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On the best proximity point theorems for interpolative proximal contractions with applications

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Abstract. We introduce some new generalized proximal interpolative contraction principles that produce corresponding proximal interpolative contraction principles and proximal contraction principles as special cases. We prove various best proximity point theorems using introduced generalized proximal interpolative contraction principles. The obtained results improve and generalize many best proximity point theorems published earlier in the literature. Some examples and applications are given to demonstrate the usefulness of our results.

1. Introduction

The principles of interpolative contraction are made up of a product of distances whose exponents meet certain requirements. Karapinar, a notable mathematician, coined the phrase interpolative contraction in his article [1], which was published in 2018. The following is a definition of the interpolative contraction:

A self-mapping S , defined on a metric space (Ω, d), satisfying the following inequality

$$d(Sx, Sy) \leq k\left(d(x, y)\right)^{\nu}, \forall x, y \in \Omega, \tag{1}$$

is called an interpolative contraction, where $\nu \in (0, 1]$ and $k \in [0, 1)$. It should be noted that S is a Banach contraction for $\nu = 1$. Recently, many classical and advanced contractions have been revisited via interpolation (see [2–4] and references therein).

On the other hand, finding an element $\hbar$ in R that is as close to $S(\hbar)$ in G as possible, is of great interest, since a non-self mapping need not have a fixed point. In other words, it is considered to find an approximation solution $\hbar$ in R such that the error $d(\hbar, S(\hbar))$ is smallest, where d is the distance function, if the fixed point equation $S(\hbar) = \hbar$ has no exact solution. In fact, *best proximity point* theorems look into the

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possibility of such *best proximity point* for approximate solutions to the fixed point equation $S(\hbar)=\hbar$ in the absence of a precise solution.

(i) The proximal contraction, contrary to the contraction, is a non-self mapping. The fundamental result on the existence of best proximity point of a proximal contraction was presented in [5]. (ii) Proinov [6](2020) presented a most general version of the contraction principle and established a method to show the existence of fixed points of this version of contraction (so called Proinov contraction). (iii) Karapinar introduced the idea of interpolation contraction in his work [1] published in 2018. (iv) Altun and Taşdemir [8] have utilized the interpolative proximal contraction to produce some *best proximity point* theorems. Motivated by ideas and investigations (i)-(iv), this article seeks to provide best proximity point theorems for contractive non-self mappings using interpolation, leading to the global optimal approximate solutions to specific fixed point equations. Iterative strategies are also provided to find such ideal approximative solutions in addition to proving the presence of *best proximity points*. Also, we extend the results appeared in [6, 8] by introducing (Ψ,Φ)-interpolative proximal contraction, which generalize and establishing the optimal proximity point theorems for them. Motivated by the contraction principles described in [6, 8]. The interpolative proximal contraction introduced in [8] are generalized by the improved interpolative proximal contraction. We look for various conditions on the functions to prove the presence of *best proximity point* of improved proximal contraction, improved Ćirić-Reich-Rus interpolative proximal contraction, improved Hardy Rogers interpolative proximal contraction. We also show non-trivial examples to demonstrate the usefulness of our results.

2. Preliminaries

We proceed with the following notations that are used in the sequel.

$$\begin{aligned} d(R,G) &= \inf\{d(\hbar,\mathfrak{q}):\hbar\in R\wedge \mathfrak{q}\in G\},\\ R_0 &= \{\hbar\in R: d(\hbar,\mathfrak{q})=d(R,G) \text{ for some } \mathfrak{q}\in G\},\\ G_0 &= \{\mathfrak{q}\in G: d(\hbar,\mathfrak{q})=d(R,G) \text{ for some } \hbar\in R\},\end{aligned}$$

where (Ω,d) is a metric space and $R,G\subseteq(\Omega,d)$.

Definition 2.1. *[9] Let $R,G\subseteq(\Omega,d)$. A mapping $S:R\to G$ satisfying*

$$\left.\begin{array}{l} d(\hbar_1,S(\mathfrak{q}_1))=d(R,G)\\ d(\hbar_2,S(\mathfrak{q}_2))=d(R,G)\end{array}\right\}\Rightarrow d(\hbar_1,\hbar_2)\le kd(\mathfrak{q}_1,\mathfrak{q}_2) \tag{2}$$

for all $\hbar_1,\hbar_2,\mathfrak{q}_1,\mathfrak{q}_2\in R$ such that $\hbar_1\neq\hbar_2$ and $k\in[0,1)$ is calledproximal contraction-I.

Every PC-I can be modified to a Banach contraction.

Definition 2.2. *[9] Let $R,G\subseteq(\Omega,d)$. A mapping $S:R\to G$ satisfying*

$$\left.\begin{array}{l} d(\hbar_1,S(\mathfrak{q}_1))=d(R,G)\\ d(\hbar_2,S(\mathfrak{q}_2))=d(R,G)\end{array}\right\}\Rightarrow d(S\hbar_1,S\hbar_2)\le kd(S\mathfrak{q}_1,S\mathfrak{q}_2),$$

for all $\hbar_1,\hbar_2,\mathfrak{q}_1,\mathfrak{q}_2\in R$ such that $S\hbar_1\neq S\hbar_2$, and $k\in[0,1)$ is said to be a proximal contraction-II.

For a self-mapping $S:R\to R$ to be a proximal contraction-II, it needs to satisfy the following inequality:

$$d\left(S^2\mathfrak{q}_1,S^2\mathfrak{q}_2\right)\le kd(S\mathfrak{q}_1,S\mathfrak{q}_2),\ \text{for all } \mathfrak{q}_1,\mathfrak{q}_2\in R.$$

Remark 2.3. *Every contraction is a proximal contraction-II but the converse is not true. Indeed, the mapping $S:[0,1]\to[0,1]$ defined by*

$$S(\hbar)=\begin{cases}0 & \text{if } \hbar \text{ is rational}\\ 1 & \text{otherwise}\end{cases}$$

is a proximal contraction-II but not a contraction in $(\mathbb{R},d)$.

Definition 2.4. *[8] Let R,G be any non-empty subsets of (Ω, d). We say that G is approximately compact with respect to R, if every sequence $\{\hbar_n\}$ in G satisfying the following condition*

$$d(\mathfrak{q}, \hbar_n) \to d(\mathfrak{q}, G),$$

for some $\mathfrak{q} \in R$, has a convergent sub-sequence.

It is obvious that any set is roughly compact in relation to itself. $R \cap G$ is contained in both R_0 and G_0 provided

R crosses G. Furthermore, the sets R_0 and G_0 are non-empty if R is compact and G is approximately compact with respect to R.

Definition 2.5. *[8] Let $R,G \subseteq (\Omega, d)$. An element $\hbar^*$ in R is called a best proximity point of the mapping $S : R \to G$, if it satisfies the equation:*

$$d(\hbar^*, S\hbar^*) = d(R, G).$$

A best proximity point of the mapping S represents both the approximate solution of the equation $S(\hbar^*) = \hbar^*$ and the optimal solution of the minimization problem:

$$\min\{d(\hbar^*, S(\hbar^*)) : \hbar^* \in R\}.$$

3. Improved proximal contractions

In this section, we define improved proximal contraction and show that it generalizes proximal contraction (2). We state and prove some existence of best proximity point theorems for improved proximal contraction and improved interpolative proximal contractions in a complete metric space.

3.1. Improved proximal contraction-II:

Let R, G be subsets of (Ω, d). A mapping $S : R \to G$ satisfying

$$\left.\begin{array}{l} d(\hbar_1, S\mathfrak{q}_1) = d(R, G) \\ d(\hbar_2, S\mathfrak{q}_2) = d(R, G) \end{array}\right\} \Rightarrow \Psi(d(S\hbar_1, S\hbar_2)) \leq \Phi(d(S\mathfrak{q}_1, S\mathfrak{q}_2)), \tag{3}$$

for all $\hbar_1, \hbar_2, \mathfrak{q}_1, \mathfrak{q}_2 \in R$ such that $\hbar_1 \neq \hbar_2$, is called an improved proximal contraction-II, where the maps $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$.

The following example shows the significance of improved proximal contraction-II.

Example 3.1. *Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be defined by*

$$d((\hbar, \mathfrak{q}), (u, v)) = |\hbar - u| + |\mathfrak{q} - v| \text{ for all } (\hbar, y), (u, v) \in X.$$

Then (Ω, d) is a metric space. *Let R,G be the subsets of Ω defined by*

$$R = \{(0, \mathfrak{q}); \mathfrak{q} \in \mathbb{R}\}, \quad G = \{(1, \mathfrak{q}); \mathfrak{q} \in \mathbb{R}\}, \text{ then } d(R, G) = 1.$$

Define the functions $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\Psi(t) = 2t \text{ and } \Phi(t) = t, t \in \mathbb{R}^+.$$

Define the mapping $S : R \to G$ by $S((0, r)) = (1, \frac{r}{4})$ for all $(0, r) \in R$. We show that S is an improved proximal contraction-II. For $\hbar_1 = (0, \hbar)$, $\hbar_2 = (0, u)$ and $\mathfrak{q}_1 = (0, 4\hbar)$, $\mathfrak{q}_2 = (0, 4u) \in R$ we have,

$$\begin{aligned} d(\hbar_1, S\mathfrak{q}_1) &= d((0, \hbar), S(0, 4\hbar)) = 1 = d(R, G), \\ d(\hbar_2, S\mathfrak{q}_2) &= d((0, u), S(0, 4u)) = 1 = d(R, G). \end{aligned}$$

This implies that

$$\Psi\left(d\left(S\hbar_1, S\hbar_2\right)\right) \leq \Phi\left(d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right)\right),$$

This shows that S is an improved proximal contraction-II. However, the following calculations show that it is not a proximal contraction-II. We know that

$$\begin{aligned} d(\hbar_1, S\mathfrak{q}_1) &= 1 = d(R, G) \\ d(\hbar_2, S\mathfrak{q}_2) &= 1 = d(R, G). \end{aligned}$$

If there exists $k \in (0,1)$ such that

$$d\left(S\hbar_1, S\hbar_2\right) \leq kd\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right),$$

then, $k = \frac{1}{6}$, a contradiction. Hence, S is not a proximal contraction-II.

The following lemmas are integral part of this paper and have an impact on further investigations.

Lemma 3.2. *[6] Let $\{l_n\}$ be a sequence in (Ω, d) verifying $\lim_{n\to\infty} d(l_n, l_{n+1}) = 0$. If the seqence $\{l_n\}$ is not cauchy, then there are sub-sequences $\{l_{n_k}\}$, $\{l_{m_k}\}$ and $\mathfrak{e} > 0$ such that*

$$\lim_{k\to\infty} d(l_{n_k+1}, l_{m_k+1}) = \epsilon + \textit{some term}(s). \tag{4}$$

$$\lim_{k\to\infty} d(l_{n_k}, l_{m_k}) = \lim_{k\to\infty} d(l_{n_k+1}, l_{m_k}) = \lim_{k\to\infty} d(l_{n_k}, l_{m_k+1}) = \epsilon. \tag{5}$$

Lemma 3.3. *[6] Let $\Psi : (0,\infty) \to R$ be a function. Then the statements (i)-(iii) are equivalent.*

(i) $\inf_{\mathfrak{z}>\varepsilon} \Psi(\mathfrak{z}) > -\infty$ for every $\varepsilon > 0$.

(ii) $\lim_{\mathfrak{z}\to\varepsilon+} \inf \Psi(\mathfrak{z}) > -\infty$ for every $\varepsilon > 0$.

(iii) $\lim_{n\to\infty} \Psi(\mathfrak{z}_n) = -\infty$ implies that $\lim_{n\to\infty} \mathfrak{z}_n = 0$.

Lemma 3.4. *Let $\{\hbar_n\}$ be a sequence in (Ω, d) obeying the equation $\lim_{n\to\infty} d(\hbar_n, \hbar_{n+1}) = 0$. Suppose that the mapping $S : R \to G$ satisfies (3) and the maps $\Psi, \Phi : (0,\infty) \to \mathbb{R}$ such that*

$$\limsup_{t\to\epsilon+} \Phi(t) < \Psi(\epsilon+) \tag{6}$$

for any $\epsilon > 0$. Then $\{\hbar_n\}$ is a Cauchy sequence.

Proof. First, we consider $\{\hbar_n\}$ is not cauchy, then by Lemma 3.2, there exist two subsequence $\{\hbar_{n_k}\}$, $\{\hbar_{m_k}\}$ of $\{\hbar_n\}$ and $\epsilon > 0$ so that (4) and (5) hold. By (4), we get that $d(\hbar_{n_k+1}, \hbar_{m_k+1}) > \epsilon$ and

$$\begin{aligned} d(\hbar_{n_k+1}, S(\hbar_{m_k})) &= d(R, G), \\ d(\hbar_{m_k+1}, S(\hbar_{n_k})) &= d(R, G), \text{ for all } k \geq 1. \end{aligned}$$

Thus, by (3), we have

$$\Psi(d(\hbar_{n_k+1}, \hbar_{m_k+1})) \leq \Phi(d(\hbar_{n_k}, \hbar_{m_k})), \text{ for any } k \geq 1. \tag{7}$$

Putting $c_k = d(\hbar_{n_k+1}, \hbar_{m_k+1})$ and $e_k = d(\hbar_{n_k}, \hbar_{m_k})$ in (7), we have

$$\Psi(c_k) \leq \Phi(e_k), \text{ for any } k \geq 1. \tag{8}$$

By (4) and (5), $\lim_{k\to\infty} c_k = \epsilon+$ some term(s) and $\lim_{k\to\infty} e_k = \epsilon$. By (8), we get

$$\Psi(\epsilon+) = \lim_{k\to\infty} \Psi(c_k) \leq \limsup_{k\to\infty} \Phi(e_k) \leq \limsup_{p\to\epsilon} \Phi(p). \tag{9}$$

This is a contradiction to the assumption (6). Consequently, $\{\hbar_n\}$ is a cauchy sequence in G . □

Now, we are in a position to state and prove our first main theorem.

Theorem 3.5. *Let R,G be non-empty, closed subsets of* complete metric space (Ω, d) *such that R is approximately compact with respect to G and $S: R \to G$ be a continuous improved proximal contraction-II verifying conditions (i)-(ii):*

(i) Ψ *is non-decreasing function and* $\limsup_{t\to\epsilon+} \Phi(t) < \Psi(\epsilon+)$ *for any* $\epsilon > 0$,

(ii) R_0 *is non-empty subset of* R *obeying* $S(R_0) \subseteq G_0$.

Then the mapping S admits a best proximity point.

Proof. Consider $\hbar_0 \in R_0$. Since $S(\hbar_0) \in S(R_0) \subseteq G_0$, there exists $\hbar_1 \in R_0$ satisfying $d(\hbar_1, S(\hbar_0)) = d(R,G)$. Also, we have $S(\hbar_1) \in S(R_0) \subseteq G_0$, there exists $\hbar_2 \in R_0$ so that $d(\hbar_2, S(\hbar_1)) = d(R,G)$. We build a series by continuing this approach such that $\{\hbar_n\}$ in R_0 satisfies the following equation:

$$d(\hbar_n, S(\hbar_{n-1})) = d(R, G), \text{for all } n \in \mathbb{N}. \tag{10}$$

If there exists $n \in \mathbb{N}$ such that $\hbar_n = \hbar_{n+1}$, then the point $\hbar_n$ is a *best proximity point* of the mapping S. On the other hand, if $\hbar_{n-1} \neq \hbar_n$ for all $n \in \mathbb{N}$, then we have

$$\begin{aligned} d(\hbar_n, S(\hbar_{n-1})) &= d(R, G), \\ d(\hbar_{n+1}, S(\hbar_n)) &= d(R, G), \text{ for all } n \geq 1. \end{aligned}$$

Thus, by (3), we have

$$\Psi(d(S\hbar_n, S\hbar_{n+1})) \leq \Phi(d(S\hbar_{n-1}, S\hbar_n)).$$

Let $d(S\hbar_n, S\hbar_{n+1}) = \mathfrak{d}_n$. Since, $\Phi(t) < \Psi(t)$ for all $t > 0$, we have

$$\Psi(\mathfrak{d}_n) \leq \Phi(\mathfrak{d}_{n-1}) < \Psi(\mathfrak{d}_{n-1}). \tag{11}$$

Given that Ψ is non-decreasing, by (11), we have $\mathfrak{d}_n < \mathfrak{d}_{n-1}$ $\forall$ $n \in \mathbb{N}$. Thus, it converges to some element $\mathfrak{d} \geq 0$. We claim that $\mathfrak{d} = 0$. If $\mathfrak{d} > 0$, by (11), we obtain the following:

$$\Psi(\mathfrak{d}+) = \lim_{n\to\infty} \Psi(\mathfrak{d}_n) \leq \lim_{n\to\infty} \Phi(\mathfrak{d}_{n-1}) \leq \lim_{t\to\mathfrak{d}+} \sup \Phi(t).$$

This contradicts (i), hence, $\mathfrak{d} = 0$ and $\lim_{n\to\infty} d(S\hbar_n, S\hbar_{n+1}) = 0$. By using (i) and Lemma 3.4, we conclude that $\{S(\hbar_n)\}$ is a cauchy sequence. Since G is a closed subset of *complete metric space* (Ω, d), there exists $\mathfrak{q}^* \in G$ such that $\lim_{n\to\infty} d(S\hbar_n, \mathfrak{q}^*) = 0$. Moreover,

$$\begin{aligned} d(\mathfrak{q}^*, R) &\leq d(\mathfrak{q}^*, \hbar_n) \\ &\leq d(\mathfrak{q}^*, S(\hbar_{n-1})) + d(S(\hbar_{n-1}), \hbar_n) \\ &\leq d(\mathfrak{q}^*, S(\hbar_{n-1})) + d(R, G) \\ &\leq d(\mathfrak{q}^*, S(\hbar_{n-1})) + d(\mathfrak{q}^*, R). \end{aligned}$$

Thus, $d(\mathfrak{q}^*, \hbar_n) \to d(\mathfrak{q}^*, R)$ as $n \to \infty$. Since R is approximately compact with respect to G, there exists a subsequence $\{(\hbar_{n_k})\}$ of $\{(\hbar_n)\}$ converging to $\hbar^* \in R$(say). We infer the following equation:

$$d(\hbar^*, \mathfrak{q}^*) = d(\hbar_{n_k}, S(\hbar_{n_k-1})) = d(R, G). \tag{12}$$

Due to the continuity of S, we have $S(\hbar_{n_k-1}) \to S(\hbar^*)$. Thus,

$$d(\hbar^*, S(\hbar^*)) = d(R, G).$$

□

For improved proximal contraction-I, we have the following theorem.

Theorem 3.6. *Let R,G be non-empty, closed subsets of* complete metric space (Ω, d) *such that R is approximately compact with respect to G and $S\colon R \to G$ be a improved proximal contraction-I verifying conditions (i)-(ii):*

(i) Ψ *is non-decreasing function and* $\limsup_{t\to\epsilon+} \Phi(t) < \Psi(\epsilon+)$ *for any* $\epsilon > 0$*,*

(ii) R_0 *is non-empty subset of* R *obeying* $S(R_0) \subseteq G_0$*.*

Then the mapping S has a best proximity point.

We omit the proof of Theorem 3.6, as it follows from the previous one. The following theorem is second main theorem stating different conditions for the existence of a *best proximity point*.

Theorem 3.7. *Let R,G be non-empty, closed subsets of* complete metric space (Ω, d) *with the property that "R is approximately compact with respect to G" and $S\colon R \to G$ be a continuous improved PC-II verifying the conditions (i)-(ii):*

(i) Ψ *is non-decreasing and if* $\{\Psi(t_n)\}$ *and* $\{\Phi(t_n)\}$ *are convergent sequence satisfying* $\lim_{n\to\infty} \Psi(t_n) = \lim_{n\to\infty} \Phi(t_n)$*, then* $\lim_{n\to\infty} t_n = 0$*,*

(ii) R_0 *is non-empty subset of* R *obeying* $S(R_0) \subseteq G_0$*.*

Then the mapping S has a best proximity point.

Proof. Following the procedure used in the proof of Theorem 3.5, we have

$$\Psi(\mathfrak{d}_n) \leq \Phi(\mathfrak{d}_{n-1}) < \Psi(\mathfrak{d}_{n-1}). \tag{13}$$

By (13), we have $\{\Psi(\mathfrak{d}_n)\}$ is strictly decreasing sequence. We have two cases here; either the sequence $\{\Psi(\mathfrak{d}_n)\}$ is bounded below or not. If $\{\Psi(w_n)\}$ is not bounded below, then

$$\inf_{w_n>\varepsilon} \Psi(\mathfrak{d}_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

From, Lemma 3.3, then $\mathfrak{d}_n \to 0$ as $n \to \infty$. Secondly, if sequence $\{\Psi(\mathfrak{d}_n)\}$ is bounded below, then, it is a convergent sequence. By (13), the sequence $\{\Phi(\mathfrak{d}_n)\}$ also converges, moreover, both have same limit. By (i), we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$, or $\lim_{n\to\infty} d(S\hbar_n, S\hbar_{n+1}) = 0$, for any sequence $\{\hbar_n\}$ in R. Now, as in the proof of Theorem 3.5, we have

$$d(\hbar^*, S(\hbar^*)) = d(R, G).$$

This shows that the point $\hbar^*$ is a *best proximity point* of the mapping S. □

3.2. Improved Ćirić-Reich-Rus Interpolative proximal contraction-II

Let (Ω, d) be a *complete metric space*, and R,G be a pair of non-empty subsets of Ω. Let $\Psi, \Phi : (0, \infty) \to \mathbb{R}$ be two functions. A mapping $S : R \to G$ is said to be an improved Ćirić-Reich-Rus interpolative PC-II if there exist $\alpha, \beta \in (0, 1)$; $\alpha + \beta < 1$ satisfying

$$\Psi(d(S\hbar_1, S\hbar_2)) \leq \Phi\begin{pmatrix} (d(S\mathfrak{q}_1, S\mathfrak{q}_2))^{\alpha} (d(S\mathfrak{q}_1, S\hbar_1))^{\beta} \\ (d(S\mathfrak{q}_2, S\hbar_2))^{1-\alpha-\beta} \end{pmatrix}, \tag{14}$$

whenever $d(\hbar_1, S\mathfrak{q}_1) = d(R, G)$ and $d(\hbar_2, S\mathfrak{q}_2) = d(R, G)$ for all distinct $\hbar_1, \hbar_2, \mathfrak{q}_1, \mathfrak{q}_2 \in R$.

Example 3.8. *Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the Euclidean metric on $\mathbb{R}^2$ and R,G be the subsets of $\mathbb{R}^2$ defined by*

$$R = \left\{(\hbar, \mathfrak{q}) : \mathfrak{q} = \sqrt{9 - \hbar^2}\right\}; G = \left\{(\hbar, v) : \mathfrak{q} = \sqrt{16 - \hbar^2}\right\} \text{ then } d(R, G) = 1.$$

Define the functions $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$ and $S : R \to G$ by

$$\Phi(\mathfrak{z}) = \sqrt{\mathfrak{z}} \text{ and } \Psi(\mathfrak{z}) = \mathfrak{z}, \text{ for all } \mathfrak{z} \in \mathbb{R}^+.$$

$$S(\zeta) = S(\hbar, v) = \begin{cases} \left(\frac{\hbar}{2}, \frac{\mathfrak{q}}{2}\right) & \text{for } \hbar \geq 0, \\ (-1, 0) & \text{for } \hbar < 0, \end{cases} \text{ for all } (\hbar, y) \in R.$$

The following information shows that S generalizes the interpolative Ćirić-Reich-Rus type proximal contraction [8]. Indeed, for $\hbar_1 = (1,0), \hbar_2 = (1,2), \mathfrak{q}_1 = (2,2), \mathfrak{q}_2 = (0,4)$, we have $d(\hbar_1, S\mathfrak{q}_1) = 1 = d(R,G)$, $d(\hbar_2, S\mathfrak{q}_2) = 1 = d(R,G)$, and for $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$,

$$\begin{aligned} \Psi(d(S(1,0), S(1,2))) &\leq \Phi\left(\begin{array}{l} (d(S(0,4), S(2,2)))^{\frac{1}{2}} (d(S(2,2), S(1,0)))^{\frac{1}{3}} \\ (d(S(0,4), S(1,2)))^{1-\frac{1}{2}-\frac{1}{3}} \end{array} \right), \\ \Psi(1) &\leq \Phi(1.2573) \Rightarrow 1 \leq 1.1213. \end{aligned}$$

Thus,

$$\Psi(d(S\hbar_1, S\hbar_2)) \leq \Phi\left(\begin{array}{l} (d(S\mathfrak{q}_1, S\mathfrak{q}_2))^{\alpha} (d(S\mathfrak{q}_1, S\hbar_1))^{\beta} \\ (d(S\mathfrak{q}_2, S\hbar_2))^{1-\alpha-\beta} \end{array} \right).$$

This shows that S is an improved interpolative Ćirić-Reich-Rus PC-II. However, for $\hbar_1 = (1,0), \hbar_2 = (1,2), \mathfrak{q}_1 = (2,2), \mathfrak{q}_2 = (0,4)$, if there exists some k satisfying the following inequality:

$$\begin{aligned} d(S\hbar_1, S\hbar_2) &\leq k (d(S\mathfrak{q}_1, S\mathfrak{q}_2))^{\alpha} (d(S\mathfrak{q}_1, S\hbar_1))^{\beta} (d(S\mathfrak{q}_2, S\hbar_2))^{1-\alpha-\beta} \\ d(S(1,0), S(1,2)) &\leq k (d(S(0,4), S(2,2)))^{\frac{1}{2}} (d(S(2,2), S(1,0)))^{\frac{1}{3}} \\ &\quad (d(S(0,4), S(1,2)))^{1-\frac{1}{2}-\frac{1}{3}}. \end{aligned}$$

Then, $k \in \left[\frac{1}{1.2573}, \infty\right)$, a contradiction. Hence, S is not interpolative Ćirić-Reich-Rus PC-II. We note that for $\hbar \geq 0$, there is $\zeta = (\hbar, v) \in R$ such that $d(\zeta, S(\zeta)) = d(R,G) = 1$.

The criteria for the existence of *best proximity point* of the improved Ćirić-Reich-Rus interpolative PC-II are stated in the following two theorems.

Theorem 3.9. *Let $R,G \subseteq (\Omega, d)$ with the property that "R is approximately compact with respect to G" and (Ω, d) be a complete metric space. If $S\colon R \to G$ is a continuous improved Ćirić-Reich-Rus type interpolative PC-II satisfying the following assumptions:*

- *(i) Ψ is non-decreasing function and $\limsup_{t \to \epsilon+} \Phi(t) < \Psi(\epsilon+)$ for any $\epsilon > 0$.*
- *(ii) R_0 is non-empty subset of R such that $S(R_0) \subseteq G_0$.*

Then S has a best proximity point*.*

Proof. Consider an arbitrary initial guess $\hbar_0 \in R_0$. Since $S(\hbar_0) \in S(R_0) \subseteq G_0$, there exists $\hbar_1 \in R_0$ such that

$$d(\hbar_1, S(\hbar_0)) = d(R, GG).$$

Also, $S(\hbar_1) \in S(R_0) \subseteq G_0$, there exists $\hbar_2 \in R_0$ such that

$$d(\hbar_2, S(\hbar_1)) = d(R, G).$$

We build a series by continuing this approach such that $\{\hbar_n\}$ in R_0 satisfies the following equation:

$$d(\hbar_{n+1}, S(\hbar_n)) = d(R, G), \text{ for all } n \in \mathbb{N}. \tag{15}$$

Now, if there exists $n \in \mathbb{N}$ such that $\hbar_n = \hbar_{n+1}$, then the point $\hbar_n$ is a *best proximity point* of the mapping S. Assume that $\hbar_n \neq \hbar_{n+1}$ for all $n \in \mathbb{N}$ and using (15), we have

$$d(\hbar_n, S(\hbar_{n-1})) = d(R, G),$$

and

$$d(\hbar_{n+1}, S(\hbar_n)) = d(R, G), \text{ for all } n \geq 1.$$

By (14), we have

$$\Psi(d(S\hbar_n, S\hbar_{n+1})) \leq \Phi\left((d(S\hbar_{n-1}, S\hbar_n))^{\alpha} (d(S\hbar_{n-1}, S\hbar_n))^{\beta} (d(S\hbar_n, S\hbar_{n+1}))^{1-\alpha-\beta}\right), \tag{16}$$

for all distinct $\hbar_{n-1}, \hbar_n, \hbar_{n+1} \in R$. Given that $\Phi(t) < \Psi(t)$ for all $t > 0$, by (16), we have

$$\Psi(d(S\hbar_n, S\hbar_{n+1})) < \Psi\left((d(S\hbar_{n-1}, S\hbar_n))^{\alpha} (d(S\hbar_{n-1}, S\hbar_n))^{\beta} (d(S\hbar_n, S\hbar_{n+1}))^{1-\alpha-\beta}\right).$$

Since Ψ is a non-decreasing function, we have

$$d(S\hbar_n, S\hbar_{n+1}) < (d(S\hbar_{n-1,}S\hbar_n))^{\alpha+\beta} (d(S\hbar_n, S\hbar_{n+1}))^{1-\alpha-\beta}.$$

This implies that

$$(d(S\hbar_n, S\hbar_{n+1}))^{\alpha+\beta} < (d(S\hbar_{n-1}, S\hbar_n))^{\alpha+\beta}.$$

This shows that the sequence $\{d(S\hbar_n, S\hbar_{n+1}) = \mathfrak{d}_n\}$ converges to some element $\mathfrak{d} \geq 0$. We claim that $\mathfrak{d} = 0$. If $\mathfrak{d} > 0$, by (16), we obtain the following:

$$\Psi(\mathfrak{d}+) = \lim_{n\to\infty} \Psi(\mathfrak{d}_n) \leq \lim_{n\to\infty} \Phi\left((\mathfrak{d}_{n-1})^{\alpha+\beta}(\mathfrak{d}_n)^{1-\alpha-\beta}\right) \leq \lim_{\mathfrak{z}\to\mathfrak{d}+} \sup \Phi(\mathfrak{z}).$$

This contradicts (i), hence, $\mathfrak{d} = 0$ and $\lim_{n\to\infty} d(S\hbar_n, S\hbar_{n+1}) = 0$. By using (i) and Lemma 3.4, we conclude that $\{S\hbar_n\}$ is a cauchy sequence. Since G is a closed subset of *complete metric space* (Ω, d), there exists $\mathfrak{q}^* \in G$, such that $\lim_{n\to\infty} d(S\hbar_n, \mathfrak{q}^*) = 0$. Now, we can obtain the desired result by following the reasoning used in the proof of Theorem 3.5. □

Theorem 3.10. *Let* $R, G \subseteq (\Omega, d)$ *with the property that "R is approximately compact with repect to G " and* (Ω, d) *be a complete metric space. If* $S\colon R \to G$ *is a continuous improved Ćirić-Reich-Rus type interpolative PC-II verifying (i)-(ii)*

(i) Ψ *is non-decreasing and if* $\{\Psi(\mathfrak{z}_n)\}$ *and* $\{\Phi(\mathfrak{z}_n)\}$ *are convergent sequences satisfying* $\lim_{n\to\infty} \Psi(\mathfrak{z}_n) = \lim_{n\to\infty} \Phi(\mathfrak{z}_n)$, *then* $\lim_{n\to\infty} \mathfrak{z}_n = 0$,

(ii) R_0 *is non-void subset of R obeying* $S(R_0) \subseteq G_0$.

Then the mapping S has a best proximity point.

Proof. Following the procedure used in the proof of Theorem 3.9, we have

$$\Psi(\mathfrak{d}_n) \leq \Phi\left((\mathfrak{d}_{n-1})^{\alpha+\beta}(\mathfrak{d}_n)^{1-\alpha-\beta}\right) < \Psi\left((\mathfrak{d}_{n-1})^{\alpha+\beta}(\mathfrak{d}_n)^{1-\alpha-\beta}\right). \tag{17}$$

By (17), we infer that $\{\Psi(\mathfrak{d}_n)\}$ is strictly decreasing sequence. We have two cases here; either the sequence $\{\Psi(\omega_n)\}$ is bounded below or not. If $\{\Psi(\mathfrak{d}_n)\}$ is not bounded below, then

$$\inf_{\mathfrak{d}_n>\varepsilon} \Psi(\mathfrak{d}_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

It follows by Lemma 3.3, that $\mathfrak{d}_n \to 0$ as $n \to \infty$. Secondly, if the sequence $\{\Psi(\mathfrak{d}_n)\}$ is bounded below, then, it is convergent sequence. By (17) the sequence $\{\Phi(\mathfrak{d}_n)\}$ also converges, moreover, both have same limit. By (i), we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ for any sequence $\{\hbar_n\}$ in R. The proof of Theorem 3.9 leads to the rest of the proof. □

Note that, if S is a self-mapping defined on R, then best proximity point is a fixed point of S.

Remark 3.11. *The following observations indicate the generality of improved interpolative Ćirić-Reich-Rus type proximal contraction for the specific definitions of the mappings* Ψ, Φ.

1. *If* $\Phi(\ell) = \Psi(\ell) - \tau$ *for all* $\ell \in (0,\infty)$ *in (14), then L is an interpolative Ćirić-Reich-Rus type F-proximal contraction.*
2. *If* $\Phi(\ell) = \Psi(\ell) - \tau(\ell)$ *for all* $\ell \in (0,\infty)$ *in (14), then L is an interpolative Ćirić-Reich-Rus type* (τ, F_T) *proximal-contraction.*
3. *If* $\Psi(\ell) = \ell$ *and* $\Phi(\mathfrak{s}) = \lambda\mathfrak{s}$ *for all* $\ell, \mathfrak{s} \in (0,\infty)$ *in (14), then we obtain the contraction introduced in [8].*
4. *For* $\nu = 0$, *we obtain improved interpolative Kannan type proximal contraction from (14) .*

3.3. Improved Hardy Rogers Interpolative PC-II:

Let $R,G \subseteq (\Omega, d)$. A mapping $S : R \to G$ satisfying

$$\Psi(d(S\hbar_1, S\hbar_2)) \le \Phi\left(\begin{array}{l} d(S\mathfrak{q}_1, S\mathfrak{q}_2)^{\alpha}\, d(S\mathfrak{q}_1, S\hbar_1)^{\beta}\, d(Sv_2, S\hbar_2)^{\gamma} \\ \left(\frac{1}{2}\left(d(S\mathfrak{q}_1, S\hbar_2) + d(Sv_2, S\hbar_1)\right)\right)^{1-\alpha-\beta-\gamma} \end{array} \right), \tag{18}$$

whenever, $d(\hbar_1, S\mathfrak{q}_1) = d(R,G)$; $d\left(\hbar_2, Sq_2\right) = d(R,G)$, is called an improved Hardy Rogers interpolative PC-II, where $\alpha, \beta, \gamma \in (0,1)$ such that $\alpha+\beta+\gamma < 1$, $\hbar_1, \hbar_2, \mathfrak{q}_1, \mathfrak{q}_2 \in R$ and $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$.
The following example shows that improved Hardy Rogers type interpolative PC-II generalizes the Hardy Rogers type interpolative PC-II appeared in [8].

Example 3.12. *Let* $d : \mathbb{R}^2 \to \mathbb{R}$ *be a usual metric and* R,G *be subsets of* Ω *defined as*

$$R = \{1,2,3,4,5,6,7\}, G = \{0,1,2,3,4,5\} \text{ then } d(R,G) = 0.$$

Define the functions $\Psi, \Phi : \mathbb{R}^+ \to \mathbb{R}$ *and* $S : R \to G$ *by*

$$\Psi(\hbar) = \begin{cases} \hbar + 1 \text{ for } \hbar = 2, \\ \hbar + 10 \text{ for } \hbar \neq 2, \end{cases} \qquad \Phi(\hbar) = \begin{cases} \frac{\hbar}{2} \text{ for } \hbar = 2, \\ \hbar + 5 \text{ otherwise}, \end{cases}$$

and $S(\hbar) = \hbar - 1$ *for all* $\hbar \in R$. *We show that S is an improved interpolative Hardy Rogers PC-II. Indeed, for* $\hbar_1 = 2, \hbar_2 = 4, y_1 = 3, y_2 = 5$, *and* $\alpha = 0.2, \beta = 0.3, \gamma = 0.4$ *we have* $d(\hbar_1, S\mathfrak{q}_1) = 0 = d(R,G)$, $d(\hbar_2, S\mathfrak{q}_2) = 0 = d(R,G)$ *and*

$$\begin{aligned} \Psi(2) &\le \Phi\left((2)^{\alpha}(1)^{\beta}(1)^{\gamma}\left(\tfrac{1}{2}(3+1)\right)^{1-\alpha-\beta-\gamma} \right) \\ \Psi(2) &\le \Phi\left((2)^{0.2}(1)^{0.3}(1)^{0.4}(2)^{0.1}\right) \\ &= \Phi(0.7764) \Rightarrow 3 < 5.7764. \end{aligned}$$

Hence,

$$\Psi\left(d\left(S\hbar_1, S\hbar_2\right)\right) \leq \Phi\left(\begin{array}{l} d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right)^{\alpha} d\left(S\mathfrak{q}_1, S\hbar_1\right)^{\beta} d\left(S\mathfrak{q}_2, S\hbar_2\right)^{\gamma} \\ \left(\frac{1}{2}\left(d\left(S\mathfrak{q}_1, S\hbar_2\right)+d\left(S\mathfrak{q}_2, S\hbar_1\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array}\right).$$

This shows that S is an improved interpolative Hardy Rogers PC-II. However, the following calculation shows that it is not an interpolative Hardy Rogers PC-II.

Suppose there is some k satisfying the following inequality:

$$d\left(S\hbar_1, S\hbar_2\right) \leq k\left(\begin{array}{l} d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right)^{\alpha} d\left(S\mathfrak{q}_1, S\hbar_1\right)^{\beta} d\left(S\mathfrak{q}_2, S\hbar_2\right)^{\gamma} \\ \left(\frac{1}{2}\left(d\left(S\mathfrak{q}_1, S\hbar_2\right)+d\left(S\mathfrak{q}_2, S\hbar_1\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array}\right).$$

Then, $k \in \left[\frac{2}{0.7764}, \infty\right)$, which is a contradiction to the assumption that $k \in (0,1)$. Hence, S is not an interpolative Hardy Rogers PC-II.

The criteria for the existence of the best proximity point of improved interpolative Hardy Rogers proximal contraction S are stated in the following two theorems. The proofs are very identical to the proofs of Theorems 3.7 and 3.9. We only write the distinct parts of the proof.

Theorem 3.13. *Let $S: R \rightarrow G$ be a continuous improved interpolative Hardy Rogers PC-II defined on a* complete metric space (Ω, d) *verifying conditions (i)-(ii), where R,G are non-empty, closed subsets of Ω with the property that R is approximately compact with respect to G .*

(i) *Ψ is a non-decreasing function and $\lim_{\mathfrak{z}\to\varepsilon+} \Phi(\mathfrak{z}) < \Psi(\varepsilon+)$ for any $\varepsilon > 0$,*

(ii) *R_0 is a non-void subset of R such that $S(R_0) \subseteq G_0$.*

Then the mapping S has a best proximity point.

Proof. Starting with the initial input as in the proof of Theorem 3.7, we have

$$d(\hbar_n, S(\hbar_{n-1})) = d(R, G),$$

$$d(\hbar_{n+1}, S(\hbar_n)) = d(R, G), \text{ for all } n \geq 1.$$

Thus by (18) we can write

$$\begin{aligned}
\Psi(d(S\hbar_n, S\hbar_{n+1})) &\leq \Phi\left(\begin{array}{l} \left(d\left(S\hbar_{n-1}, S\hbar_n\right)\right)^{\alpha}\left(d(S\hbar_{n-1}, S\hbar_n)\right)^{\beta}\left(d\left(S\hbar_n, S\hbar_{n+1}\right)\right)^{\gamma} \\ \left(\frac{1}{2}\left(d\left(S\hbar_{n-1}, S\hbar_{n+1}\right)+d\left(S\hbar_n, S\hbar_n\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array}\right) \\
\Psi\left(d\left(S\hbar_n, S\hbar_{n+1}\right)\right) &\leq \Phi\left(\begin{array}{l} \left(d\left(S\hbar_{n-1}, S\hbar_n\right)\right)^{\alpha}\left(d\left(S\hbar_{n-1}, S\hbar_n\right)\right)^{\beta}\left(d\left(S\hbar_n, S\hbar_{n+1}\right)\right)^{\gamma} \\ \left(\frac{1}{2} d\left(S\hbar_{n-1}, S\hbar_{n+1}\right)\right)^{1-\alpha-\beta-\gamma} \end{array}\right) \\
\Psi\left(d\left(S\hbar_n, S\hbar_{n+1}\right)\right) &\leq \Phi\left(\begin{array}{l} \left(d\left(S\hbar_{n-1}, S\hbar_n\right)\right)^{\alpha}\left(d(S\hbar_{n-1}, S\hbar_n)\right)^{\beta}\left(d\left(S\hbar_n, S\hbar_{n+1}\right)\right)^{\gamma} \\ \left(\frac{1}{2}\left(d\left(S\hbar_{n-1}, S\hbar_n\right)+d\left(S\hbar_n, S\hbar_{n+1}\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array}\right) \\
\Psi\left(d\left(S\hbar_n, S\hbar_{n+1}\right)\right) &\leq \Phi\left(\begin{array}{l} \left(d\left(S\hbar_{n-1}, S\hbar_n\right)\right)^{\alpha+\beta}\left(d\left(S\hbar_n, S\hbar_{n+1}\right)\right)^{\gamma} \\ \left(\frac{1}{2}\left(d\left(S\hbar_{n-1}, S\hbar_n\right)+d\left(S\hbar_n, S\hbar_{n+1}\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array}\right),
\end{aligned}$$

for all distinct$\hbar_{n-1}, \hbar_n, \hbar_{n+1} \in R$. Let $d(S\hbar_n, S\hbar_{n+1}) = \omega_n$. Since $\Phi(t) < \Psi(t)$ for all $t > 0$, we get

$$\Psi\left(\mathfrak{d}_n\right) < \Psi\left(\left(\mathfrak{d}_{n-1}\right)^{\alpha+\beta}\left(\mathfrak{d}_n\right)^{\gamma}\left(\frac{1}{2}\left(\mathfrak{d}_{n-1}+\mathfrak{d}_n\right)\right)^{1-\alpha-\beta-\gamma}\right). \tag{19}$$

Suppose that $\mathfrak{d}_{n-1} < \mathfrak{d}_n$ for some $n \geq 1$ and by monotonicity of Ψ, we have

$$(\mathfrak{d}_n)^{\alpha+\beta} < (\mathfrak{d}_n)^{\alpha+\beta},$$

which is a false statement. Consequently, we have $\mathfrak{d}_n < \mathfrak{d}_{n-1}$ for all $n \in \mathbb{N}$. This implies $\mathfrak{d}_n < \mathfrak{d}_{n-1}$ for all $n \in \mathbb{N}$. Thus, it converges to some element $\mathfrak{d} \geq 0$. Suppose $\mathfrak{d} > 0$, then

$$\Psi(\mathfrak{d}+) = \lim_{n\to\infty} \Psi(\mathfrak{d}_n) \leq \lim_{n\to\infty} \Phi\left((\mathfrak{d}_{n-1})^{\alpha+\beta} (\mathfrak{d}_n)^{\gamma} \left(\frac{1}{2} (\mathfrak{d}_n + \mathfrak{d}_{n-1}) \right)^{1-\alpha-\beta-\gamma} \right) \leq \lim_{t\to\omega+} \Phi(t).$$

This contradicts (i), hence, $\mathfrak{d} = 0$ and $\lim_{n\to\infty} d(S\hbar_n, S\hbar_{n+1}) = 0$. We omit the remaining details as they are similar to proof of Theorem 3.7. □

Theorem 3.14. *Every continuous improved interpolative Hardy Rogers PC-II $S: R \to G$ defined on* complete metric space (Ω, d) *and verifying conditions (i)-(ii) admits a* best proximity point *provided that R,G are non-empty, closed subsets of Ω with the property that R is approximately compact with respect to G .*

(i) Ψ *is non-decreasing and* $\{\Psi(\mathfrak{z}_n)\}$ *and* $\{\Phi(\mathfrak{z}_n)\}$ *are convergent sequences obeying* $\lim_{n\to\infty} \Psi(\mathfrak{z}_n) = \lim_{n\to\infty} \Phi(\mathfrak{z}_n)$, *then* $\lim_{n\to\infty} \mathfrak{z}_n = 0$.

(ii) R_0 *is non-void subset of* R *obeying* $S(R_0) \subseteq G_0$.

Proof. This proof follows from the proof of Theorem 3.9 and Theorem 3.10. □

Remark 3.15. *If $S: R \to R$ ($G = R$), then the best proximity point is a fixed point and Theorem 3.5, Theorem 3.7, Theorem 3.9, Theorem 3.10, Theorem 3.13 and Theorem 3.14 are fixed point theorems.*

4. Application to integral equations

The theory of integral equations may be traced back at least to Fourier's discovery of the theorem concerning integrals that bears his name; indeed, while not Fourier's point of view, this theorem can be seen as a statement of the solution of a certain first-order integral equation. However, Abel and Liouville, as well as others after them, began to study exceptional integral equations in a fully conscious manner, and many of them recognised the critical role the theory was destined to play. We are intend to apply Theorem 3.6 (for $R \subseteq G$) to show the existence of the solution to the following nonlinear Volterra type integral equations:

$$f(k) = \int_0^k H_\varsigma(k, h, f)\, dh, \tag{20}$$

for all $k \in [0,1]$, $\varsigma \in \Theta$, and H_ς is a function defined on $[0,1]^2 \times C([0,1], \mathbb{R}_+)$ to $\mathbb{R}$. We show the existence to the solution of (20). For $f \in C([0,1], \mathbb{R}_+)$, the norm as: $\|f\|_\tau = \sup_{k\in[0,1]} \left| f(k) \right| e^{-\tau k}$, $\tau > 0$. Define

$$\eta_\tau(f, \varkappa) = \left[\sup_{k\in[0,1]} \left| f(k) - \varkappa(k) \right| e^{-\tau k} \right] = \|f - \varkappa\|_\tau$$

for all $f, \varkappa \in C([0,1], \mathbb{R}_+)$, with these settings, $(C([0,1], \mathbb{R}_+), \eta_\tau)$ represents a complete metric space.
Now, we show the following theorem to clarify that the solution of integral equation exists.

Theorem 4.1. *Suppose that the mapping $H_\varsigma : [0,1] \times [0,1] \times C([0,1], \mathbb{R}_+) \to \mathbb{R}$ is a continuous mapping:*

$$\left|H_\varsigma(k,h,f) - H_\varsigma(k,h,c)\right| \leq \frac{\tau\eta_\tau(f,c)}{\tau\eta_\tau(f,c)+1}e^{\tau h} \tag{21}$$

for every $h, k \in [0,1]$ and $f, c \in C([0,1], \mathbb{R})$. Then, integral equation (20) has at most one solution in $C([0,1], \mathbb{R}_+)$ or equivalently the associated operator $L_\varsigma : \mathbb{R} \to \mathbb{R}$ defined by

$$(L_\varsigma f)(k) = \int_0^k H_\varsigma(k,h,f)\,dh, \tag{22}$$

admits a best proximity point.

Proof. By (21) and (22), we have the following information.

$$\begin{aligned}
\left|L_\varsigma f - L_\varsigma \varkappa\right| &= \int_0^k \left|H_\varsigma(k,h,f) - H_\varsigma(k,h,\varkappa)\right| dh, \\
&\leq \int_0^k \frac{\tau\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1}e^{\tau h}dh \\
&\leq \frac{\tau\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1}\int_0^k e^{\tau h}dh \\
&\leq \frac{\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1}e^{\tau k}.
\end{aligned}$$

This implies

$$\left|L_\varsigma f - L_\varsigma \varkappa\right| e^{-\tau k} \leq \frac{\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1}$$

$$\left\|L_\varsigma f - L_\varsigma \varkappa\right\|_\tau \leq \frac{\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1}$$

$$\frac{\tau\eta_\tau(f,\varkappa)+1}{\eta_\tau(f,\varkappa)} \leq \frac{1}{\left\|L_\varsigma f - L_\varsigma \varkappa\right\|_\tau}$$

$$\tau + \frac{1}{\eta_\tau(f,\varkappa)} \leq \frac{1}{\left\|L_\varsigma f - L_\varsigma \varkappa\right\|_\tau}$$

which further implies

$$\tau - \frac{1}{\left\|L_\varsigma f - L_\varsigma \varkappa\right\|_\tau} \leq \frac{-1}{\eta_\tau(f,\varkappa)}.$$

So all the conditions of Theorem 3.6 are satisfied for $\Psi(\varkappa) = \frac{-1}{\varkappa}$; $\varkappa > 0$ and $\Phi(\varkappa) = \Psi(\varkappa) - \tau$. Hence, the integral equation (20) admits a solution. □

5. Application to functional equations

Equations in which the unknowns are functions rather than traditional variables are known as functional equations. The methods for solving functional equations, on the other hand, can differ significantly from those for isolating a classical variable. Here, we present an application of Theorem 3.6 (for $R \subseteq G$) to show the existence of the solution to a functional equation in dynamical programming.

Let Γ be a Banach space, $£, \mho \subseteq \Gamma$ and

$$\begin{aligned} \varphi &: £ \times \mho \to £ \\ \hbar, y &: £ \times \mho \to \mathbb{R} \\ C, K &: £ \times \mho \times \mathbb{R} \to \mathbb{R}. \end{aligned}$$

We assume that £ and $\mho$ show only for the decisions spaces. The problem related to dynamical programming is to find the solution of the given equations:

$$p(\alpha) = \sup_{\alpha \in \mho}\{\hbar(\alpha, \theta) + C(\alpha, \theta, p(\varphi(\alpha, \theta)))\}, \tag{23}$$

for $\alpha \in £$. We want to show the equations (23) have unique solution. Suppose $R(£)$ represents the class of all positive valued functions on £. Consider,

$$\eta_\tau(\mathfrak{q}, w) = \|\mathfrak{q} - w\|_\infty = \sup_{\alpha \in \mho} |\mathfrak{q}(\alpha) - w(\alpha)| \tag{24}$$

for all $\mathfrak{q}, w \in R(£)$, and $(R(£), \eta)$ becomes a complete metric space. Assume that
($\hat{C}1$): $C, K, \hbar$, and y are bounded;
($\hat{C}2$): for $\alpha \in £$, $\mathfrak{q} \in R(£)$, $\Upsilon_\varsigma : R(£) \to R(£)$, take

$$C\mathfrak{q}(\alpha) = \sup_{\theta \in \mho}\{\hbar(\alpha, \theta) + C(\alpha, \theta, \mathfrak{q}(\varphi(\alpha, \theta)))\}. \tag{25}$$

Furthermore, for each $(\alpha, \theta) \in £ \times \mho$, $\mathfrak{q}, w \in R(£)$, $t \in £$ and $\tau > 0$,

$$|C(\alpha, \theta, \mathfrak{q}(t)) - C(\alpha, \theta, w(t)| \leq \eta_\tau(\mathfrak{q}, w)e^{-\tau}. \tag{26}$$

Theorem 5.1. *Suppose that ($\hat{C}1$), ($\hat{C}2$) and (26) hold. Then, the equation (23) admits a unique bounded solution in $R(£)$.*

Proof. Take any $c > 0$. By (25) and (26), there are $\mathfrak{q}_1 \in R(£)$, and $\theta_1 \in \mho$ such that

$$(\Upsilon_\varsigma \mathfrak{q}_1) < \hbar(\alpha, \theta_1) + C(\alpha, \theta_1, \mathfrak{q}_1(\varphi(\alpha, \theta_1))) + c, \tag{27}$$

Using the definition of supremum, we get

$$(\Upsilon_\varsigma \mathfrak{q}_1) \geq \hbar(\alpha, \theta_2) + C(\alpha, \theta_2, \mathfrak{q}_1(\varphi(\alpha, \theta_2))). \tag{28}$$

Then, from (26), (27) and (28), we have

$$\begin{aligned} & (\Upsilon_\varsigma \mathfrak{q}_1)(\alpha) - (\Upsilon_\varsigma \mathfrak{q}_2)(\alpha) \\ \leq \;& C(\alpha, \theta_1, \mathfrak{q}_1(\varphi(\alpha, \theta_1))) - C(\alpha, \theta_1, \mathfrak{q}_2(\varphi(\alpha, \theta_1))) + c \\ \leq \;& \left|C(\alpha, \theta_1, \mathfrak{q}_1(\varphi(\alpha, \theta_1))) - C(\alpha, \theta_1, \mathfrak{q}_2(\varphi(\alpha, \theta_1)))\right| + c \\ \leq \;& \eta_\tau(\mathfrak{q}, w)e^{-\tau} + c. \end{aligned}$$

Since, $c > 0$ is arbitrary, we obtain

$$\begin{aligned} \left|\Upsilon_\varsigma \mathfrak{q}_1(\alpha) - \Upsilon_\varsigma \mathfrak{q}_2(\alpha)\right| &\leq \eta_\tau(\mathfrak{q}, w)e^{-\tau} \\ e^\tau \left|\Upsilon_\varsigma \mathfrak{q}_1(\alpha) - \Upsilon_\varsigma \mathfrak{q}_2(\alpha)\right| &\leq \eta_\tau(\mathfrak{q}, w). \end{aligned}$$

It implies that,

$$\tau + \ln \left| \Upsilon_{\varsigma} \mathfrak{q}_1(\alpha) - \Upsilon_{\varsigma} \mathfrak{q}_2(\alpha) \right| \leq \ln(\eta_{\tau}(\mathfrak{q}, w)).$$

Thus, the requirements of Theorem 3.6 are hold for $\Psi(\varkappa) = \ln(\varkappa)$; $\varkappa > 0$ and $\Phi(\varkappa) = \Psi(\varkappa) - \tau$. Hence, C admits a best proximity point $\mathfrak{q}^* \in R(£)$. □

6. Conclusion and future work

The theorems provided here establish a broad criterion for the existence of a best proximity point of improved interpolative PC-II. The results will extend earlier results of Basha [5], Altun and Taşdemir [8], Beg *et al.* [10], Espinola *et al.* [11], Suzuki [13] and others. Future work: The study presented in this paper can be revisited to demonstrate the existence of PPF-dependent optimum proximity sites of non-self mappings (for more information, see [12] and references).

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Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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The authors declare that they do not have any competing interests. All authors read and approved the final manuscript.

Authors' contributions

Each author equally contributed to this paper, read and approved the final manuscript.

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