents the problem; Section 4 estimates the error functional of the quadrature formula; Section 5 finds the extremum of the error functional norm; Section 6 discusses the algorithm for solving the system of equations; Section 7 obtains the analytical forms of the optimal coefficients; and Section 8 presents some discussion and numerical results.

2. Auxiliary results

In [29], the following weighted optimal quadrature formula in $L_2^{(1)}(0, 1)$ space:

$$\int_0^1 p(x)\varphi(x)\mathrm{d}x \cong \sum_{\beta=0}^{N} C_{\beta,0}\varphi(x_\beta) \tag{1}$$

was constructed, where $x_\beta = h\beta \in [0, 1]$, $h = \frac{1}{N}$ is the step size, weight function $p(x)$ is the integrable over the interval $[0, 1]$, and a function $\varphi(x)$ belongs to the Sobolev space

$$L_2^{(1)}(0, 1) = \{\varphi : [0, 1] \to \mathbb{R} \mid \varphi \text{ is abs. cont. and } \varphi' \in L_2(0, 1)\}.$$

The coefficients $C_{\beta,0}$ of the quadrature formula (1) have the following form ($\beta = \overline{0, N}$):

$$C_{\beta,0} = \frac{h^{-1}}{2}\int_0^1 p(x)\Big(|x - h(\beta - 1)| - 2|x - h\beta| + |x - h(\beta + 1)|\Big)\mathrm{d}x. \tag{2}$$

The quadrature formula of the form (1) was constructed in the work [17] in the space $W_2^{(1,0)}(0, 1)$.

To achieve high accuracy, we are developing an optimal quadrature formula that includes a derivative.

3. Statement of the problem

Given a table of function values $\varphi(x_\beta)$ and its first derivative $\varphi'(x_\beta)$ at points $x_\beta \in [0,1]$, where $\beta = 0,1,\dots,N$, we assume that this function belongs to the Hilbert space $W_2^{(2,1)}(0,1) = \{\varphi : [0,1] \to \mathbb{R} \mid \varphi'$ is abs. cont. and $\varphi'' \in L_2(0,1)\}$, equipped with the norm

$$\left\|\varphi | W_2^{(2,1)}(0,1)\right\| = \left\{\int\limits_0^1 (\varphi''(x) + \varphi'(x))^2 dx\right\}^{1/2}, \tag{3}$$

where $\int\limits_0^1 (\varphi''(x) + \varphi'(x))^2 dx < \infty$. We can easily verify that (3) is a semi-norm.

We denote by $P = span\{1, e^{-x}\}$ the space consisting of all possible linear combinations of functions 1 and e^{-x}. We form a factor space of $W_2^{(2,1)}(0,1)$ space with respect to P. That is, if $f_1 - f_2 \in P$, we take $f_1 \sim f_2$ and say that they belong to one class, and we define this class as $\overline{f}(x) = f(x) + P$. Here

$$f_1(x) - f_2(x) = a_0 + a_1 e^{-x} \ \Rightarrow \ f_1(x) = f_2(x) + a_0 + a_1 e^{-x},$$

a_0 and a_1 are real numbers. In particular, the zero element $\overline{\theta}(x)$ of this factor space has the following form $\overline{\theta}(x) = a_0 + a_1 e^{-x}$.

For convenience, let's denote the resulting factor space $W_2^{(2,1)}(0,1)/P$ as $W_2^{(2,1)}(0,1)$ again. Furthermore, when working with elements of this space, such as the element $\overline{f}(x)$, it is sufficient to work with any function $f(x)$ belonging to this class. For the elements of the resulting factor space, the relationship defined by equation (3) satisfies all conditions of a norm. In particular, $\|\varphi\|_{W_2^{(2,1)}} = 0$ if and only if $\varphi(x) = \overline{\theta}(x) = a_0 + a_1 e^{-x}$.

And the inner product in this space is defined by the following formula

$$\langle \varphi, \psi \rangle = \int\limits_0^1 (\varphi''(x) + \varphi'(x))\,(\psi''(x) + \psi'(x))\,dx. \tag{4}$$

We consider a quadrature formula in the space $W_2^{(2,1)}(0,1)$ given by

$$\int_0^1 p(x)\varphi(x)\mathrm{d}x \cong \sum_{\beta=0}^{N} C_{\beta,0}\varphi(x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\varphi'(x_\beta). \tag{5}$$

Constructing an optimized quadrature formula by both coefficients $C_{\beta,0}$ and $C_{\beta,1}$ can be a complex problem. The process involves successive optimizations. Initially, we take the coefficients (2) rather than $C_{\beta,0}$ and optimize the quadrature formula (5) with respect to the coefficients $C_{\beta,1}$. It is worth noting that in [18], the optimal quadrature formula was constructed using this sequence while considering the constant weight function in the $W_2^{(2,1)}(0,1)$ space. In this construction, the coefficients of the trapezoidal formula are obtained as the coefficients $C_{\beta,0}$. Furthermore, the coefficients $C_{\beta,1}$ are found, leading to the optimal quadrature formula of the Euler-Maclaurin type.

Furthermore, we studied the difference

$$(\ell, \varphi) = \int_0^1 p(x)\varphi(x)dx - \sum_{\beta=0}^{N} C_{\beta,0}\varphi(x_\beta) - \sum_{\beta=0}^{N} C_{\beta,1}\varphi'(x_\beta). \tag{6}$$

The difference (6) is called *the error* of the quadrature formula (5). Here

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx$$

and

$$\ell(x) = p(x)\varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_{\beta,0}\delta(x - x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(x - x_\beta) \tag{7}$$

is *the error functional* of the quadrature formula (5) and belongs to the space $W_2^{(2,1)*}(0,1)$. The space $W_2^{(2,1)*}(0,1)$ is dual to the space $W_2^{(2,1)}(0,1)$. Here, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, and $\delta(x)$ is Dirac's delta-function.

From (7), it is clear that the error functional ℓ defined in the space $W_2^{(2,1)}(0,1)$ satisfies the following equalities: (see [25, 26])

$$(\ell, e^{-x}) = 0 \tag{8}$$

$$(\ell, 1) = 0. \tag{9}$$

Applying the Cauchy-Schwarz inequality, we estimate the absolute value of the error (6) as follows:

$$|(\ell,\varphi)| \le \|\varphi\|_{W_2^{(2,1)}} \cdot \|\ell\|_{W_2^{(2,1)*}},$$

where

$$\|\ell\|_{W_2^{(2,1)*}} = \sup_{\varphi,\ \|\varphi\|\neq 0} \frac{|(\ell,\varphi)|}{\|\varphi\|}.$$

From this, we derive the following problem.

Problem 1. Find the norm of the error functional ℓ of the quadrature formula (5) in the space $W_2^{(2,1)*}(0,1)$.

It is evident that the norm of the error functional ℓ depends on the coefficients $C_{\beta,1}$ and the nodes x_β. Minimizing $\|\ell\|$ by adjusting the coefficients $C_{\beta,1}$, is a straightforward linear problem. However, when we try to minimize $\|\ell\|$ by the coefficients $C_{\beta,1}$ and the nodes x_β, this becomes a more complicated and nonlinear problem. Our goal is to minimize $\|\ell\|$ by finding the optimal values for the coefficients $C_{\beta,1}$ while keeping the nodes x_β fixed.

The primary objective of this study is outlined as follows:

Problem 2. Find the coefficients $\mathring{C}_{\beta,1}$ that minimize the value of $\|\ell\|_{W_2^{(2,1)*}}$, and compute

$$\left\|\mathring{\ell}\right\|_{W_2^{(2,1)*}} = \inf_{C_{\beta,1}} \|\ell\|_{W_2^{(2,1)*}}. \tag{10}$$

The functional $\mathring{\ell}$ in (7) is termed the error functional corresponding to *the optimal quadrature formula* in $W_2^{(2,1)}(0,1)$, and $\mathring{C}_{\beta,1}$ are called the optimal coefficients. For convenience, we will retain the optimal coefficients $\mathring{C}_{\beta,1}$ as $C_{\beta,1}$.

4. Estimation of the error functional

To solve Problem 1, our initial step is to determine the norm of the error functional. According to the Riesz theorem, any linear continuous functional ℓ in a Hilbert space is represented in the form of inner product (4). Therefore, in our case, for any function φ from $W_2^{(2,1)}(0,1)$ space, we have

$$(\ell,\varphi) = \langle \psi_\ell, \varphi\rangle.$$

Here, ψ_ℓ is a function from $W_2^{(2,1)}(0,1)$ that is defined uniquely by the functional ℓ and is an extremal function (see, [10])

$$\psi_\ell(x) = \ell(x) * G_2(x) + de^{-x} + p_0. \tag{11}$$

Herein

$$G_2(x) = \frac{\mathrm{sign}(x)}{2}\left(\frac{e^x - e^{-x}}{2} - x\right). \tag{12}$$

Now we obtain the norm of the error functional ℓ. Since the space $W_2^{(2,1)}(0,1)$ is the Hilbert space then by the Riesz theorem we have

$$(\ell, \psi_\ell) = \|\ell\| \cdot \|\psi_\ell\| = \|\ell\|^2.$$

Specifically, using (11) and (12), we derive

$$\begin{aligned} \|\ell\|^2_{W_2^{(2,1)*}} &= \int_0^1\int_0^1 p(x)p(y)G_2(x-y)\mathrm{d}x\mathrm{d}y - 2\sum_{\beta=0}^N C_{\beta,0}\int_0^1 p(x)G_2(x-x_\beta)\mathrm{d}x \\ &+\sum_{\beta=0}^N\sum_{\gamma=0}^N C_{\beta,0}C_{\gamma,0}G_2(x_\beta - x_\gamma) - \sum_{\beta=0}^N\sum_{\gamma=0}^N C_{\beta,1}C_{\gamma,1}G_2''(x_\beta - x_\gamma) \\ &+2\sum_{\beta=0}^N C_{\beta,1}\left(\int_0^1 p(x)G_2'(x-x_\beta)\mathrm{d}x + \sum_{\gamma=0}^N C_{\gamma,0}G_2'(x_\beta - x_\gamma)\right). \end{aligned} \tag{13}$$

5. The error functional norm

We consider the Lagrange function

$$\Phi(C_{0,1}, C_{1,1}, \ldots, C_{N,1}, \lambda) = \|\ell\|^2_{W_2^{(2,1)*}} - 2\lambda(\ell, e^{-x})$$

to find the minimum of (13) under the conditions (8) and (9).

We take the first partial derivatives with respect to $C_{\beta,1}$ ($\beta = 0, 1, \ldots, N$) and λ of the function Φ and equating them to zero, we obtain the following system of linear equations

$$\sum_{\gamma=0}^N C_{\gamma,1}G_2''(x_\beta - x_\gamma) + \lambda e^{-x_\beta} = F_2(x_\beta), \ \ \beta = 0, 1, \ldots, N, \tag{14}$$

$$\sum_{\gamma=0}^N C_{\gamma,1}e^{-x_\gamma} = g, \tag{15}$$

where

$$F_2(h\beta) = \sum_{\gamma=0}^N C_{\gamma,0}G_2'(x_\beta - x_\gamma) + \int_0^1 p(x)G_2'(x - x_\beta)\mathrm{d}x, \tag{16}$$

$$g = \sum_{\gamma=0}^N C_{\gamma,0}e^{-x_\gamma} - \int_0^1 p(x)e^{-x}\mathrm{d}x, \tag{17}$$

and the function $G_2(x)$ is defined from (12). Additionally, generalized derivatives of this function have the following forms

$$G_2'(x) = \frac{\mathrm{sign}(x)}{2}\left(\frac{e^x + e^{-x}}{2} - 1\right) \text{ and } G_2''(x) = \frac{\mathrm{sign}(x)}{2}\left(\frac{e^x - e^{-x}}{2}\right).$$

$G_2''(x)$ is the function $G_1(x)$ which is a fundamental solution to the differential operator $\frac{d^2}{dx^2}-1$ (see [11]), that is,

$$G_1(x) = \frac{\text{sign}(x)}{2}\left(\frac{e^x - e^{-x}}{2}\right). \tag{18}$$

In equations (14) and (15), there are $N+1$ unknowns and $N+1$ equations. For a solution to exist for this system of equations, it is sufficient that $N \geq 1$.

This system of linear equations (14) and (15) has a unique solution, as shown in [3], and its solution gives a minimum value to the expression (13).

Next, an analytical solution to the system of equations (14) and (15) is derived.

Assuming $C_{\beta,1} = 0$ for $\beta < 0$ and $\beta > N$, and using the convolution of discrete functions [34], we rewrite the system of equations (14) and (15) as equations in convolutions

$$\begin{cases} C_{\beta,1} * G_1(x_\beta) + \lambda e^{-x_\beta} = F_2(x_\beta), & 0 \leq \beta \leq N, \\ C_{\beta,1} = 0, & \beta < 0 \text{ and } \beta > N, \\ \sum_{\gamma=0}^{N} C_{\gamma,1} e^{-x_\gamma} = g, & \end{cases} \tag{19}$$

where $x_\beta = h\beta$.

6. The algorithm for solving the system of linear equations

Here, we use the method suggested by Sobolev [34–36] for the discrete analogue of the differential operator $\frac{d^2}{dx^2}-1$. The discrete analogue of the operator $\frac{d^2}{dx^2}-1$ has the following form [10]:

$$D_1(h\beta) = \frac{1}{1-e^{2h}} \begin{cases} 0, & |\beta| \geq 2, \\ -2e^h, & |\beta| = 1, \\ 2(1+e^{2h}), & \beta = 0. \end{cases} \tag{20}$$

The solution of the system (19) is found by introducing the function $U_2(h\beta)$ in place of $C_{\beta,1}$

$$U_2(h\beta) = C_{\beta,1} * G_1(h\beta) + \lambda e^{-h\beta}, \tag{21}$$

and the operator $D_1(h\beta)$ satisfying $D_1(h\beta) * G_1(h\beta) = \delta_d(h\beta)$, where $\delta_d(h\beta)$ is the discrete delta-function. The coefficients are expressed as

$$C_{\beta,1} = D_1(h\beta) * U_2(h\beta). \tag{22}$$

If we determine the function $U_2(h\beta)$ for all integer values β, we can find the coefficients $C_{\beta,1}$ from equation (22).

From expression (21), taking into account (18), we have

$$\begin{aligned} U_2(h\beta) &= C_{\beta,1} * G_1(h\beta) + \lambda e^{-h\beta} \\ &= \sum_{\gamma=-\infty}^{\infty} C_{\gamma,1} \frac{\text{sign}(h\beta - h\gamma)}{4}\left(e^{h\beta-h\gamma} - e^{h\gamma-h\beta}\right) + \lambda e^{-h\beta}. \end{aligned}$$

According to the assumption that $C_{\beta,1} = 0$ when $\beta = -1, -2, \ldots,$ and $\beta = N+1, N+2, \ldots$, we obtain

$$U_2(h\beta) = \sum_{\gamma=0}^{N} C_{\gamma,1} \frac{\text{sign}(h\beta - h\gamma)}{4}\left(e^{h\beta-h\gamma} - e^{h\gamma-h\beta}\right) + \lambda e^{-h\beta}. \tag{23}$$

From expression (23) when $\beta < 0$ and $\beta > N$, taking into account the last equation of the system (19), we obtain

$$U_2(h\beta) = \begin{cases} -\frac{e^{h\beta}}{4}g + (\lambda + L)e^{-h\beta}, & \beta < 0, \\ F_2(h\beta), & 0 \le \beta \le N, \\ \frac{e^{h\beta}}{4}g + (\lambda - L)e^{-h\beta}, & \beta > N, \end{cases} \tag{24}$$

where, $L = \frac{1}{4}\sum_{\gamma=0}^{N} C_{\gamma,1}e^{h\gamma}$. Using (20) and (22) we find the unknowns $\lambda + L$ and $\lambda - L$ when $\beta = -1$ and $\beta = N + 1$ in (24), respectively. Therefore, we obtain the following

$$\lambda + L = F_2(0) + \frac{g}{4} \text{ and } \lambda - L = F_2(1)e - \frac{e^2 g}{4}.$$

We rewrite the function $U_2(h\beta)$ taking into account the last equalities

$$U_2(h\beta) = \begin{cases} -\frac{e^{h\beta}}{4}g + \left(F_2(0) + \frac{g}{4}\right)e^{-h\beta}, & \beta < 0, \\ F_2(h\beta), & 0 \le \beta \le N, \\ \frac{e^{h\beta}}{4}g + \left(F_2(1) - \frac{e}{4}g\right)e^{1-h\beta}, & \beta > N. \end{cases} \tag{25}$$

7. The optimal coefficients

Now, utilizing (20) and (25), we find the optimal coefficients $C_{\beta,1}$ for $\beta = 0, 1, \ldots, N$:

$$C_{\beta,1} = D_1(h\beta) * U_2(h\beta) = \sum_{\gamma=-\infty}^{\infty} D_1(h\gamma)U_2(h\beta - h\gamma) \tag{26}$$

where $D_1(h\beta)$ and $U_2(h\beta)$ are defined by expressions (20) and (25), respectively.

For $\beta = 0$, the coefficient $C_{0,1}$ is given by:

$$C_{0,1} = \frac{2}{1 - e^{2h}}\left(\frac{g}{4}(1 - e^{2h}) + F_2(0) - e^h F_2(h)\right). \tag{27}$$

Now we calculate the coefficients $C_{\beta,1}$ from (26) when $\beta = 1, 2, \ldots, N - 1$

$$C_{\beta,1} = \frac{2}{1 - e^{2h}}\left(-e^h(F_2(h(\beta - 1)) + F_2(h(\beta + 1))) + (1 + e^{2h})F_2(h\beta)\right).$$

Similarly, the coefficient $C_{N,1}$ is found as:

$$C_{N,1} = \frac{2}{1 - e^{2h}}\left(\frac{eg}{4}(1 - e^{2h}) + e^h(e^h F_2(1) - F_2(1 - h))\right).$$

Thus, the following theorem is proven.

Theorem 7.1. *The coefficients of the optimal quadrature formula (5) with equidistant nodes in the space $W_2^{(2,1)}(0,1)$ have the following forms:*

$$\begin{aligned} C_{0,1} &= \frac{2}{1 - e^{2h}}\left(\frac{g}{4}(1 - e^{2h}) + F_2(0) - e^h F_2(h)\right), \\ C_{\beta,1} &= \frac{2}{1 - e^{2h}}\left(-e^h(F_2(h(\beta - 1)) + F_2(h(\beta + 1))) + (1 + e^{2h})F_2(h\beta)\right), \\ &\quad \beta = 1, 2, \ldots, N - 1, \\ C_{N,1} &= \frac{2}{1 - e^{2h}}\left(\frac{eg}{4}(1 - e^{2h}) + e^h(e^h F_2(1) - F_2(1 - h))\right). \end{aligned}$$

Remark 1. *The quadrature formula* (5) *with the coefficients given in Theorem 7.1 is exact for the functions* 1, e^{-x}, e^x, $\sinh(x)$, *and* $\cosh(x)$.

8. Numerical results

In this section, we present the numerical results that validate the theoretical findings obtained in the previous sections.

Now, let us explore the following example. We numerically validate the optimal quadrature formula by utilizing Python. The computations encompass a range of functions with distinct values of N. We can estimate the convergence order (rate) of our quadrature formula using Big O notation based on the error analysis.

Ratio of Errors: Take the ratio of the errors obtained for consecutive node numbers (assuming the errors monotonically decrease as n increases). This ratio helps compare how much the error shrinks when the number of nodes doubles.

Logarithmic Relationship: Apply the logarithm (base 2 is commonly used) to the error ratio. This transformation helps establish a relationship between the error and the number of nodes (n) on a logarithmic scale. The Big O notation addresses asymptotic behavior, and the logarithmic transformation aligns well with that concept.

Interpretation based on the limit: As the number of nodes (n) tends to infinity, the limit of the logarithmic error ratio should be a constant value (ideally positive). This constant reflects the convergence order of the quadrature formula.

Understanding the Limit:

- A limit close to 1 indicates linear convergence (the error is proportional to $1/n$).
- A limit close to 2 suggests quadratic convergence (the error is proportional to $1/n^2$), which is generally faster than linear convergence.
- Higher limits imply even faster convergence rates (the error is proportional to $1/n^3$ or even higher powers of $1/n$).

Steps to Estimate Convergence Order:

1. After calculating errors for different node numbers (n), compute the error ratio between consecutive n values (n_i/n_{i-1}) for $i = 2$ to the last data point.
2. Take the logarithm (base 2) of each error ratio.
3. Analyse the limiting behavior of these logarithmic error ratios as n tends to infinity. A constant value (positive ideally) indicates convergence, and the value itself suggests the order (based on closeness to 1, 2, and so on).

Example 1. Consider the integral

$$\int_0^1 p(x)f_1(x)\mathrm{d}x$$

where the function $f_1(x) = x^3 + sin(2x)$, $x \in [0, 1]$, and $p(x) = 1$.

We calculate numerically the integral of the function $f(x)$. This is accomplished using the optimal quadrature formula (1) in $L_2^{(1)}(0, 1)$ space, and the optimal quadrature formula with derivative (5) in $W_2^{(2,1)}(0, 1)$ space. We have tabulated the absolute errors of the optimal quadrature formulas in Table 1. The absolute error of the optimal quadrature formula in the space $L_2^{(1)}$ (Err1), and the absolute error of the optimal quadrature formula with the derivative in the space $W_2^{(2,1)}$ (Err2)) are shown in the table. The interval [0; 1] is divided into N segments. As mentioned above, the order of convergence of the quadrature formulas is reflected in the last line of Table 1.

N	Err1	Err2
2	$0.24864 \cdot 10^{-2}$	$0.99298 \cdot 10^{-3}$
4	$0.81164 \cdot 10^{-3}$	$0.60924 \cdot 10^{-4}$
8	$0.21452 \cdot 10^{-3}$	$0.37904 \cdot 10^{-5}$
16	$0.54352 \cdot 10^{-4}$	$0.23663 \cdot 10^{-6}$
32	$0.13633 \cdot 10^{-4}$	$0.14785 \cdot 10^{-7}$
64	$0.34111 \cdot 10^{-5}$	$0.92402 \cdot 10^{-9}$
128	$0.85294 \cdot 10^{-6}$	$0.57749 \cdot 10^{-10}$
256	$0.21324 \cdot 10^{-6}$	$0.36094 \cdot 10^{-11}$
	$O(h^2)$	$O(h^4)$

Table 1: The absolute errors of the optimal quadrature formulas for numerical integration $f_1(x)$ function

Example 2. Consider the integral

$$\int_0^1 p(x) f_2(x) \mathrm{d}x$$

where the function $f_2(x) = sin(5x)$, $x \in [0, 1]$, and $p(x) = e^{x-2}$.

N	Err1	Err2
2	$0.61990 \cdot 10^{-3}$	$0.86055 \cdot 10^{-3}$
4	$0.81842 \cdot 10^{-3}$	$0.71326 \cdot 10^{-4}$
8	$0.23726 \cdot 10^{-3}$	$0.46586 \cdot 10^{-5}$
16	$0.61293 \cdot 10^{-4}$	$0.29407 \cdot 10^{-6}$
32	$0.15443 \cdot 10^{-4}$	$0.18424 \cdot 10^{-7}$
64	$0.35682 \cdot 10^{-5}$	$0.11522 \cdot 10^{-8}$
128	$0.96750 \cdot 10^{-6}$	$0.72022 \cdot 10^{-10}$
256	$0.24191 \cdot 10^{-6}$	$0.45015 \cdot 10^{-11}$
	$O(h^2)$	$O(h^4)$

Table 2: The absolute errors of the optimal quadrature formulas for numerical integration $f_2(x)$ function

In both examples above, increasing the number of nodes $N + 1$ leads to a corresponding decrease in absolute errors.

Conclusion

In this article, the derivation and analysis of a weighted optimal quadrature formula in the Hilbert space $W_2^{(2,1)}(0, 1)$ were addressed. The quadrature formula is expressed as a linear combination of function values and their first-order derivatives at equidistant nodes in the interval $[0, 1]$. The coefficients are determined by minimizing the norm of the error functional in the dual space $W_2^{(2,1)*}(0, 1)$. The error functional represents the difference between the integral of a function over the interval and the quadrature approximation. The key results include explicit expressions for the coefficients and the norm of the error functional.

The optimization problem was formulated and solved, leading to a system of linear equations for the coefficients. Analytical solutions of the system were obtained using the Sobolev method, providing an explicit expression for the optimal coefficients. The convergence with the exact values of the integrals was analyzed via numerical experiments.

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