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An explicit Milstein-type scheme for simulation of SDEs

Hassan Ranjbar[a]

[a]*Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P. O. Box 35195-363, Semnan, Iran*

Abstract. In the present paper, we developed a new explicit Milstein-type integrator for stiff SDEs. Theoretically, we indicate that the scheme converges to the true value with a strong order of 1.0. For linear scalar SDE, the asymptotic mean square stability of our method is investigated. For all time steps, we prove that the presented integrator is asymptotically mean square stable. In addition, the A-stability and L-stability of the scheme was discussed in the mean square sense. For dimension two of the submitted scheme, the asymptotic mean square stability of two test systems has been analyzed. Numerical simulations confirm the theoretical results.

1. Introduction

In recent decades, SDEs have been shown to be more powerful in modeling real-life problems than deterministic differential equations [7, 21–23]. Unfortunately, the analytical solution of a few SDEs leads to an explicit solution. So, solution procedures has been an exciting area for researchers, in this last half-century [10, 26, 32].

In this investigation, our goal is to provide a new explicit numerical scheme based on the Milstein approach for the solution of SDEs

$$\mathrm{d}Z(t) = a(Z(t))\mathrm{d}t + \sum_{n=1}^{k} b_n(Z(t))\mathrm{d}W_n(t), \tag{1}$$

with initial condition $Z(0) = Z_0 \in \mathbb{R}^d$ and $k, d \in \mathbb{N}$. In (1), $a, b_n : \mathbb{R}^d \to \mathbb{R}^d$ and $\{W_n\}_{n=1}^{k}$ are standard Wiener processes. In the last half-century, Milstein's approach has inspired many numerical schemes [27, 40]. Despite the utility of implicit schemes in addressing stiff stochastic problems and high-dimensional stochastic systems, the increased CPU time and computational cost associated with solving implicit equations in per step via the Newton-Raphson iterative algorithm has become a significant drawback for these numerical techniques. The researchers found the solution in developing of explicit integrators with extended stability regions. A balancing strategy to increasing the efficiency and accuracy of the explicit Euler-Maruyama approach was introduced in 1998, designed to address the challenges of stiff and high-dimensional stochastic

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Email address: hranjbar@semnan.ac.ir; h.r.hassanranjbar@gmail.com (Hassan Ranjbar)
ORCID iD: https://orcid.org/0000-0001-9374-1091 (Hassan Ranjbar)

problems [20]. In 2006, Kahl and Schurz [15] extend the balanced Milstein scheme to SDE (1). Wang and Gan [39] by taming strategy achieved a new one-step explicit Milstein scheme. Also, in [42] designed a new one-step explicit balanced Milstein scheme using the hyperbolic tangent function. Erdǒgan and G.J. Lord [11] examined the strong convergence of an explicit exponential Milstein integrator for semi-linear SDEs. In 2015, Yin and Gan [41] enhanced the explicit Milstein method with an error correction term to address the challenges posed by stiff SDEs. Papers [24, 37] recently proposed same new numerical schemes with the combination of the exponential integrator and the explicit Milstein technique for (1). Motivated by the above note, the authors shall develop the explicit Milstein approach for stiff SDEs and explore the strong convergence rate of the scheme, in this paper. Also, we achieve the stability properties of the scheme for linear scalar test equation and for the bi-dimensional of the integrator, we examine the mean square (MS) stability of multi-dimensional systems with scalar noise. For linear test SDE with multiplicative noise, we show that the scheme is MS A-stable. Moreover, the L-stability of the scheme is studied in the MS sense.

Set $h = (T - t_0)/N$ on $[t_0, T]$ and $t_l = t_0 + lh$, $0 \le l \le N$. To solve SDE (1), we submit the following explicit exponential Milstein (EXM, for short) integrator

$$\begin{aligned} Z_{l+1} = Z_l &+ hf\left(\mathcal{J}_a, \mathcal{J}_{L^n b_n}\right)\left\{a(Z_l) - \frac{1}{2}\sum_{n=1}^{k} L^n b_n(Z_l)\right\} + g\left(\mathcal{J}_a, \mathcal{J}_{L^n b_n}\right)\left\{\sum_{n=1}^{k} b_n(Z_l) I_{(n)}^{t_l,t_{l+1}}\right. \\ &\left. + \sum_{n=1}^{k} L^n b_n(Z_l) J_{(n,n)}^{t_l,t_{l+1}} + \sum_{\substack{n,n_1=1 \\ n\neq n_1}}^{k} L^n b_{n_1}(Z_l) I_{(n,n_1)}^{t_l,t_{l+1}}\right\}, \end{aligned} \tag{2}$$

where $\mathcal{J}_a$ and $\mathcal{J}_{L^n b_n}$ denoting the Jacobian matrices of $a(\cdot)$ and $L^n b_n(\cdot)$, respectively. Furthermore,

$$\begin{aligned} &f\left(\mathcal{J}_a, \mathcal{J}_{L^n b_n}\right) = \frac{g\left(v\left(\mathcal{J}_a, \mathcal{J}_{L^n b_n}\right)\right) - I_{\mathrm{id}}}{v\left(\mathcal{J}_a, \mathcal{J}_{L^n b_n}\right)}, \quad g\left(\mathcal{J}_a, \mathcal{J}_{L^n b_n}\right) = \exp\left(v\left(\mathcal{J}_a, \mathcal{J}_{L^n b_n}\right)\right), \\ &v\left(\mathcal{J}_a, \mathcal{J}_{L^n b_n}\right) = h\left(\mathcal{J}_a - \frac{1}{2}\sum_{n=1}^{k}\mathcal{J}_{L^n b_n}\right), \\ &I_{(n)}^{t_l,t_{l+1}} = \int_{t_l}^{t_{l+1}} \mathrm{d}W_n(\tau) = W_n(t_{l+1}) - W_n(t_l), \\ &J_{(n,n)}^{t_l,t_{l+1}} = \int_{t_l}^{t_{l+1}}\left(\int_{t_l}^{\tau_1} \circ \mathrm{d}W_n(\tau_2)\right)\circ \mathrm{d}W_n(\tau_1), \quad I_{(n,n_1)}^{t_l,t_{l+1}} = \int_{t_l}^{t_{l+1}}\left(\int_{t_l}^{\tau_1} \mathrm{d}W_n(\tau_2)\right)\mathrm{d}W_{n_1}(\tau_1), \end{aligned}$$

where I_{id} represents the identity matrix. Also,

$$\begin{aligned} L^n &= \sum_{i=1}^{d} b_{n,i}\frac{\partial}{\partial Z_i}, \\ \sum_{n,n_1=1}^{k} L^n b_{n_1}(Z_l) I_{(n,n_1)}^{t_l,t_{l+1}} &= \sum_{\substack{n,n_1=1 \\ n\neq n_1}}^{k}\sum_{i=1}^{d}\frac{\partial b_{n_1}(Z_l)}{\partial Z_i} b_{n,i} I_{(n,n_1)}^{t_l,t_{l+1}} + \sum_{n=1}^{k}\sum_{i=1}^{d}\frac{\partial b_n(Z_l)}{\partial Z_i} b_{n,i} J_{(n,n)}^{t_l,t_{l+1}} \\ &= \sum_{i=1}^{d}\begin{bmatrix} \frac{\partial b_{1,1}}{\partial Z_i} & \cdots & \frac{\partial b_{k,1}}{\partial Z_i} \\ \vdots & \ddots & \vdots \\ \frac{\partial b_{1,d}}{\partial Z_i} & \cdots & \frac{\partial b_{k,d}}{\partial Z_i} \end{bmatrix}\begin{bmatrix} \mathcal{A}_{1,i} \\ \vdots \\ \mathcal{A}_{k,i} \end{bmatrix} + \sum_{i=1}^{d}\begin{bmatrix} \frac{\partial b_{1,1}}{\partial Z_i} & \cdots & \frac{\partial b_{k,1}}{\partial Z_i} \\ \vdots & \ddots & \vdots \\ \frac{\partial b_{1,d}}{\partial Z_i} & \cdots & \frac{\partial b_{k,d}}{\partial Z_i} \end{bmatrix}\begin{bmatrix} \mathcal{B}_{1,i} \\ \vdots \\ \mathcal{B}_{k,i} \end{bmatrix}, \end{aligned}$$

where $\mathcal{A} = b(\cdot) I_{(n,n_1)}^{t_l,t_{l+1}} \in \mathbb{R}^{d\times k}$ and $\mathcal{B} = b(\cdot) J_{(n,n)}^{t_l,t_{l+1}} \in \mathbb{R}^{d\times k}$, with $(b(\cdot))_{d\times k} \in \mathbb{R}^{d\times k}$, are a product of $d \times k$ diffusion matrices $b(\cdot)$ and $k \times k$ matrices of double stochastic integrals $I_{(n,n_1)}^{t_l,t_{l+1}}$ and $J_{(n,n)}^{t_l,t_{l+1}}$, respectively. $\mathcal{A}_i$ and $\mathcal{B}_i$ denote the i^{th} row of matrices $\mathcal{A}$ and $\mathcal{B}$, respectively. Note that, the symbol $\circ$ in above notation indicates the Stratonovich stochastic integral.

The work is organized as follows. Section 2 provides some preliminaries and applies the EXM scheme to (1). We carry out the analysis of the strong convergence of the proposed scheme in Section 3. In Section 4, has been studied stability properties of our method for linear test equation. Also, the MS A-stable and MS L-stable of the EXM method have been analyzed in this section. Furthermore, the MS stability of a two-dimensional EXM scheme for multi-dimensional systems with one Wiener noise has been investigated. In Section 5, we present numerical results to validate our theoretical findings and, a brief summary is made in Section 6.

2. Notations and preliminaries

Using the truncating idea for general stochastic Itô-Taylor expansions [17, Theorem 5.5.1], we can obtain the following explicit order 1.0 strong Itô-Taylor scheme

$$\begin{aligned} Z(t) = Z_0 &+ \int_{t_0}^{t} \left(a(Z(\tau)) - \frac{1}{2}\sum_{n=1}^{k} L^n b_n(Z(\tau)) \right) \mathrm{d}\tau + \sum_{n=1}^{k} \int_{t_0}^{t} b_n(Z(\tau)) \mathrm{d}W_n(\tau) \\ &+ \sum_{\substack{n,n_1=1 \\ n\neq n_1}}^{k} \int_{t_0}^{t} \int_{t_0}^{\tau_1} L^n b_{n_1}(Z(\tau_2)) \mathrm{d}W_n(\tau_2) \mathrm{d}W_{n_1}(\tau_1) \\ &+ \sum_{n=1}^{k} \int_{t_0}^{t} \int_{t_l}^{\tau_1} L^n b_n(Z(\tau_2)) \circ \mathrm{d}W_n(\tau_2) \circ \mathrm{d}W_n(\tau_1) \\ &+ \sum_{n=1}^{k} \int_{t_0}^{t} \int_{t_0}^{\tau_1} c(Z(\tau_2)) \mathrm{d}\tau_2 \mathrm{d}W_n(\tau_1), \end{aligned} \tag{3}$$

where $c(Z(\cdot))$ is the vector function in terms of a, b_n and $L^n b_{n_1}$, $n, n_1 = 1, 2, \ldots, k$. Similar to [11], we take $t = t_{l+1}$, $t_0 = t_l$ and adapt the Itô-Taylor expansion (3) to analyse scheme (2) as follows

$$\begin{aligned} Z(t_{l+1}) = Z(t_l) &+ \int_{t_l}^{t_{l+1}} \left(a(Z(\tau)) - \frac{1}{2}\sum_{n=1}^{k} L^n b_n(Z(\tau)) \right) \mathrm{d}\tau + \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} b_n(Z_l) \mathrm{d}W_n(\tau) \\ &+ \sum_{\substack{n,n_1=1 \\ n\neq n_1}}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} L^n b_{n_1}(Z(\tau_2)) \mathrm{d}W_n(\tau_2) \mathrm{d}W_{n_1}(\tau_1) \\ &+ \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} L^n b_n(Z(\tau_2)) \circ \mathrm{d}W_n(\tau_2) \circ \mathrm{d}W_n(\tau_1) \\ &+ \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} c(Z(\tau_2)) \mathrm{d}\tau_2 \mathrm{d}W_n(\tau_1). \end{aligned} \tag{4}$$

We let $\|\cdot\|_2$ denote the standard Euclidean norm for both vectors and matrices and $\|\cdot\|^2_{L^2(\Omega,\mathbb{R}^d)} = \mathbb{E}\left[\|\cdot\|_2^2\right]$. Also, let us impose the following assumptions and proposition.

Assumption 2.1. *The functions* $a, \{b_n\}_{n=1}^{k} : \mathbb{R}^d \to \mathbb{R}^d$ *are twice continuously differentiable.*

Assumption 2.2. *There exists a positive constant ℓ for $a, b_n \in \mathbb{R}^d$, $n = 1, 2, \ldots, k$ such that*

- *global Lipschitz condition:*

$$\left\|a(p) - a(q)\right\|_2 \vee \left\|b_n(p) - b_n(q)\right\|_2 \vee \left\|L^n b_n(p) - L^n b_n(q)\right\|_2 \vee \left\|L^n b_{n_1}(p) - L^n b_{n_1}(q)\right\|_2 \leq \ell \|p - q\|_2,$$

- *linear growth condition:*

$$\left\|\left(\vartheta(\mathfrak{J}_a, \mathfrak{J}_{L^n b_n})\right)^r a(p)\right\|_2^2 \vee \left\|\left(\vartheta(\mathfrak{J}_a, \mathfrak{J}_{L^n b_n})\right)^r b_n(p)\right\|_2^2 \vee \left\|\left(\vartheta(\mathfrak{J}_a, \mathfrak{J}_{L^n b_n})\right)^r L^n b_n(p)\right\|_2^2 \vee \left\|\left(\vartheta(\mathfrak{J}_a, \mathfrak{J}_{L^n b_n})\right)^r L^n b_{n_1}(p)\right\|_2^2 \leq \ell\left(1 + \|p\|_2^2\right), r = 0, 1, 2, \ldots,$$

where $\vartheta(\mathfrak{J}_a, \mathfrak{J}_{L^n b_n}) = \left(\mathfrak{J}_a - \frac{1}{2}\sum_{n=1}^{k} \mathfrak{J}_{L^n b_n}\right)$ *and $\vee$ is a maximal operator.*

We recall the following proposition before presenting the strong convergence analysis of our approach.

Proposition 2.3. *([11]). Let Assumption 2.2 holds. For each $T > 0$ and $Z(0) = Z_0 \in \mathbb{R}^d$ there exists a unique Z satisfying* (1) *such that*

$$\sup_{0\leq t\leq T} \|Z(t)\|_{L^2(\Omega,\mathbb{R}^d)} = \sup_{0\leq t\leq T} \mathbb{E}\left[\|Z(t)\|_2^2\right]^{1/2} < +\infty.$$

Furthermore, there exists, $\mathcal{L}$ such that for $0 \leq t, \tau \leq T$,

$$\|Z(t) - Z(\tau)\|_{L^2(\Omega,\mathbb{R}^d)} \leq \mathcal{L}|t - \tau|^{1/2}. \tag{5}$$

3. Convergence analysis of the proposed scheme

In the present section, we analyze the strong convergence of the numerical integrator (2). To achieve this goal, we first prove the following lemma.

Lemma 3.1. *Let Assumptions 2.1, 2.2 and Proposition 2.3 hold. Then*

$$\left\|\sum_{l=0}^{N-1} (Z(t_l) - Z_l)\right\|_{L^2(\Omega,\mathbb{R}^d)}^2 = O(h^2).$$

Proof. We derive from (2) and (4), local error $\epsilon_l = Z(t_l) - Z_l$, that

$$\left\|\sum_{l=0}^{N-1} \epsilon_l\right\|_{L^2(\Omega,\mathbb{R}^d)}^2 \leq 9 \sum_{j=1}^{9} \mathfrak{I}_j,$$

where

$$\mathcal{J}_1 = \left\| \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} (a(Z(\tau)) - a(Z_l))\, \mathrm{d}\tau \right\|^2_{L^2(\Omega,\mathbb{R}^d)},$$

$$\mathcal{J}_2 = \frac{1}{2}\left\| \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} (L^n b_n(Z(\tau)) - L^n b_n(Z_l))\, \mathrm{d}\tau \right\|^2_{L^2(\Omega,\mathbb{R}^d)},$$

$$\mathcal{J}_3 = \left\| \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} (I_{\mathrm{id}} - f(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l))) \left(a(Z_l) - \frac{1}{2}\sum_{n=1}^{k} L^n b_n(Z_l) \right) \mathrm{d}\tau \right\|^2_{L^2(\Omega,\mathbb{R}^d)},$$

$$\mathcal{J}_4 = \left\| \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} (I_{\mathrm{id}} - g(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)))\, b_n(Z_l) \mathrm{d}W(\tau) \right\|^2_{L^2(\Omega,\mathbb{R}^d)},$$

$$\mathcal{J}_5 = \left\| \sum_{l=0}^{N-1} \sum_{\substack{n,n_1=1\\ n\neq n_1}}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} (L^n b_{n_1}(Z(\tau_2)) - L^n b_{n_1}(Z_l))\, \mathrm{d}W_n(\tau_2) \mathrm{d}W_{n_1}(\tau_1) \right\|^2_{L^2(\Omega,\mathbb{R}^d)},$$

$$\mathcal{J}_6 = \left\| \sum_{l=0}^{N-1} \sum_{\substack{n,n_1=1\\ n\neq n_1}}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} (I_{\mathrm{id}} - g(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)))\, L^n b_{n_1}(Z_l) \mathrm{d}W_n(\tau_2) \mathrm{d}W_{n_1}(\tau_1) \right\|^2_{L^2(\Omega,\mathbb{R}^d)},$$

$$\mathcal{J}_7 = \left\| \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} (L^n b_n(Z(\tau_2)) - L^{n_1} b_n(Z_l)) \circ \mathrm{d}W_n(\tau_2) \circ \mathrm{d}W_n(\tau_1) \right\|^2_{L^2(\Omega,\mathbb{R}^d)},$$

$$\mathcal{J}_8 = \left\| \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} (I_{\mathrm{id}} - g(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)))\, L^n b_n(Z_l) \circ \mathrm{d}W_n(\tau_2) \circ \mathrm{d}W_n(\tau_1) \right\|^2_{L^2(\Omega,\mathbb{R}^d)},$$

$$\mathcal{J}_9 = \left\| \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} c(Z(\tau_2)) \mathrm{d}\tau_2 \mathrm{d}W_n(\tau_1) \right\|^2_{L^2(\Omega,\mathbb{R}^d)}.$$

Using the Taylor formula, we get

$$a(Z(\tau)) = a(Z_l) + \mathcal{J}_a(Z_l) \sum_{n=1}^{k} b_n(Z_l) I_{(n)}^{t_l,\tau} + \epsilon_a,$$

with $\epsilon_a = O(\tau - t_l)$. From $\mathbb{E}\left[\left\langle I_{(i)}^{t_l,\tau}, I_{(j)}^{t_l,\tau} \right\rangle\right]=0$, $i \neq j$, and Jensen's inequality, we have

$$\begin{aligned}\mathcal{J}_1 &\leq 2\left\| \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \mathcal{J}_a(Z_l) \sum_{n=1}^{k} b_n(Z_l) I_{(n)}^{t_l,\tau} \mathrm{d}\tau \right\|^2_{L^2(\Omega,\mathbb{R}^d)} + 2\left\| \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \epsilon_a \mathrm{d}\tau \right\|^2_{L^2(\Omega,\mathbb{R}^d)} \\ &\leq 2kK_{\mathcal{J}_1} h \sup_{n\in\{1,2,\dots,k\}} \mathbb{E}\left[\|\mathcal{J}_a(Z_l) b_n(Z_l)\|_2^2 \right] \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| I_{(n)}^{t_l,\tau} \right\|_2^2 \right] \mathrm{d}\tau + 2N\widetilde{K}_{\mathcal{J}_1} h \sup_{n\in\{1,2,\dots,k\}} \mathbb{E}\left[\|\epsilon_a\|_2^2 \right] \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \mathrm{d}r\end{aligned}$$

$$= O(h^2).$$

Similarly, for $\mathcal{I}_2$, we have

$$L^n b_n(Z(\tau)) = L^n b_n(Z_l) + \mathcal{J}_{L^n b_n}(Z_l) \sum_{n=1}^{k} b_n(Z_l) I_{(n)}^{t_l,\tau} + \epsilon_{L^n b_n},$$

with $\epsilon_{L^n b_n} = O(\tau - t_l)$. Then

$$\begin{aligned}
\mathcal{I}_2 &\leq \left\| \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \mathcal{J}_{L^n b_n}(Z_l) \sum_{n=1}^{k} b_n(Z_l) I_{(n)}^{t_l,\tau} \mathrm{d}\tau \right\|_{L^2(\Omega,\mathbb{R}^d)}^2 + \left\| \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \epsilon_{L^n b_n} \mathrm{d}\tau \right\|_{L^2(\Omega,\mathbb{R}^d)}^2 \\
&\leq kK_{\mathcal{I}_2} h \sup_{n\in\{1,2,\dots,k\}} \mathbb{E}\left[\left\| \mathcal{J}_{L^n b_n}(Z_l) b_n(Z_l) \right\|_2^2 \right] \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| I_{(n)}^{t_l,\tau} \right\|_2^2 \right] \mathrm{d}\tau + N\widetilde{K}_{\mathcal{I}_2} h \sup_{n\in\{1,2,\dots,k\}} \mathbb{E}\left[\left\| \epsilon_{L^n b_n} \right\|_2^2 \right] \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \mathrm{d}\tau \\
&= O(h^2).
\end{aligned}$$

For $\mathcal{I}_3$, in view of Assumption 2.2 and Jensen's inequality, we obtain that

$$\begin{aligned}
\mathcal{I}_3 &\leq K_{\mathcal{I}_3} \mathbb{E}\left[\left\| \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \left(I_{\mathrm{id}} - f\left(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)\right)\right) \left(a(Z_l) - \frac{1}{2} \sum_{n=1}^{k} L^n b_n(Z_l) \right) \mathrm{d}\tau \right\|_2^2 \right] \\
&\leq NK_{\mathcal{I}_3} h \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| \left(I_{\mathrm{id}} - f\left(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)\right)\right) \left(a(Z_l) - \frac{1}{2} \sum_{n=1}^{k} L^n b_n(Z_l) \right) \right\|_2^2 \right] \mathrm{d}\tau \\
&\leq 2TK_{\mathcal{I}_3} \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| \left(I_{\mathrm{id}} - f\left(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)\right)\right) a(Z_l) \right\|_2^2 \right] \mathrm{d}\tau \\
&\quad + TkK_{\mathcal{I}_3} \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| \left(I_{\mathrm{id}} - f\left(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)\right)\right) L^n b_n(Z_l) \right\|_2^2 \right] \mathrm{d}\tau \\
&\leq 2TK_{\mathcal{I}_3} \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| \sum_{r=1}^{\infty} \frac{\left(\vartheta(\mathcal{J}_a, \mathcal{J}_{L^n b_n})\right)^r a(Z_l)}{(r+1)!} |\tau - t_l|^r \right\|_2^2 \right] \mathrm{d}\tau \\
&\quad + TkK_{\mathcal{I}_3} \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| \sum_{r=1}^{\infty} \frac{\left(\vartheta(\mathcal{J}_a, \mathcal{J}_{L^n b_n})\right)^r L^n b_n(Z_l)}{(r+1)!} |\tau - t_l|^r \right\|_2^2 \right] \mathrm{d}\tau \\
&\leq 2TK_{\mathcal{I}_3} \sup_{n\in\{1,2,\dots,k\}} \mathbb{E}\left[\left\| \left(\vartheta(\mathcal{J}_a, \mathcal{J}_{L^n b_n})\right)^r a(Z_l) \right\|_2^2 \right] \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \left\| \sum_{r=1}^{\infty} \frac{|\tau - t_l|^r}{(r+1)!} \right\|_2^2 \mathrm{d}\tau \\
&\quad + TkK_{\mathcal{I}_3} \sup_{n\in\{1,2,\dots,k\}} \mathbb{E}\left[\left\| \left(\vartheta(\mathcal{J}_a, \mathcal{J}_{L^n b_n})\right)^r L^n b_n(Z_l) \right\|_2^2 \right] \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \left\| \sum_{r=1}^{\infty} \frac{|\tau - t_l|^r}{(r+1)!} \right\|_2^2 \mathrm{d}\tau \\
&\leq 2TK_{\mathcal{I}_3} \ell \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \left(\frac{\mathrm{e}^{|\tau - t_l|} - |\tau - t_l| - 1}{|\tau - t_l|} \right)^2 \mathrm{d}\tau + TkK_{\mathcal{I}_3} \ell \sum_{l=0}^{N-1} \sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \left(\frac{\mathrm{e}^{|\tau - t_l|} - |\tau - t_l| - 1}{|\tau - t_l|} \right)^2 \mathrm{d}\tau
\end{aligned}$$

$$\leq 2TK_{\mathcal{I}_3}\ell \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} |\tau - t_l|^2 \mathrm{d}\tau + TkK_{\mathcal{I}_3}\ell \sum_{l=0}^{N-1}\sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} |\tau - t_l|^2 \mathrm{d}\tau$$
$$= O(h^2).$$

Similarly, for $\mathcal{I}_4$ we have

$$\mathcal{I}_4 \leq NkK_{\mathcal{I}_4} \sum_{l=0}^{N-1}\sum_{n=1}^{k} \mathbb{E}\left[\left\| \int_{t_l}^{t_{l+1}} \left(I_{\mathrm{id}} - g\left(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)\right)\right) b_n(Z_l) \mathrm{d}W(\tau)\right\|_2^2\right]$$
$$\leq NkK_{\mathcal{I}_4} \sum_{l=0}^{N-1}\sum_{n=1}^{k} \mathbb{E}\left[\int_{t_l}^{t_{l+1}} \left\|\left(I_{\mathrm{id}} - g\left(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)\right)\right) b_n(Z_l)\right\|_2^2\right] \mathrm{d}\tau$$
$$\leq NkK_{\mathcal{I}_4} \sum_{l=0}^{N-1}\sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| \sum_{r=1}^{\infty} \frac{\left(\vartheta(\mathcal{J}_a, \mathcal{J}_{L^n b_n})\right)^r b_n(Z_l)}{(r+1)!} |\tau - t_l|^r \right\|_2^2\right] \mathrm{d}\tau$$
$$\leq NkK_{\mathcal{I}_4}\ell \sum_{l=0}^{N-1}\sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} \left(\frac{\mathrm{e}^{|\tau - t_l|} - |\tau - t_l| - 1}{|\tau - t_l|}\right)^2 \mathrm{d}\tau$$
$$\leq NkK_{\mathcal{I}_4}\ell \sum_{l=0}^{N-1}\sum_{n=1}^{k} \int_{t_l}^{t_{l+1}} |\tau - t_l|^2 \mathrm{d}\tau$$
$$= O(h^2).$$

For $\mathcal{I}_5$, Itô isometry implies that

$$\mathcal{I}_5 = \sum_{l=0}^{N-1}\sum_{n_1=1}^{k} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| \sum_{\substack{n=1\\ n\neq n_1}}^{k} \int_{t_l}^{\tau_1} \left(L^n b_{n_1}(Z(\tau_2)) - L^n b_{n_1}(Z_l)\right) \mathrm{d}W_n(\tau_2)\right\|_2^2\right] \mathrm{d}\tau_1$$
$$= \sum_{l=0}^{N-1} \sum_{\substack{n,n_1=1\\ n\neq n_1}}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} \mathbb{E}\left[\left\| L^n b_{n_1}(Z(\tau_2)) - L^n b_{n_1}(Z_l)\right\|_2^2\right] \mathrm{d}\tau_2 \mathrm{d}\tau_1.$$

It follows from Assumption 2.2 and Proposition 2.3 that

$$\mathcal{I}_5 \leq (\ell\mathcal{L})^2 K_{\mathcal{I}_5} \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} |\tau_2 - t_l| \mathrm{d}\tau_2 \mathrm{d}\tau_1$$
$$= O(h^2).$$

Using virtue of Itô isometry and Assumption 2.2 for $\mathcal{I}_6$, we yield

$$\mathcal{I}_6 \leq NK_{\mathcal{I}_6} \sum_{l=0}^{N-1}\sum_{n_1=1}^{k} \int_{t_l}^{t_{l+1}} \mathbb{E}\left[\left\| \sum_{\substack{n=1\\ n\neq n_1}}^{k} \int_{t_l}^{\tau_1} \left(I_{\mathrm{id}} - g\left(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)\right)\right) L^n b_{n_1}(Z_l) \mathrm{d}W_n(\tau_2)\right\|_2^2\right] \mathrm{d}\tau_1$$
$$\leq NK_{\mathcal{I}_6} \sum_{l=0}^{N-1} \sum_{\substack{n,n_1=1\\ n\neq n_1}}^{k} \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_1} \mathbb{E}\left[\left\| \left(I_{\mathrm{id}} - g\left(\mathcal{J}_a(Z_l), \mathcal{J}_{L^n b_n}(Z_l)\right)\right) L^n b_{n_1}(Z_l)\right\|_2^2\right] \mathrm{d}\tau_2 \mathrm{d}\tau_1$$

$$
\begin{aligned}
&\le NK_{\mathcal{I}_6}\sum_{l=0}^{N-1}\sum_{\substack{n,n_1=1\\ n\neq n_1}}^{k}\int_{t_l}^{t_{l+1}}\int_{t_l}^{\tau_1}\mathbb{E}\left[\left\|\sum_{r=1}^{\infty}\frac{(\vartheta(\mathcal{J}_a,\mathcal{J}_{L^n b_n}))^r L^n b_{n_1}(Z_l)}{(r+1)!}|\tau_2-t_l|^r\right\|_2^2\right]\mathrm{d}\tau_2\mathrm{d}\tau_1\\
&\le NK_{\mathcal{I}_6}\ell\sum_{l=0}^{N-1}\sum_{\substack{n,n_1=1\\ n\neq n_1}}^{k}\int_{t_l}^{t_{l+1}}\int_{t_l}^{\tau_1}\left(\frac{\mathrm{e}^{|\tau_2-t_l|}-|\tau_2-t_l|-1}{|\tau_2-t_l|}\right)^2\mathrm{d}\tau_2\mathrm{d}\tau_1\\
&\le NK_{\mathcal{I}_6}\ell\sum_{l=0}^{N-1}\sum_{\substack{n,n_1=1\\ n\neq n_1}}^{k}\int_{t_l}^{t_{l+1}}\int_{t_l}^{\tau_1}|\tau_2-t_l|^2\mathrm{d}\tau_2\mathrm{d}\tau_1\\
&=O(h^2).
\end{aligned}
$$

In a similar way, it can be shown that $\mathcal{I}_7 = O(h^2)$ and $\mathcal{I}_8 = O(h^2)$. And for $\mathcal{I}_9$, recalling Jensen's inequality and Itô isometry, one observes

$$
\begin{aligned}
\mathcal{I}_9 &\le kNK_{\mathcal{I}_9}\sup_{n\in\{1,2,\dots,k\}}\sum_{l=0}^{N-1}\sum_{n=1}^{k}\mathbb{E}\left[\|c(Z(\tau_2))\|_2^2\right]\mathbb{E}\left[\left\|\int_{t_l}^{t_{l+1}}\int_{t_l}^{\tau_1}\mathrm{d}\tau_2\mathrm{d}W_n(\tau_1)\right\|_2^2\right]\\
&\le \widetilde{K}_{\mathcal{I}_9}\sum_{l=0}^{N-1}\int_{t_l}^{t_{l+1}}|\tau_1-t_l|^2\mathrm{d}\tau_1\\
&=O(h^2).
\end{aligned}
$$

The proof is hence completed. □

The strong order of convergence of the EXM integrator (2) is presented in the following theorem.

Theorem 3.2. *Let Assumptions 2.1, 2.2 and Proposition 2.3 hold. Also, let Z_l be the approximation to the solution of* (4) *using the scheme* (2)*. Then, for $T > 0$, there exists $K > 0$ independent of h such that*

$$
\sup_{0\le t_l\le T}\|Z(t_l)-Z_l\|_{L^2(\Omega,\mathbb{R}^d)}\le Kh.
$$

Proof. First, by induction, we express the EXM scheme (2) at $t = t_N$ as

$$
\begin{aligned}
Z_N = Z_0 &+ \sum_{l=0}^{N-1}\int_{t_l}^{t_{l+1}} f\left(\mathcal{J}_a(Z_l),\mathcal{J}_{L^n b_n}(Z_l)\right)\left(a(Z_l)-\frac{1}{2}\sum_{n=1}^{k}L^n b_n(Z_l)\right)\mathrm{d}\tau\\
&+\sum_{l=0}^{N-1}\left(\sum_{n=1}^{k}\int_{t_l}^{t_{l+1}} g\left(\mathcal{J}_a(Z_l),\mathcal{J}_{L^n b_n}(Z_l)\right)b_n(Z_l)\mathrm{d}W_n(\tau)\right.\\
&+\sum_{\substack{n,n_1=1\\ n\neq n_1}}^{k}\int_{t_l}^{t_{l+1}}\int_{t_m}^{\tau_1} g\left(\mathcal{J}_a(Z_l),\mathcal{J}_{L^n b_n}(Z_l)\right)L^n b_{n_1}(Z_l)\mathrm{d}W_n(\tau_2)\mathrm{d}W_{n_1}(\tau_1)\\
&\left.+\sum_{n=1}^{k}\int_{t_l}^{t_{l+1}}\int_{t_l}^{\tau_1} g\left(\mathcal{J}_a(Z_l),\mathcal{J}_{L^n b_n}(Z_l)\right)L^n b_n(Z_l)\circ\mathrm{d}W_n(\tau_2)\circ\mathrm{d}W_n(\tau_1)\right).
\end{aligned}
\tag{6}
$$

To continue, we rewrite the EXM integrator (2) in continuous time process form $Z_h(t)$ that agrees with approximation Z_l at $t = t_l$. By introducing the variable $t = \tilde{t}_l$ for $t_l \le t < t_{l+1}$,

$$\begin{aligned} Z_h(t) = Z_0 &+ \int_0^t f\left(\mathcal{J}_a(Z(\tilde{\tau})), \mathcal{J}_{L^n b_n}(Z(\tilde{\tau}))\right)\left(a(Z(\tilde{\tau})) - \frac{1}{2}\sum_{n=1}^k L^n b_n(Z(\tilde{\tau}))\right)\mathrm{d}\tau \\ &+ \Bigg(\sum_{n=1}^k \int_0^t g\left(\mathcal{J}_a(Z(\tilde{\tau})), \mathcal{J}_{L^n b_n}(Z(\tilde{\tau}))\right) b_n(Z(\tilde{\tau}))\mathrm{d}W_n(\tau) \\ &+ \sum_{\substack{n,n_1=1\\ n\neq n_1}}^k \int_0^t \int_{\tilde{\tau}_2}^{\tau_1} g\left(\mathcal{J}_a(Z(\tilde{\tau}_2)), \mathcal{J}_{L^n b_n}(Z(\tilde{\tau}_2))\right) L^n b_{n_1}(Z(\tilde{\tau}_2))\mathrm{d}W_n(\tilde{\tau}_2)\mathrm{d}W_{n_1}(\tau_1) \\ &+ \sum_{n=1}^k \int_0^t \int_{\tilde{\tau}_2}^{\tau_1} g\left(\mathcal{J}_a(Z(\tilde{\tau}_2)), \mathcal{J}_{L^n b_n}(Z(\tilde{\tau}_2))\right) L^n b_n(Z(\tilde{\tau}_2)) \circ \mathrm{d}W_n(\tau_2) \circ \mathrm{d}W_n(\tau_1)\Bigg). \end{aligned} \tag{7}$$

This continuous version has the property that $Z_h(t_k) = Z_k$. The iterated sum of the actual value at $t = t_N$ is gained inductively to be

$$\begin{aligned} Z(t_N) = Z_0 &+ \int_0^{t_N} f\left(\mathcal{J}_a(Z(\tilde{\tau})), \mathcal{J}_{L^n b_n}(Z(\tilde{\tau}))\right)\left(a(Z(\tilde{\tau})) - \frac{1}{2}\sum_{n=1}^k L^n b_n(Z(\tilde{\tau}))\right)\mathrm{d}\tau \\ &+ \Bigg(\sum_{n=1}^k \int_0^{t_N} g\left(\mathcal{J}_a(Z(\tilde{\tau})), \mathcal{J}_{L^n b_n}(Z(\tilde{\tau}))\right) b_n(Z(\tilde{\tau}))\mathrm{d}W_n(\tau) \\ &+ \sum_{\substack{n,n_1=1\\ n\neq n_1}}^k \int_0^{t_N} \int_{\tilde{\tau}_2}^{\tau_1} g\left(\mathcal{J}_a(Z(\tilde{\tau})), \mathcal{J}_{L^n b_n}(Z(\tilde{\tau}))\right) L^n b_{n_1}(Z(\tilde{\tau}_2))\mathrm{d}W_n(\tilde{\tau}_2)\mathrm{d}W_{n_1}(\tau_1) \\ &+ \sum_{n=1}^k \int_0^{t_N} \int_{\tilde{\tau}_2}^{\tau_1} g\left(\mathcal{J}_a(Z(\tilde{\tau})), \mathcal{J}_{L^n b_n}(Z(\tilde{\tau}))\right) L^n b_n(Z(\tilde{\tau}_2)) \circ \mathrm{d}W_n(\tau_2) \circ \mathrm{d}W_n(\tau_1)\Bigg) \\ &+ \sum_{l=0}^{N-1} \epsilon_l. \end{aligned} \tag{8}$$

Denote $\mathsf{Err}(\tilde{t}) = Z(\tilde{t}) - Z_h(\tilde{t})$. So, by (7) and (8), we can prove that

$$\left\|\mathsf{Err}(\tilde{t})\right\|^2_{L^2(\Omega,\mathbb{R}^d)} \le 6\ell^2\left(2 + k + k\tilde{t}(1+k)\right)\int_0^{\tilde{t}} \mathbb{E}\left[\|\mathsf{Err}(\tau)\|_2^2\right]\mathrm{d}\tau + 6Kh^2.$$

Finally, Gronwall's inequality gives the desired assertion. □

4. Stability analysis of the proposed scheme

Nowadays, the study of the stability of SDEs has become one of the important parts of the literature. For one-dimensional numerical schemes, MS stability of the linear test equation has been discussed in many works, see [5, 8, 25, 35, 36] for instance. Also, the MS stability of the stochastic test systems has attracted the attention of some researchers see, for example, [3, 6, 30]. While for two-dimensional numerical integrators, few monographs [4, 28, 31, 33, 34] analyze the MS stability behavior of stochastic test systems.

Consider the linear scalar test equation as follows

$$\mathrm{d}Z(t) = \lambda Z(t)\mathrm{d}t + \sum_{n=1}^{k} \mu_n Z(t)\mathrm{d}W_n(t), \quad Z_0 = 1, \tag{9}$$

with $\lambda, \mu_1, \dots, \mu_k \in \mathbb{C}$. It has been demonstrated that SDE (9) is asymptotically MS stable if and only if

$$2\Re(\lambda) + \sum_{n=1}^{k} |\mu_n|^2 < 0, \tag{10}$$

see [2, 16, 19]. We would like to highlight that in inequality (10), $\Re(\lambda)$ denotes the real part of λ. For (9), the MS stability domain is defined in the following form

$$\mathfrak{D}^{\mathrm{MS}}_{(10)} = \left\{(\lambda, \mu_1, \dots, \mu_k) \in \mathbb{C}^{k+1} : \text{condition (10) holds}\right\}.$$

If the test equation (9) is solved by the presented scheme (2), we obtain the recurrence equation

$$Z_{l+1} = D^{\mathrm{MS}}_{\mathrm{num}}\left(h, \lambda, \{\mu_n\}_{n=1}^{k}, \{\zeta_{n,l}\}_{n=1}^{k}\right) Z_l, \quad l = 0, 1, \dots, \tag{11}$$

where $\zeta_{n,l} = \frac{\Delta W_{n,l}}{\sqrt{h}} \sim \mathcal{N}(0,1)$. Note that set $D^{\mathrm{MS}}_{\mathrm{num}}(\cdot)$ is called a stochastic function of a numerical scheme. In order to peruse the stability analysis of numerical techniques, the following definitions are provided, see [1, 13, 14, 29].

Definition 4.1. *The numerical scheme applied to the SDE* (9) *is asymptotically MS stable if and only if*

$$\widetilde{D}^{MS}_{num}\left(h, \lambda, \{\mu_n\}_{n=1}^{k}\right) = \mathbb{E}\left[\left|D^{MS}_{num}\left(h, \lambda, \{\mu_n\}_{n=1}^{k}, \{\zeta_{n,l}\}_{n=1}^{k}\right)\right|^2\right] < 1. \tag{12}$$

Now, we can identify the MS stability domain of the unknown numerical integrator for SDE (9) with

$$\mathfrak{D}^{\mathrm{MS}}_{\mathrm{num}} = \left\{(\lambda, \mu_1, \dots, \mu_k) \in \mathbb{C}^{k+1} : \text{condition (12) holds}\right\}.$$

Definition 4.2. *A numerical integrator is said to be MS A-stable if*

$$\mathfrak{D}^{MS}_{SDE} \subseteq \mathfrak{D}^{MS}_{num}.$$

Definition 4.3. *Assume that the numerical scheme is MS A-stable. If we have*

$$\lim_{\Re(\lambda_i)\to-\infty} \widetilde{D}^{MS}_{num}\left(h, \lambda_i, \{\mu_{i,n}\}_{n=1}^{k}\right) = 0,$$

for all sequences $(\lambda_i, \{\mu_{i,n}\}_{n=1}^{k}) \in \widetilde{D}^{MS}_{SDE}$, *then the scheme is called MS L-stable.*

The following theorem demonstrates that our method for the SDE (9) exhibits MS stability for any step-size.

Theorem 4.4. *For any time step* $h > 0$, $\mathfrak{D}^{MS}_{(9)} \subset \mathfrak{D}^{MS}_{(2)}$.

Proof. By choosing commutative noise terms in test equation (9), for method (2), we drive from (11) that

$$\begin{aligned} &D^{\mathrm{MS}}_{(2)}\left(h, \lambda, \{\mu_n\}_{n=1}^{k}, \{\zeta_{n,l}\}_{n=1}^{k}\right) \\ &= \left(1 + \sqrt{h}\sum_{n=1}^{k} \mu_n \zeta_{n,l} + \frac{1}{2}h \sum_{n=1}^{k} \mu_n^2 \zeta_{n,l}^2 + \frac{1}{2}h \sum_{\substack{n=1 \\ n\neq n_1}}^{k} \sum_{n_1=1}^{k} \mu_n \mu_{n_1} \zeta_{n,l} \zeta_{n_1,l}\right) \times \\ &\quad \exp\left(\lambda h - \frac{1}{2}h \sum_{n=1}^{k} \mu_n^2\right), \quad l = 0, 1, \dots, \end{aligned}$$

and consequently

$$\begin{aligned}
&\left|D_{(2)}^{\mathrm{MS}}\left(h,\lambda,\{\mu_n\}_{n=1}^{k},\{\zeta_{n,l}\}_{n=1}^{k}\right)\right|^2\\
&=D_{(2)}^{\mathrm{MS}}\left(h,\lambda,\{\mu_n\}_{n=1}^{k},\{\zeta_{n,l}\}_{n=1}^{k}\right)\cdot\overline{D_{(2)}^{\mathrm{MS}}\left(h,\lambda,\{\mu_n\}_{n=1}^{k},\{\zeta_{n,l}\}_{n=1}^{k}\right)}\\
&=\left(1+\sqrt{h}\sum_{n=1}^{k}\mu_n\zeta_{n,l}+\frac{1}{2}h\sum_{n=1}^{k}\mu_n^2\zeta_{n,l}^2+\frac{1}{2}h\sum_{\substack{n=1\\n\neq n_1}}^{k}\sum_{n_1=1}^{k}\mu_n\mu_{n_1}\zeta_{n,l}\zeta_{n_1,l}\right)\exp\left(\lambda h-\frac{1}{2}h\sum_{n=1}^{k}\mu_n^2\right)\times\\
&\quad\left(1+\sqrt{h}\sum_{n=1}^{k}\bar{\mu}_n\zeta_{n,l}+\frac{1}{2}h\sum_{n=1}^{k}\bar{\mu}_n^2\zeta_{n,l}^2+\frac{1}{2}h\sum_{\substack{n=1\\n\neq n_1}}^{k}\sum_{n_1=1}^{k}\bar{\mu}_n\bar{\mu}_{n_1}\zeta_{n,l}\zeta_{n_1,l}\right)\exp\left(\bar{\lambda}h-\frac{1}{2}h\sum_{n=1}^{k}\bar{\mu}_n^2\right)\\
&=\Bigg(1+2\sqrt{h}\sum_{n=1}^{k}\Re(\mu_n)\zeta_{n,l}+h\sum_{n=1}^{k}|\mu_n|^2\zeta_{n,l}^2+h\sum_{n=1}^{k}\sum_{n_1=1}^{k}\mu_n\bar{\mu}_{n_1}\zeta_{n,l}\zeta_{n_1,l}\\
&\quad+h\sum_{\substack{n=1\\n\neq n_1}}^{k}\sum_{n_1=1}^{k}|\mu_n\mu_{n_1}|^2\zeta_{n,l}\zeta_{n_1,l}+\frac{1}{2}h^{3/2}\sum_{n=1}^{k}\sum_{n_1=1}^{k}\left(\mu_n\bar{\mu}_{n_1}^2+\bar{\mu}_n\mu_{n_1}^2\right)\zeta_{n,l}\zeta_{n_1,l}^2\\
&\quad+\frac{1}{2}h^{3/2}\sum_{\substack{n=1\\n\neq n_1}}^{k}\sum_{n_1=1}^{k}\sum_{n_2=1}^{k}\left(\mu_n\mu_{n_1}\mu_{n_2}+\bar{\mu}_n\bar{\mu}_{n_1}\bar{\mu}_{n_2}\right)\zeta_{n,l}\zeta_{n_1,l}\zeta_{n_2,l}\\
&\quad+\frac{1}{4}h^2\sum_{\substack{n=1\\n\neq n_1}}^{k}\sum_{n_1=1}^{k}\sum_{n_2=1}^{k}\left(\mu_n\mu_{n_1}\mu_{n_2}^2+\bar{\mu}_n\bar{\mu}_{n_1}\bar{\mu}_{n_2}^2\right)\zeta_{n,l}\zeta_{n_1,l}\zeta_{n_2,l}^2\\
&\quad+\frac{1}{4}h^2\sum_{n=1}^{k}\sum_{n_1=1}^{k}\mu_n^2\mu_{n_1}^2\zeta_{n,l}^2\zeta_{n_1,l}^2\\
&\quad+\frac{1}{4}h^2\sum_{\substack{n=1\\n\neq n_1}}^{k}\sum_{n_1=1}^{k}\sum_{\substack{n'=1\\n'\neq n'_1}}^{k}\sum_{n'_1=1}^{k}\mu_n\mu_{n_1}\mu_{n'}\mu_{n'_1}\zeta_{n,l}\zeta_{n_1,l}\zeta_{n',l}\zeta_{n'_1,l}\Bigg)\exp\left(2\Re(\lambda)h-h\sum_{n=1}^{k}|\mu_n|^2\right).
\end{aligned}$$

On the basis of Definition 4.1, we can get

$$\begin{aligned}
\widetilde{D}_{(2)}^{\mathrm{MS}}\left(h,\lambda,\{\mu_n\}_{n=1}^{k}\right)=\Bigg(1+2h\sum_{n=1}^{k}|\mu_n|^2+\frac{3}{4}h^2\sum_{n=1}^{k}|\mu_n|^4\\
+\frac{1}{4}h^2\sum_{\substack{n=1\\n\neq n_1}}^{k}\sum_{n_1=1}^{k}|\mu_n|^2|\mu_{n_1}|^2\Bigg)\exp\left(2\Re(\lambda)h-h\sum_{n=1}^{k}|\mu_n|^2\right).
\end{aligned}\tag{13}$$

From (12), we know that inequality $\widetilde{D}_{(2)}^{\mathrm{MS}}\left(h,\lambda,\{\mu_n\}_{n=1}^{k}\right)<1$ is always true if

$$1+2h\sum_{n=1}^{k}|\mu_n|^2+\frac{3}{4}h^2\sum_{n=1}^{k}|\mu_n|^4+\frac{1}{4}h^2\sum_{\substack{n=1\\n\neq n_1}}^{k}\sum_{n_1=1}^{k}|\mu_n|^2|\mu_{n_1}|^2-\exp\left(-2\Re(\lambda)h+h\sum_{n=1}^{k}|\mu_n|^2\right)<0.$$

The above inequality can be written as

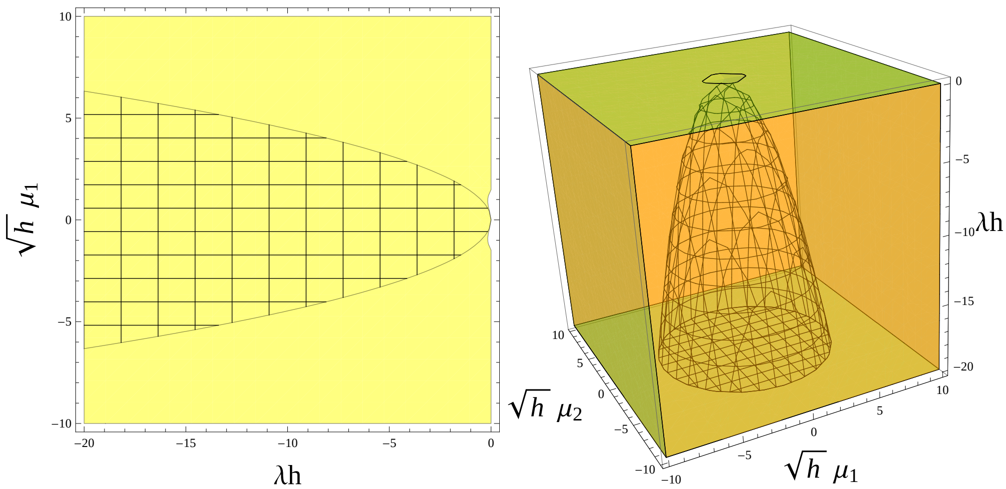
$$2\Re(\lambda)h + h\sum_{n=1}^{k}|\mu_n|^2 - 2(\Re(\lambda)h)^2 - \frac{3}{4}h^2\left(\sum_{n=1}^{k}|\mu_n|^2\right)^2 - \frac{1}{2}h^2\sum_{\substack{n=1\\ n\neq n_1}}^{k}\sum_{n_1=1}^{k}|\mu_n|^2|\mu_{n_1}|^2$$
$$+ h^2\sum_{n=1}^{k}|\mu_n|^2\left(2\Re(\lambda) + \sum_{n=1}^{k}|\mu_n|^2\right) + \sum_{j=3}^{\infty}(-1)^{j+1}\frac{\left(2\Re(\lambda)h - h\sum_{n=1}^{k}|\mu_n|^2\right)^j}{j!} < 0.$$

With the aid of condition (10) and $\Re(\lambda) < 0$, we complete the proof. □

In view of Theorem 4.4, it is easy to show that the EXM approach is MS stable for test equation (9) with one noise term. In Figure 1, has been shows a graphical comparison of the domain of MS stability of the scheme (2) and the test equation (9). From Figure 1, it is clear that our scheme is MS stable for all time step $h > 0$. Thereby, EXM method (2) is MS A-stable, see Definition 4.2. Furthermore, we can obtain from the (13) and Definition 4.3

$$\lim_{\Re(\lambda_i)\to-\infty} \widetilde{D}_{(2)}^{\mathrm{MS}}\left(h, \lambda_i, \{\mu_{i,n}\}_{n=1}^{k}\right) = 0,$$

for all $(\lambda_i, \{\mu_{i,n}\}_{n=1}^{k}) \in \widetilde{D}_{(9)}^{\mathrm{MS}}$. Hence, scheme (2) is MS L-stable.



(a) One noise term. (b) Two commutative noise terms.

Figure 1: Comparison of MS stability regions: scheme (2) (shaded) and test SDE (9)(gridded).

For real parameters $\lambda_1, \lambda_2, \sigma$, we consider the two-dimensional test system [33]

$$\mathrm{d}Z(t) = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} Z(t)\mathrm{d}t + \begin{bmatrix}0 & \sigma\\ \sigma & 0\end{bmatrix} Z(t)\mathrm{d}W(t), \tag{14}$$

where $Z(t) = [Z_1(t), Z_2(t)]^T$.

Lemma 4.5. *([33]). The SDE* (14) *is asymptotically MS-stable if and only if*

$$\lambda_1 + \lambda_2 + \sqrt{\sigma^4 + (\lambda_1 - \lambda_2)^2} < 0. \tag{15}$$

For test system (14), the domain of MS stability is provided in the form

$$\mathsf{M}_{(14)}^{\mathrm{MS}} = \left\{(\lambda_1, \lambda_2, \sigma) \in \mathbb{R}^3 : \text{condition (15) holds}\right\}.$$

By applying the bi-dimensional numerical method (2) to (14), we can get

$$\begin{bmatrix} Z_{1,l+1} \\ Z_{2,l+1} \end{bmatrix} = \mathcal{M}_{(2)} \begin{bmatrix} Z_{1,l} \\ Z_{2,l} \end{bmatrix}, \tag{16}$$

where

$$\mathcal{M}_{(2)} = \begin{bmatrix} (1 + \frac{1}{2}h\sigma^2\zeta^2)\mathrm{e}^{h\left(\lambda_1 - \frac{1}{2}\sigma^2\right)} & \sqrt{h}\sigma\zeta\mathrm{e}^{h\left(\lambda_1 - \frac{1}{2}\sigma^2\right)} \\ \sqrt{h}\sigma\zeta\mathrm{e}^{h\left(\lambda_2 - \frac{1}{2}\sigma^2\right)} & \left(1 + \frac{1}{2}h\sigma^2\zeta^2\right)\mathrm{e}^{h\left(\lambda_2 - \frac{1}{2}\sigma^2\right)} \end{bmatrix}.$$

According to [33], the EXM scheme (2) is asymptotically MS stable if the numerical solutions Z_l satisfy

$$\lim_{l\to\infty} \mathbb{E}\left[\|(Z_l)^2\|\right] = 0.$$

This condition can be written as follows

$$\lim_{l\to\infty} \mathsf{Z}_l = 0,$$

with $\mathsf{Z}_l = \left[\mathbb{E}[(Z_{1,l})^2], \mathbb{E}[(Z_{2,l})^2], \mathbb{E}[Z_{1,l}Z_{2,l}]\right]^T$, see [2] for more details. Thus, we rewrite (16) as

$$\mathsf{Z}_{l+1} = \overline{\mathcal{M}}_{(2)}\mathsf{Z}_l, \tag{17}$$

where

$$\overline{\mathcal{M}}_{(2)} = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix},$$

with

$$\begin{aligned} m_{11} &= \left(1 + \sigma^2 h + \frac{3}{4}\sigma^4 h^2\right)\mathrm{e}^{h\left(2\lambda_1 - \sigma^2\right)}, \\ m_{12} &= \sigma^2 h\mathrm{e}^{h\left(2\lambda_1 - \sigma^2\right)}, \\ m_{21} &= \sigma^2 h\mathrm{e}^{h\left(2\lambda_2 - \sigma^2\right)}, \\ m_{22} &= \left(1 + \sigma^2 h + \frac{3}{4}\sigma^4 h^2\right)\mathrm{e}^{h\left(2\lambda_2 - \sigma^2\right)}, \\ m_{33} &= \left(1 + 2\sigma^2 h + \frac{3}{4}\sigma^4 h^2\right)\mathrm{e}^{h\left(\lambda_1 + \lambda_2 - \sigma^2\right)}. \end{aligned}$$

Since $\mathsf{Z}_l \to 0$ if and only if $\rho(\overline{\mathcal{M}}_{(2)}) < 1$, see [9]. Note that $\rho(\overline{\mathcal{M}}_{(2)})$ stands for the spectral radius of the matrix $\overline{\mathcal{M}}_{(2)}$. So, numerical approach (2) is asymptotically MS stable if for eigenvalues of matrix $\overline{\mathcal{M}}_{(2)}$ we have

$$\max\{|m_{33}|, |\Lambda_\pm|\} < 1,$$

where

$$\Lambda_\pm = \frac{1}{2}\left(m_{11} + m_{22} \pm \sqrt{(m_{11} - m_{22})^2 + 4m_{12}m_{21}}\right).$$

Since $m_{33} \leq \Lambda_+$ and $|\Lambda_-| \leq \Lambda_+$, the asymptotic MS stability of a scheme (2) becomes

$$-1 < m_{33}, \quad \Lambda_+ < 1. \tag{18}$$

We specify the MS stability domain of the EXM scheme applied to the test system (14) with

$$\mathsf{M}^{\mathrm{MS}}_{(2)} = \left\{(\lambda_1, \lambda_2, \sigma) \in \mathbb{R}^3 : \text{condition (18) holds}\right\}.$$

The asymptotic MS stability of method (2) applied to the stochastic system (14) is investigated in the following theorem.

Theorem 4.6. *For all step-size $h > 0$, $M^{MS}_{(14)} \subset M^{MS}_{(2)}$.*

Proof. The first condition of (18) is always satisfied because

$$1 + 2\sigma^2 h + \frac{3}{4}\sigma^4 h^2 + \mathrm{e}^{h\left(-(\lambda_1+\lambda_2)+\sigma^2\right)} > 0.$$

For second condition of (18), we have

$$(m_{11} - m_{22})^2 + 4m_{12}m_{21} < (2 - (m_{11} + m_{22}))^2.$$

This is equivalent to showing

$$(1 - m_{11})(1 - m_{22}) > m_{12}m_{21}. \tag{19}$$

Since

$$(1 - m_{11})(1 - m_{22}) > \left(\mathrm{e}^{h\left(2\lambda_1-\sigma^2\right)} - m_{11}\right)\left(\mathrm{e}^{h\left(2\lambda_2-\sigma^2\right)} - m_{22}\right),$$

inequality (19) is equivalent to

$$\left(\mathrm{e}^{h\left(2\lambda_1-\sigma^2\right)} - m_{11}\right)\left(\mathrm{e}^{h\left(2\lambda_2-\sigma^2\right)} - m_{22}\right) - m_{12}m_{21} > 0,$$

which can be written as

$$\begin{aligned}\left(\mathrm{e}^{h\left(2\lambda_1-\sigma^2\right)} - m_{11}\right)\left(\mathrm{e}^{h\left(2\lambda_2-\sigma^2\right)} - m_{22}\right) - m_{12}m_{21} &= \mathrm{e}^{2h\left(\lambda_1+\lambda_2-\sigma^2\right)}\sigma^2 h\left(\left(1 + \frac{3}{4}\sigma^2 h\right)^2 - \sigma^2 h\right) \\ &> 0.\end{aligned}$$

This completes the proof. □

The MS stability domains of the EXM integrator and test system (14), $\mathsf{M}^{\mathrm{MS}}_{(2)}$ and $\mathsf{M}^{\mathrm{MS}}_{(14)}$ respectively, are plotted in Figure 2. Based on our finding from Figure 2, numerical method (2) is MS stable for all step-size $h > 0$. Therefore, the findings of Theorem 4.6 are confirmed.

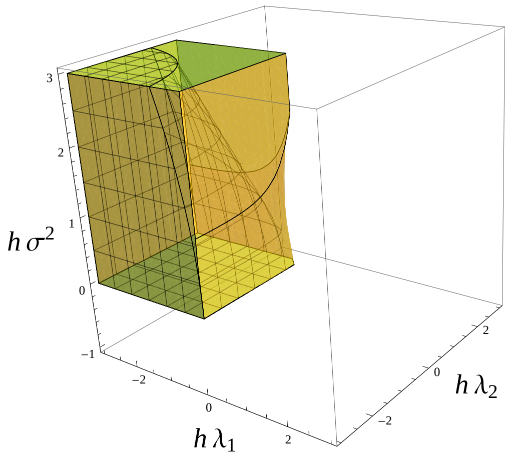


Figure 2: Comparison of MS stability regions: scheme (2) (shaded) and test system (14)(gridded).

For $\lambda, \epsilon, \sigma > 0$, consider the following two-dimensional test system [33]

$$\mathrm{d}Z(t) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} Z(t)\mathrm{d}t + \begin{bmatrix} \epsilon & \sigma \\ \sigma & \epsilon \end{bmatrix} Z(t)\mathrm{d}W(t). \tag{20}$$

Lemma 4.7. *([33]). The SDE* (20) *is asymptotically MS-stable if and only if*

$$\max\left\{2\lambda + (\epsilon - \sigma)^2, 2\lambda + (\epsilon + \sigma)^2\right\} < 0. \tag{21}$$

The MS stability domain of the test system (20) can be identified with

$$\mathsf{S}^{\mathrm{MS}}_{(20)} = \left\{(\lambda, \epsilon, \sigma) \in \mathbb{R}^3 : \text{condition (21) holds}\right\}.$$

If the test system (20) is solved by the scheme (2), we obtain the recurrence equation as (16) with the stochastic function of the form

$$\mathcal{S}_{(2)} = \frac{1}{2}\mathrm{e}^{\frac{1}{2}h\left(2\lambda - \epsilon^2 - \sigma^2\right)} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_1 \end{bmatrix},$$

with

$$s_1 = \left(\mathrm{e}^{-h\epsilon\sigma} + \mathrm{e}^{h\epsilon\sigma}\right)\left(1 + \sqrt{h}\epsilon\zeta + \frac{1}{2}h\left(\epsilon^2 + \sigma^2\right)\zeta^2\right) + \left(\mathrm{e}^{-h\epsilon\sigma} - \mathrm{e}^{h\epsilon\sigma}\right)\left(\sqrt{h}\sigma\zeta + h\epsilon\sigma\zeta^2\right)$$
$$s_2 = \left(\mathrm{e}^{-h\epsilon\sigma} - \mathrm{e}^{h\epsilon\sigma}\right)\left(1 + \sqrt{h}\epsilon\zeta + \frac{1}{2}h\left(\epsilon^2 + \sigma^2\right)\zeta^2\right) + \left(\mathrm{e}^{-h\epsilon\sigma} + \mathrm{e}^{h\epsilon\sigma}\right)\left(\sqrt{h}\sigma\zeta + h\epsilon\sigma\zeta^2\right).$$

Similar to the (17), for SDE (20), we have

$$\overline{\mathcal{S}}_{(2)} = \frac{1}{8}\mathrm{e}^{h\left(2\lambda - \epsilon^2 - \sigma^2\right)} \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{11} & s_{13} \\ \frac{1}{2}s_{13} & \frac{1}{2}s_{13} & s_{11} + s_{12}, \end{bmatrix}, \tag{22}$$

where

$$
\begin{aligned}
s_{11} &= \frac{1}{2}\Big\{\mathrm{e}^{-2h\epsilon\sigma}\left(2+h(\epsilon+\sigma)^2\right)\left(2+3h(\epsilon+\sigma)^2\right) \\
&\quad + \mathrm{e}^{2h\epsilon\sigma}\left(2+h(\epsilon-\sigma)^2\right)\left(2+3h(\epsilon-\sigma)^2\right)\Big\} + 4\left(1+2h\epsilon^2\right) + 3h^2\left(\epsilon^2-\sigma^2\right)^2, \\
s_{12} &= \frac{1}{2}\Big\{\mathrm{e}^{-2h\epsilon\sigma}\left(2+h(\epsilon+\sigma)^2\right)\left(2+3h(\epsilon+\sigma)^2\right) \\
&\quad + \mathrm{e}^{2h\epsilon\sigma}\left(2+h(\epsilon-\sigma)^2\right)\left(2+3h(\epsilon-\sigma)^2\right)\Big\} - 4\left(1+2h\epsilon^2\right) - 3h^2\left(\epsilon^2-\sigma^2\right)^2, \\
s_{13} &= \mathrm{e}^{-2h\epsilon\sigma}\left(2+h(\epsilon+\sigma)^2\right)\left(2+3h(\epsilon+\sigma)^2\right) - \mathrm{e}^{2h\epsilon\sigma}\left(2+h(\epsilon-\sigma)^2\right)\left(2+3h(\epsilon-\sigma)^2\right).
\end{aligned}
$$

For the stability matrix of the scheme EXM (22), we calculate the eigenvalues

$$
\begin{aligned}
\Gamma &= \frac{1}{8}\mathrm{e}^{h\left(2\lambda-\epsilon^2-\sigma^2\right)}(s_{11}-s_{12}) \\
&= \frac{1}{4}\mathrm{e}^{h\left(2\lambda-\epsilon^2-\sigma^2\right)}\left(4\left(1+2h\epsilon^2\right)+3h^2\left(\epsilon^2-\sigma^2\right)^2\right), \\
\Gamma_\pm &= \frac{1}{8}\mathrm{e}^{h\left(2\lambda-\epsilon^2-\sigma^2\right)}(s_{11}+s_{12}\pm s_{13}) \\
&= \frac{1}{4}\mathrm{e}^{h\left(2\lambda-(\epsilon\pm\sigma)^2\right)}\left(2+h(\epsilon\pm\sigma)^2\right)\left(2+3h(\epsilon\pm\sigma)^2\right).
\end{aligned}
$$

Now, the numerical MS stability condition test system (20) becomes

$$
\max\{|\Gamma|, |\Gamma_\pm|\} < 1.
$$

Since, $\Gamma_\pm \geq 0$ and $|\Gamma| \leq \max\{\Gamma_\pm\}$, the asymptotic MS stability of an EXM scheme is converted to

$$
\max\{\Gamma_\pm\} < 1. \tag{23}
$$

Then, the MS stability domain of our integrator was applied to solve the test system (20) become

$$
\mathsf{S}_{(2)}^{\mathrm{MS}} = \left\{(\lambda,\epsilon,\sigma)\in\mathbb{R}^3 : \text{condition (23) holds}\right\}.
$$

The following theorem is dedicated to analyzing the asymptotic MS stability of our scheme (2) employed in equation (20).

Theorem 4.8. *For any step-size $h > 0$, $\mathsf{S}_{(20)}^{MS} \subset \mathsf{S}_{(2)}^{MS}$.*

Proof. We rewrite condition (23) as follows

$$
\begin{aligned}
\left(2+h(\epsilon\pm\sigma)^2\right)\left(2+3h(\epsilon\pm\sigma)^2\right) - 4\mathrm{e}^{h\left(-2\lambda+(\epsilon\pm\sigma)^2\right)} &= 4h\left(2\lambda+(\epsilon\pm\sigma)^2\right) + h^2\left(2\lambda+(\epsilon\pm\sigma)^2\right)\left(-2\lambda+(\epsilon\pm\sigma)^2\right) \\
&\quad + 8h^2\lambda(\epsilon\pm\sigma)^2 - 4\sum_{j=3}^{\infty}\frac{h^j\left(-2\lambda+(\epsilon\pm\sigma)^2\right)^j}{j!} \\
&< 0.
\end{aligned}
$$

Obviously, the proof could be completed by using condition (21). □

In Figure 3, the domains of MS stability of EXM method (2) and test system (20) are shown. It can be seen from Figure 3 that our approach is asymptotically MS stable for any time step $h > 0$. It can be observed that the findings of Figure 3 support the theoretical results of Theorem 4.8.

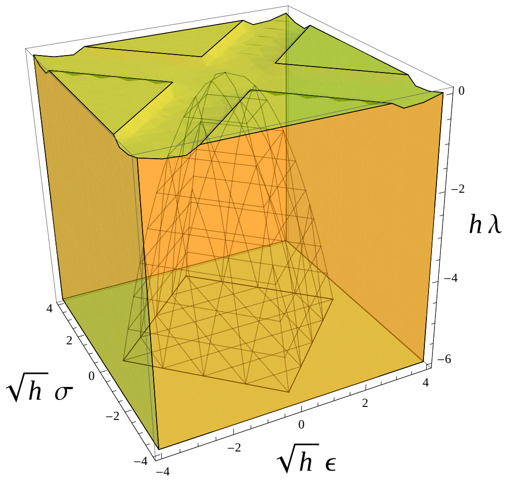


Figure 3: Comparison of MS stability regions: scheme (2) (shaded) and test system (20)(gridded).

5. Numerical illustrations

This section examines the applicability and accuracy of the EXM (2) scheme with a comparison of explicit 1.0 order methods Milstein [21] and balanced Milstein (BM) [15], by using six problems. For eight time steps $h = 2^m$, $m = -4, -5, \dots, -11$, 5000 sample trajectories are used to simulate the root of MS errors (RMSEs) at end point $T = 1$. Here, we have used the numerical approximation of schemes with a small time step $h_{\text{exact}} = 2^{-14}$ instead of the unknown exact value.

Example 5.1. *The first linear SDEs that we consider are*

$$\mathrm{d}Z(t) = \alpha Z(t)\mathrm{d}t + \beta_1 Z(t)\mathrm{d}W(t), \quad Z_0 = 1, \tag{24}$$
$$\mathrm{d}Z(t) = \alpha Z(t)\mathrm{d}t + \beta_1 Z(t)\mathrm{d}W_1(t) + \beta_2 Z(t)\mathrm{d}W_2(t), \quad Z_0 = 1. \tag{25}$$

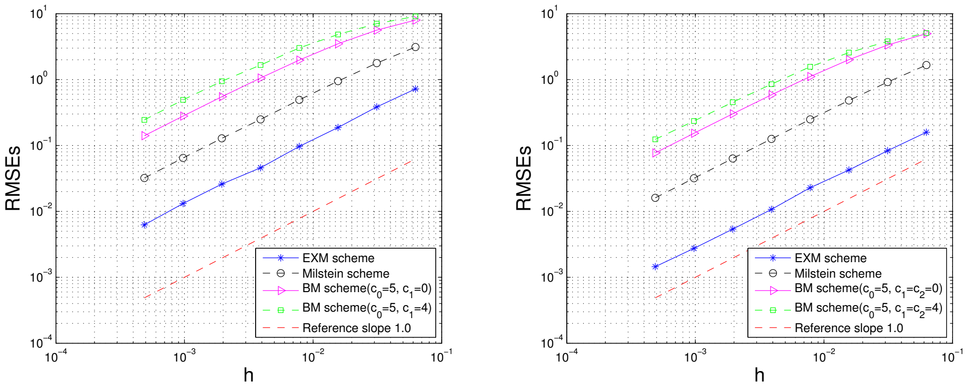
In general, the analytical solution of linear scalar SDE is

$$Z(t) = Z_0 \exp\left(\left(\alpha - \frac{1}{2}\sum_{n=1}^{k}\beta_n\right)t + \sum_{n=1}^{k}\beta_n W_n(t)\right).$$

In order to demonstrate the convergence rate of the numerical integrators, we choose coefficients $\alpha = 2$, $\beta_1 = 1$ and $\alpha = 2$, $\beta_1 = \beta_2 = \frac{1}{2}$ for (24) and (25), respectively. These results are reported in Figure 4. Further, we assume RMSEs $\approx \mathcal{K}h^{\gamma}$ for some constants $\mathcal{K}$, γ and can write

$$\ln(RMSEs) \approx \ln(\mathcal{K}) + \gamma \ln(h). \tag{26}$$

In Table 1, we presented a least squares fit for the parameters $\mathcal{K}$ and γ for linear SDEs (24) and (25). We can find from Table 1 that only the estimated convergence orders of EXM and Milstein methods are close to the theoretical result of 1.0. In the following experiment, we have changed the coefficients of SDEs (24) and (25) by $\alpha = 2$, $\beta_1 = 1$ and $\alpha = 2$, $\beta_1 = \beta_2 = \frac{1}{2}$, respectively, and compared the RMSEs of numerical schemes in Tables 2 and 3. For this aim, we simulate 5000 sample paths for eight different step-sizes $h = 2^{3-m}/100$, $m = 1, 2, \dots, 8$ in the final time $T = 1$. It can be seen from Tables 2 and 3 that the numerical solution generated with our scheme has the closest trend to an actual value in the MS sense.



(a) SDE (24) with $\alpha = 2, \beta_1 = 1$.

(b) SDE (25) with $\alpha = 2, \beta_1 = \beta_2 = \frac{1}{2}$.

Figure 4: Strong convergence order of numerical schemes for Example 5.1.

Table 1: A least squares for the parameters $\mathcal{K}$ and γ, for Example 5.1.

		EXM (2)	Milstein	BM($c_0 = 5, c_1 = 0$)	BM($c_0 = 5, c_1 = 4$)
SDE (24)	γ	0.9810	0.9497	0.8933	0.8646
	$\ln(\mathcal{K})$	0.0491	0.1204	0.2597	0.3638
SDE (25)	γ	0.9818	0.9623	0.8948	0.8163
	$\ln(\mathcal{K})$	0.0337	0.0953	0.2572	0.3798

Table 2: RMSEs of numerical schemes for SDE (24) with $\alpha = 3, \beta_1 = \frac{3}{4}$.

n	EXM (2)	Milstein	BM($c_0 = 5, c_1 = 0$)	BM($c_0 = 5, c_1 = 4$)
1	3.10e–01	5.31e+00	1.15e+01	1.45e+01
2	1.91e–01	3.28e+00	7.79e+00	1.12e+00
3	8.76e–02	1.61e+00	4.39e+00	6.56e+00
4	4.35e–02	8.78e–01	2.43e+00	3.91e+00
5	2.12e–02	4.15e–01	1.26e+00	2.00e+00
6	1.07e–02	2.12e–01	6.41e–01	1.05e+00
7	5.27e–03	1.06e–01	3.23e–01	5.35e–01
8	2.74e–03	5.44e–02	1.63e–01	2.75e–01

Table 3: RMSEs of numerical schemes for SDE (25) with $\alpha = 3, \beta_1 = \frac{1}{2}$ and $\beta_2 = \frac{1}{4}$.

n	EXM (2)	Milstein	BM($c_0 = 5, c_1 = 0$)	BM($c_0 = 5, c_1 = 4$)
1	1.06e–01	4.24e+00	1.15e+01	1.10e+01
2	6.01e–02	2.47e+00	8.65e+00	8.09e+00
3	2.95e–02	1.25e+00	5.39e+00	4.84e+00
4	1.49e–02	6.58e–01	3.59e+00	2.84e+00
5	7.44e–03	3.28e–01	2.40e+00	1.50e+00
6	3.81e–03	1.65e–01	1.71e+00	7.81e–01
7	1.88e–03	8.26e–02	1.19e+00	4.00e–01
8	9.19e–04	4.14e–02	6.49e–01	2.02e–01

Example 5.2. *Next, consider the immigration-death reaction network [12]*

$$\emptyset \xrightarrow{\bar{c}_1} U_1,$$
$$U_1 \xrightarrow{\bar{c}_2} \emptyset,$$

where $\bar{c}_1 = 4$ *and* $\bar{c}_2 = 0.8$. *The corresponding chemical Langevin equation is given by*

$$\mathrm{d}Z(t) = (\bar{c}_1 - \bar{c}_2 Z(t))\,\mathrm{d}t + \sqrt{\bar{c}_1 + \bar{c}_2 Z(t)}\mathrm{d}W(t), \quad Z_0 = 500. \tag{27}$$

To compare the RMSEs of the numerical results, one can refer to Figure 5. From Figure 5, the convergence orders 1.0 *can be detected for numerical EXM and Milstein schemes. Moreover, it is clear that our method is the closest scheme to the exact solution. Table 4 has summarized the convergence rate estimation for the numerical schemes in SDE* (27).

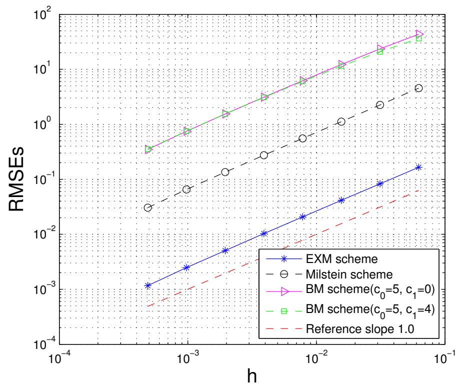


Figure 5: Strong convergence order of numerical integrators for nonlinear SDE (27).

Table 4: A least squares for the parameters $\mathcal{K}$ and γ, for Example 5.2.

	EXM (2)	Milstein	BM($c_0 = 5, c_1 = 0$)	BM($c_0 = 5, c_1 = 4$)
γ	1.0180	1.0269	0.9944	0.9558
$\ln(\mathcal{K})$	0.0579	0.0579	0.1329	0.2103

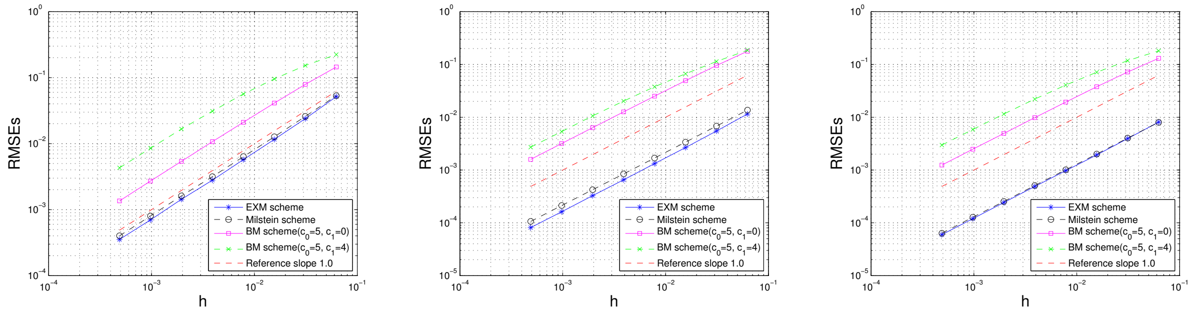
Example 5.3. *Consider the 1-dimensional nonlinear SDE*

$$\mathrm{d}Z(t) = -\left(\gamma_1 + \gamma_2^2 Z(t)\right)\left(1 - Z^2(t)\right)\mathrm{d}t + \gamma_2\left(1 - Z^2(t)\right)\mathrm{d}W(t), \quad Z_0 = 1/2, \tag{28}$$

with exact solution is given by [17]

$$Z(t) = \frac{(1 + Z_0)\exp(-2\gamma_1 t + 2\gamma_2 W(t)) + Z_0 - 1}{(1 + Z_0)\exp(-2\gamma_1 t + 2\gamma_2 W(t)) - Z_0 + 1}.$$

Figure 6 show strong convergence rates of numerical schemes applied to nonlinear SDE (28) *for stiff case with parameter* $(\gamma_1, \gamma_2) = (1, 1)$ *and non-stiff cases with parameters* $(\gamma_1, \gamma_2) = (1, 1/2), (1/2, 1/2)$. *Also, Table 5 indicate the calculated convergence rates of numerical schemes for different parameters of* (γ_1, γ_2). *From numerical results, we conclude that the obtained strong convergence rates are about 1.0, which agree with the theoretical result.*



(a) Stiff with $\gamma_1 = \gamma_2 = 1$. (b) Nonstiff with $\gamma_1 = 1, \gamma_2 = \frac{1}{2}$. (c) Nonstiff with $\gamma_1 = \gamma_2 = 1/2$.

Figure 6: Strong convergence order of numerical schemes for Example 5.3.

Table 5: A least squares for the parameters $\mathcal{K}$ and γ for nonlinear SDE (28).

		EXM (2)	Milstein	BM($c_0 = 5, c_1 = 0$)	BM($c_0 = 5, c_1 = 4$)
$\gamma_1 = \gamma_2 = 1$	γ	1.0224	1.0042	0.9676	0.8231
	$\ln(\mathcal{K})$	0.0720	0.0387	0.0794	0.3315
$\gamma_1 = 1, \gamma_2 = 1/2$	γ	1.0184	1.000	0.9763	0.8764
	$\ln(\mathcal{K})$	0.0487	0.0118	0.0661	0.2255
$\gamma_1 = \gamma_2 = 1/2$	γ	1.0086	0.9964	0.9655	0.8561
	$\ln(\mathcal{K})$	0.0259	0.0089	0.0922	0.2804

Example 5.4. *Consider the SIS epidemic model [18]*

$$dZ(t) = Z(t)\left(\beta N - \mu - \gamma - \beta Z(t)\right) dt + \sigma Z(t)\left(N - Z(t)\right) dW(t), \quad Z_0 = 10, \tag{29}$$

with two sets of parameters

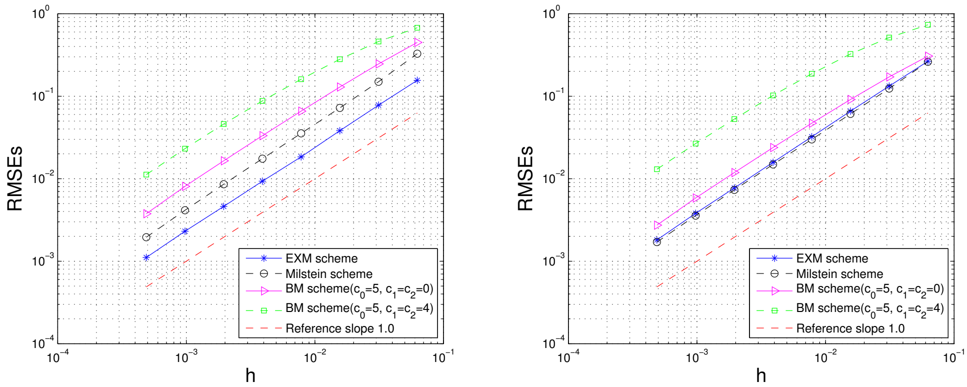
(i): $N = 10, \beta = 0.5, \sigma = 0.2$ *and* $\mu + \gamma = 4$;

(ii): $N = 10, \beta = 0.6, \sigma = 0.2$ *and* $\mu + \gamma = 2$.

The simulation outcome is depicted in Figure 7. Figure 7 demonstrates that the numerical schemes achieve the strong order 1.0 *on nonlinear SDE* (29)*. Again Table 6, confirms the estimated convergence rates.*

Table 6: A least squares for the parameters $\mathcal{K}$ and γ for (29).

		EXM (2)	Milstein	BM($c_0 = 5, c_1 = 0$)	BM($c_0 = 5, c_1 = 4$)
Case (i)	γ	1.0172	1.0460	0.9861	0.8531
	$\ln(\mathcal{K})$	0.0311	0.0569	0.1433	0.3391
Case (ii)	γ	1.0274	1.0306	0.9743	0.8427
	$\ln(\mathcal{K})$	0.0379	0.0348	0.1611	0.3802



(a) Case (i) (b) Case (ii)

Figure 7: Strong convergence order of numerical schemes for nonlinear SDE (29).

Example 5.5. *Let us consider the following stochastic Brusselator [17]*

$$\begin{aligned} \mathrm{d}Z_1(t) &= \left((\alpha-1)Z_1(t) + \alpha Z_1^2(t) + (1+Z_1(t))^2\, Z_2(t)\right)\mathrm{d}t + \sigma Z_1(t)(1+Z_1(t))\mathrm{d}W(t), \\ \mathrm{d}Z_2(t) &= \left(-\alpha Z_1(t) - \alpha Z_1^2(t) - (1+Z_1(t))^2\, Z_2(t)\right)\mathrm{d}t - \sigma Z_1(t)(1+Z_1(t))\mathrm{d}W(t), \end{aligned} \tag{30}$$

with $\alpha = 1.9$, $\sigma = 0.1$ and initial data $(Z_1(0), Z_2(0)) = (-0.1, 0)$. Figure 8 displays the RMSEs of numerical integrators. The capability of our approximation scheme can be seen in these graphs.

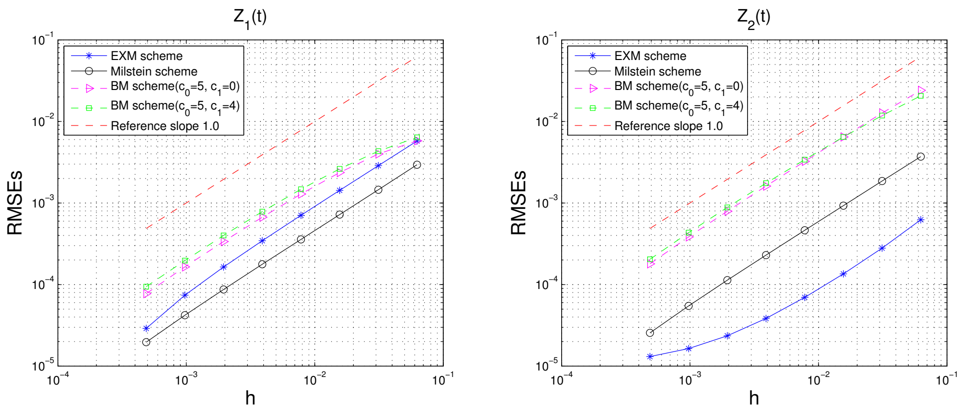


Figure 8: RMSEs of numerical schemes applied to nonlinear stochastic system (30).

Table 7: A least squares for the parameters $\mathcal{K}$ and γ for (30).

		EXM (2)	Milstein	BM($c_0 = 5, c_1 = 0$)	BM($c_0 = 5, c_1 = 4$)
$Z_1(t)$	γ	1.0736	1.0273	0.9045	0.8801
	$\ln(\mathcal{K})$	0.2263	0.0571	0.3672	0.3577
$Z_2(t)$	γ	0.8084	1.0214	1.0094	0.9528
	$\ln(\mathcal{K})$	0.6073	0.0685	0.1040	0.1889

Example 5.6. *Lastly, we consider the 2-dimensional stiff stochastic system [20]*

$$dZ(t) = \beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Z(t)dt + \frac{\sigma}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} Z(t)dW_1(t) + \frac{\rho}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} Z(t)dW_2(t). \quad (31)$$

Figure 9 presents the behavior of the numerical simulation of stochastic system (31) *with parameters* $\beta = 5$, $\sigma = 4$, $\rho = 0.5$ *for* $0 \le t \le 50$, *starting at* $Z(0) = (1, 0)^T$ *and step-size* $h = 1/10$. *We can observe that all the approximate trajectories produced by numerical schemes stay close to the origin* $(0, 0)$, *but the EXM scheme has better stability the behavior than the other methods.*

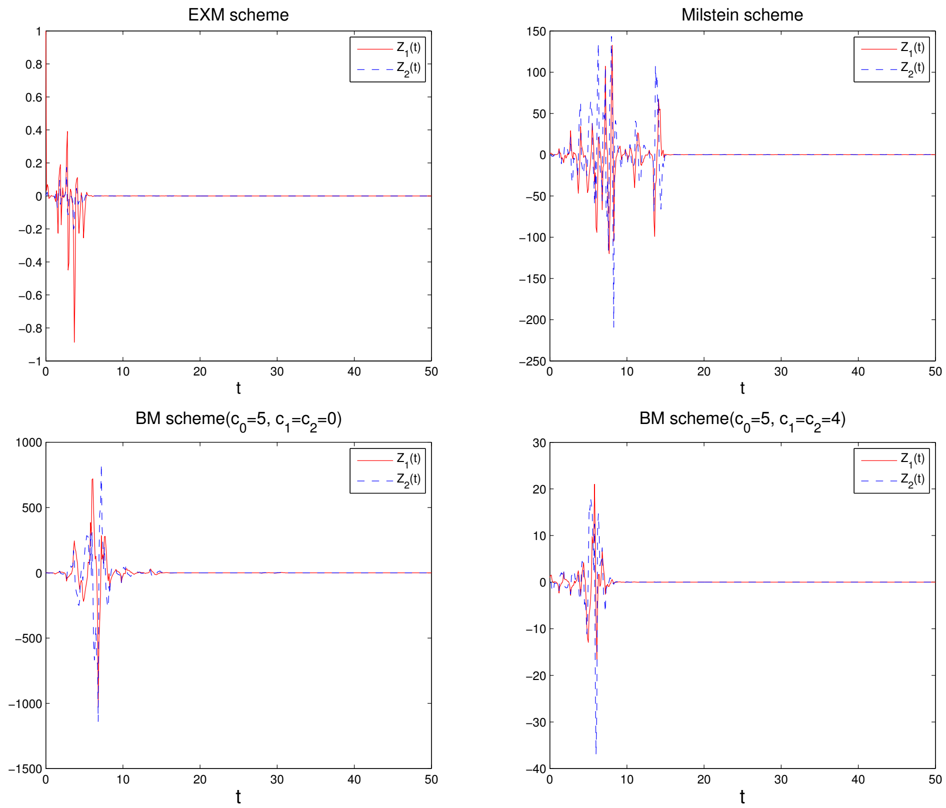


Figure 9: Numerical simulation of stochastic system (31).

6. Summary

This article has addressed the construction of an explicit Milstein-type integrator for SDEs. The strong convergence of our approach is then proven theoretically, see Theorem 3.2. For linear SDE (9), based on MS stability analysis of EXM integrator (2), the method is MS stable for all time step $h > 0$, see Theorem 4.4. Thus the EXM method is MS A-stable. So, the scheme is suitable for the solution of stiff SDEs. It was also shown that our method is MS L-stable. Furthermore, for two stochastic systems (14) and (20), we prove that the bi-dimensional EXM method (2) is MS stable for any $h > 0$. The numerical results obtained through the EXM scheme (2) and compared to Milstein [21] and BM [15] evidenced that by applying the EXM scheme (2), the accuracy of the numerical solutions is improved, see Examples 5.1-5.6.

For SDE (1) with a linearly growing and globally Lipschitz continuous drift and diffusion coefficients, we proved that our scheme strongly converges to the exact solution. In the future, we will investigate the convergence properties of the designed integrator for SDE (1) with locally Lipschitz continuous coefficient in detail. In the examination of the proposed scheme's MS stability, the stochastic systems under consideration include a single noise term. Future work will extend this analysis to systems with two noise terms to investigate the MS stability of the numerical integrator (2). In recent years, the positivity-preserving property of approximation schemes applied to financial models has garnered significant interest among researchers. In the future, we aim to explore this property for the EXM method (2).

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