Filomat 39:8 (2025), 2525–2556
https://doi.org/10.2298/FIL2508525D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

SIRS model for computer viruses propagation with time dependent delay and jumps

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Abstract. In this paper we propose stochastic SIRS model which describes propagation of two independent computer viruses. Randomness is introduced in the model by perturbing transmission rates via Brownian motion and Poisson jump. For this model, we theoretically prove that its solution is unique, global and positive. Furthermore, we derive conditions under which viruses become extinct from population. Also, conditions for stochastic strong persistence in mean of viruses will be obtained. In the end, we will present numerical simulations to illustrate our theoretical results using Euler-Maruyama method.

1. Introduction

1.1. About computer viruses

The first *computer viruses* were programs that copied themselves. Like biological viruses, computer viruses spread by taking advantage of the existing environment. Some computer viruses need a "host" program and they replicate themselves by attaching their program instructions to an ordinary host program or document, such that the virus instructions are executed during the execution of the host program. On the other hand, *computer worms* do not require host program and can carry out attacks independently.

The term "virus" for computers first appeared in 1988, but computer programs that are denoted by the term "virus" in fact existed many years before. The first computer virus, Creeper System, was created in 1971 (for more details, see [36], page 19). Fred Cohen was the first author who gave a precise definition of what computer viruses are in his doctoral thesis [8] which was published in 1986. This definition is now widely accepted and used.

The beginning of 21st century marked a turning point for development of computer viruses. Computer viruses constructed after 2000s are more sophisticated than ones before. Previous paragraphs were about development of computer viruses up to 2000s. Our mathematical models and discussion will be about computer viruses which are constructed after 2000s because they are the most representative nowadays.

2020 *Mathematics Subject Classification*. Primary 60H10; Secondary 60H30, 65C30.
Keywords. Computer viruses, stochastic model with jump, SIRS model, time dependent delay.
Received: 22 September 2024; Accepted: 07 January 2025
Communicated by Miljana Jovanović
The second author acknowledges financial support through the Ministry of Science, Technological Development and Innovation of Republic of Serbia, agreement no. 451-03-137/2025-03/ 200124.
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Definition 1.1. *([12], Definition 17, page 41)*
A virus can be described by a sequence of symbols which is able, when interpreted in a suitable environment (a machine), to modify other sequences of symbols in that environment by including a, possibly evolved, copy of itself.

Definition 1.2. *([12], Definition 36, page 83)*
A computer infection program (Malicious software or malware) is a simple or self-replicating program, which discretely installs itself in a data processing system, without user's knowledge or consent, with a view to either endangering data confidentiality, data integrity and system availability or making sure that users be framed for computer crime.

Malware is a term for any malicious software written specifically to infect and harm the host system or its user. A computer virus is just one type of malware that, when executed, tries to replicate itself in other executable codes. When it succeeds, the code is said to be *infected*. The infected code, when activated, can infect the new code in return and thus the infection grows. This self-replication into existing executable code is the key defining characteristic of a virus.

Computer viruses spread by attaching themselves to legitimate files and programs and are distributed through infected websites, flash drives, emails, etc. A victim activates a virus by opening the infected application or file. Once activated, a virus may delete or encrypt files, modify applications, disable system functions, etc.

The computer virus has three parts ([35], page 205) (i.e. its code has three parts):

1. *Infection mechanism* - Finds and infects new files i.e. it is how virus spreads. Virus spreads by modifying other code to contain a (possibly altered) copy of the virus. The exact means through which a virus spreads is referred as its *infection vector*. This doesn't have to be unique - a virus that infects in multiple ways (or several types of target at the same time) is called *multipartite*.
2. *Payload* - Malicious code to execute. It is body of the virus that executes the malicious activity. The payload may involve damage, either intentional or accidental. Accidental damage may result from bugs in the virus, encountering an unknown type of system, or perhaps unanticipated multiple viral infection.
3. *Trigger* - Determines when to activate payload. This is part of virus which determines conditions for which the payload is activated. This conditions may be a particular date, time, presence of another program, opening specific file, etc.

There are four phases in spreading computer viruses ([35], page 205):

1. *Dormant phase* - The virus is idle (inactive) in the dormant phase. It has accessed the target's device but does not take any action. The virus will eventually be activated by the trigger. Not all viruses have dormant phase.
2. *Propagation phase* - The virus starts propagating (spreading) by replicating itself and places a copy of itself into other programs or into certain system areas on the disk. The copy may not be identical to the propagating version because viruses often change in order to evade detection. Each infected program will contain a clone of the virus which will enter its own propagation phase as well.
3. *Triggering phase* - The virus is activated to perform actions it is supposed to accomplish. This phase can be caused by various system events such as the number of times the virus has cloned, after a set time interval has elapsed, etc.
4. *Execution phase* - In the execution phase the payload will be released. It can harm deleting files, crashing the system, etc or it can be harmless too and pop some humorous messages on screen.

Since computer viruses pose a serious problem to individual and cooperative computer systems, a lot of effort has been dedicated into studying how to avoid their deleterious actions, trying to create antivirus programs acting as vaccines in personal computers or in strategic network nodes.

Definition 1.3. *([4], page 66)*
Antivirus software is a computer program created to prevent, detect, identify and remove computer viruses and malware.

As hardware and software technology developed and computer networks became an essential tool for daily life, viruses started to be a major threat. Today, these programs have more complex codes, are capable of producing mutations of themselves and their detection and removal by antivirus programs became more difficult [15]. In addition, they are capable of acquiring personal data from network users, such as passwords and bank accounts, causing severe damage to individuals and corporations. Consequently, better understanding computer viruses spreading dynamics is an important matter in order to improve the safety and reliability in computer systems and networks.

Some computer viruses have a *latent period*. Latent period is time period in life cycle of computer virus during which individual computers are exposed to a computer virus but are not yet infected (for example, downloaded file may be corrupted, but virus does not spread until user activates corrupted file [16]). Hence, latent period can be interpreted as dormant phase in life cycle of computer virus. An infected computer with latency, called an *exposed computer*, will not infect other computers immediately. Based on these characteristics (virus activation after some time), different types of time delay are used in some mathematical models to in order to precisely explain computer virus propagation.

Infections in computer networks are complex. However, unlike contagious diseases, viruses in computers or computer networks can spread and cause destruction in a few minutes. Dynamics of computer viruses has always drawn resemblance to modeling of biological epidemiology. Mathematical modeling has proven to be an important tool in analysis of virus spread and control in computer networks. So far, different mathematical models have been discussed. In [7], the authors proposed a deterministic SIR model which describes the evolution of self-propagating malware in computer networks, analyzed the stability of equilibrium points and illustrated results with numerical examples. Essouifi et al. in [11] proposed a deterministic and stochastic SIR-SIS model for the spread of the virus through the two-degree network and analyze its dynamics. In [22] author analyzed stochastic SIR model for computer virus propagation and introduced control variables, while in [26] authors analyzed dynamic of stochastic SEIR model. Such models take into account key factors that administer the virus spread and prognosticate how the infection will behave over time period.

As it was stated in the previous paragraph, mathematical modeling is very important tool in analyzing the dynamic of computer viruses. It is possible to use mathematical models which were constructed for biological viruses for computer viruses, because computer viruses were constructed according to biological viruses i.e. biological viruses were inspiration for construction of computer viruses. Analogy between biological and computer viruses is given in following table (this analogy was given in ([12], page 92) but last column about modern viruses was given by authors)

Biological viruses	Computer viruses (up to 2000s)	Computer viruses (nowadays)
Attack on specific cells	Attack on specific file formats	Attack on specific host vulnerabilities
Infected cells produce new offsprings	Infected programs (codes) produce new viral codes	Infected hosts produce new viral codes or are used to launch new attacks
Modification of cell's genome	Modification of program's functions	Modification of host's functions
Viruses use cell's structure to replicate	Viruses use file's structures for copy mechanisms	Viruses use host's abilities to spread
Viral interactions	Combined viruses	Combined viruses
Viruses replicate only in living cells	Execution is required for virus spread	Host must be powered on or connected to network for virus spread
Already infected cells are not reinfected	Virus use an infection marker to prevent overinfection	Virus use an infection marker to prevent overinfection

Retrovirus - virus with ability to suppress immune response of organism	Virus specifically bypasses a given antivirus software - Source code viruses	Source code viruses
Viral mutation	Viral polymorphism	Viral polymorphism
Healthy virus carriers	Latent or dormant viruses	Latent or dormant viruses

Table 1. Analogy between biological and computer viruses.

As it was states in previous table, the difference between modern viruses and viruses which were developed up to 2000s is that for modern viruses host is computer, not specific file. Also, modern viruses are far more sophisticated in a way of infiltrating and attacking its targets, although basic idea for constructing and infecting stays the same. Our mathematical models and analysis will consider nowadays viruses. Also, viruses with multipartite will not be analyzed.

The most frequent viruses and malware in 2023. according to [17] were: financial malware, ransomware, Trojans, miners etc.

1.2. **Mathematical SIRS model for describing the spread of computer viruses**

Mathematical modeling of biological viruses became a very important tool to understand their propagation and control. Different mathematical (epidemiological) models are used for modeling different type of diseases. Kerman and McKendric were pioneers in the field of mathematical epidemiology. They proposed now famous SIR model in articles ([18], [19] and [20]) which is a model based on ordinary differential equations. The model's name comes from words "Susceptible" (S), "Infected" (I) and "Recovered" (R) because total population of people had been divided into three mutually independent compartments. Model has form

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta S(t)I(t),\\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - \gamma I(t),\\
\frac{dR(t)}{dt} &= \gamma I(t),
\end{aligned}$$

for $t \geq 0$ where $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$ is initial value, $\beta > 0$ is transmission rate and $\gamma > 0$ is recovery rate. Total population size is constant N. Besides these three mentioned compartments, the whole population can be divided into more compartments such as: "Exposed" (E) or "Vaccinated" (V) individuals. In that way SEIR, SVIR, SVEIR models are obtained. Also, there are SIRS models (in these models after recovery the person does not obtain permanent immunity and after some time again becomes susceptible to disease).

Previously mentioned models belong to the class of deterministic models. Deterministic models were firstly introduced in the field of mathematical epidemiology and, for some time, they were only tool for modeling epidemics. However, deterministic models cannot describe the spread of diseases in the best way due to unpredictable human behavior and random contact, which leads us to the conclusion that deterministic models have to be improved via environmental uncertainty. Environmental uncertainty or random effects in deterministic models can be introduced in several ways, such as: centering around equilibrium of deterministic model, parameter perturbation, perturbation proportional to the variables and using Markov chains [31]. In that way stochastic epidemiological models are obtained. The first three previously mentioned ways introduce randomness in a deterministic model using Brownian motion. Beside Brownian motion, randomness can be introduce in models in other ways (for example, using Poisson jumps, colored noise or via mean-reverting processes [10]). Due to the nature of diseases (randomness of spread caused by unpredictable and accidental contact between people), stochastic models are more realistic and appropriate in modeling epidemics than deterministic ones, as they introduce the environmental uncertainty. In recent years, many epidemiological models have been proposed in order to better understand and control the spread of diseases (see, for example, [10], [14], [21], [29], [31]).

Due to their similarity to biological viruses (which was stated in Table 1), dynamic of computer viruses can also be described using mathematical models which can be deterministic and stochastic. Deterministic and stochastic models offer different advantages (for example, deterministic models are intuitive but do not describe reality precisely, while stochastic models capture environmental randomness). A lot of models (mainly deterministic, but also some of stochastic type) were proposed, such as [7], [11], [22], [26]. The authors analyzed different problems (existence and uniqueness of solution, boundedness, different types of stabilities of equilibrium points, extinction of disease, stationary distributions, persistence of disease, etc).

J. Zhao et al. in [38] introduced the stochastic SIRS model for two different viruses and analyzed its dynamics. In Table 1 we can see analogy between biological viruses and computer viruses. The biggest similarities are how they reproduce (by replication), fast spreading of infection, how they attack their targets, mutations, etc. Furthermore, we can interpret previously mentioned model for computer viruses because computer viruses have similar (almost same) dynamics of spreading as biological viruses (spread via infecting its target). The model was constructed assuming that the total population of computer units N included in the system is a constant and is divided into four mutually independent groups:

- S - susceptible computers,
- I_1 - computers infected with first virus,
- I_2 - computers infected with second virus,
- R - recovered or removed computers.

Let $B_1(t)$ and $B_2(t), t \geq 0$ be two independent Brownian motions defined on complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying usual conditions (i.e. right continuous and increasing, while $\mathcal{F}_0$ contains all P-null sets), where $\mathcal{F}_t = \mathcal{F}_t^{B_1} \cup \mathcal{F}_t^{B_2}$ and $\mathcal{F}_t^{B_i}, i = 1, 2$ are σ-algebras derived from natural filtrations of Brownian motions B_1 and B_2. The model which authors proposed in [38] has the form:

$$\begin{aligned}
dS(t) &= \Big(\mu - \beta_1 S(t)I_1(t) - \beta_2 S(t)I_2(t) - \mu S(t) + \delta R(t)\Big)dt - \alpha S(t)I_1(t)dB_1(t) - \sigma S(t)I_2(t)dB_2(t),\\
dI_1(t) &= \Big(\beta_1 S(t)I_1(t) - (\mu + \gamma_1)I_1(t)\Big)dt + \alpha S(t)I_1(t)dB_1(t),\\
dI_2(t) &= \Big(\beta_2 S(t)I_2(t) - (\mu + \gamma_2)I_2(t)\Big)dt + \sigma S(t)I_2(t)dB_2(t),\\
dR(t) &= \Big(\gamma_1 I_1(t) + \gamma_2 I_2(t) - (\mu + \delta)R(t)\Big)dt,
\end{aligned} \tag{1}$$

for $t \geq 0$ where $(S(0), I_1(0), I_2(0), R(0)) \in \mathbb{R}_+^4$ is initial value. All constants are positive and have the following meaning:

- μ is the rate of new units of computers included in system and also a death rate due to computer break-down or energy crash;

- δ is the rate according to which recovered computers become susceptible because after some time computers can be again infected with the same virus, i.e. it is transition rate from R to S;

- β_1 and β_2 are transmission rates (i.e. rates according to which viruses spread through nodes due to contact between infected and susceptible nodes) for the first and second virus, respectively;

- γ_1 and γ_2 are recovery rates for infected computers with the first virus and second virus, respectively, due to antivirus ability of the network;

- α and σ are intensities of independent environmental noises $B_1(t)$ and $B_2(t)$.

It is assumed that Brownian motions are independent because it is possible to find two independent computer viruses (for example, Trojans and worms are independent). Hence, previously mentioned model can describe dynamic of two independent computer viruses.

For system (1) authors proved the existence and uniqueness of a global positive solution, gave sufficient conditions for stochastically asymptotic stability of disease-free equilibrium, proved the boundedness of solution and illustrated given results with numerical examples.

Environmental fluctuations can be introduced in model in other ways beside previously mentioned. Brownian motion is an example of white noise, but beside white noise, colored noise can also be introduced in system via Markov chains [5]. Furthermore, environmental noise can be introduced using driven jump processes (Poisson jump or Lévy jump), mean-reverting processes, etc. (see, for example, [1], [10], [21], [27], [34]).

Different viruses have different behaviour due to their construction and purpose [16]. For example, after downloading corrupted file, that file has to be opened in order to activate virus, i.e. virus will be activated after some period of time during which it stays in latency. Hence, time dependent delay can be good way for describing latent period (due to difference in viruses' construction and hence behaviour). Furthermore, sudden environmental changes such as power outage or guided cyber attacks can manifest as very strong shocks for system and lead to discontinuous sample paths. Therefore, Brownian motion cannot anymore describe such sudden changes and hence driven jump processes are introduced.

Inspired by everything mentioned (latent period and sudden environmental changes), model (1) will be improved by introducing time dependent delay and Poisson jump in order to get more realistic results because only white noise cannot describe real situation (latency and strong environmental shocks) properly, i.e. latent period can mathematically be described using time dependent delay (time delay as function because latent period varies due to construction and purpose of viruses), while sudden environmental shocks such as guided cyber attacks or power outage can be explained via jump processes (Poisson jump or Lévy jump).

In order to adapt model (1) to the spread of computer viruses, we will introduce new features in the mentioned model step by step. First, new constants and coefficients will be introduced in order to better explain the functioning of antivirus. Also, total population size will not be constant, it will be dependent of time. Second, a time dependent delay and a Poisson jump will be introduced. All that mentioned leads to a more realistic model that can better explain real situations. For this new model, the existence and uniqueness of the global positive solution will be proven, sufficient conditions for the extinction of disease and the strong stochastic persistence in the mean of diseases will be derived and theoretical results will be illustrated with numerical simulations using Euler-Maruyama method.

Lets introduce following stochastic model for the spread of computer viruses (shorter SM) with additional parameters comparing to model (1):

$$\begin{aligned}
dS(t) &= \Big(\Lambda - \beta_1 S(t)I_1(t) - \beta_2 S(t)I_2(t) - (\mu+\gamma)S(t) + \delta R(t)\Big)dt - \sigma_1 S(t)I_1(t)dB_1(t) - \sigma_2 S(t)I_2(t)dB_2(t),\\
dI_1(t) &= \Big(\beta_1 S(t)I_1(t) - (\mu+\varepsilon_1+\gamma_1)I_1(t)\Big)dt + \sigma_1 S(t)I_1(t)dB_1(t),\\
dI_2(t) &= \Big(\beta_2 S(t)I_2(t) - (\mu+\varepsilon_2+\gamma_2)I_2(t)\Big)dt + \sigma_2 S(t)I_2(t)dB_2(t),\\
dR(t) &= \Big(\gamma S(t) + \gamma_1 I_1(t) + \gamma_2 I_2(t) - (\mu+\delta)R(t)\Big)dt,
\end{aligned} \tag{2}$$

for $t \geq 0$, where positive constants are:

- Λ is number of new units of computers included in system (network) which have already been equipped with antivirus software,
- ε_1 and ε_2 are death rates caused by first and second virus, respectively,
- μ is death rate due to computer break-down or loss energy [1)],
- γ is recovery rate for susceptible computers due to antivirus ability of network [2)],
- $\beta_1, \beta_2, \delta, \gamma_1, \gamma_2$ have the same meaning as mentioned before,
- σ_1 and σ_2 are intensities of Brownian motions $B_1(t)$ and $B_2(t)$, respectively.

[1)] In this framework μ represents the rate of removal of computers from the system, specifically due to irreparable breakdowns or depletion of energy resources.

[2)] How does antivirus software work? Antivirus software has list of files/codes which are recognized and classified as malicious, i.e. viruses. With every software update, new viruses and/or their variants are added to the list. Each time when antivirus software is activated, it scans network or computer for potential threats. If some file is on the list, it is removed from system (if it is not on the list, it stays in the computer or system). In that way susceptible computers are treated. If some file/code is not recognized as virus and it turns out to be, that file/code will infect computer or system after its activation. Next time when antivirus software is activated (after new update) it again scans computer or network and if recognizes infected file as virus it will remove it. In that way infected computers are treated.

Model (2) is improved version of model (1) because:

-model (2) has different parameters for death rate and number of new units of computers included in system, while model (1) does not have,

-model (2) includes recovery rate for susceptible computers which model (1) does not include and

-model (2) includes different death rates caused by different viruses which model (1) does not include.

Time dependent delay will be introduced in our model because corrupted file may be activated after some time (i.e. virus is in dormant phase or latent period). It is natural to assume that different viruses have different delays. Thus, let

$$\delta_1 : \mathbb{R}_+ \to [0,\tau], \quad \delta_2 : \mathbb{R}_+ \to [0,\tau], \quad \tau > 0$$

be continuous and differentiable functions (with continuous and bounded first derivatives) such that

$$t-\tau \le t-\delta_i(t) \le t, \quad i = 1,2.$$

Depending on the construction and purpose of viruses, time delay functions can have different physical interpretation. For example, if virus is constructed to be activated after certain time period, than delay can be interpreted as constant or linear function, or delay can be interpreted as exponential decay function if virus is constructed to be activated on file opening.

Our model with the delay now has the form:

$$\begin{aligned}
dS(t) &= \Big(\Lambda - \beta_1 S(t)I_1(t-\delta_1(t)) - \beta_2 S(t)I_2(t-\delta_2(t)) - (\mu+\gamma)S(t) + \delta R(t)\Big)dt \\
&\quad - \sigma_1 S(t)I_1(t-\delta_1(t))dB_1(t) - \sigma_2 S(t)I_2(t-\delta_2(t))dB_2(t), \\
dI_1(t) &= \Big(\beta_1 S(t)I_1(t-\delta_1(t)) - (\mu+\varepsilon_1+\gamma_1)I_1(t)\Big)dt + \sigma_1 S(t)I_1(t-\delta_1(t))dB_1(t), \\
dI_2(t) &= \Big(\beta_2 S(t)I_2(t-\delta_2(t)) - (\mu+\varepsilon_2+\gamma_2)I_2(t)\Big)dt + \sigma_2 S(t)I_2(t-\delta_2(t))dB_2(t), \\
dR(t) &= \Big(\gamma S(t) + \gamma_1 I_1(t) + \gamma_2 I_2(t) - (\mu+\delta)R(t)\Big)dt,
\end{aligned}$$

for $t \ge -\tau$.

Beside time delay, sudden environmental perturbations such as power outage, guided cyber attacks or antivirus treatments may affect epidemic model and manifest as extreme changes in transmission coefficients of viruses [15]. Disturbances like this can lead to discontinuous sample path trajectories and cannot be described only with Brownian motion. This phenomenon can be mathematically described using driven jump processes. We will use Poisson process as driven jump process to describe this phenomena. In case of computer viruses, the new update of antivirus can be interpreted as a negative jump, while guided cyber attacks can be interpreted as positive jump in spreading of viruses.

Let $\bar{N}$ be Poisson counting process with characteristic measure λ defined of finite subsets Y of $(0,+\infty)$ such that $\lambda(Y) < +\infty$. Then $\tilde{N}(dt,du) = \bar{N}(dt,du) - \lambda(du)dt$ denotes the compensated Poisson process which is martingale.

Following Mao's method for introducing white noise in models (see [6], [9] and [37]) we will introduce jump in our model as

$$\beta_i dt + \sigma_i dB_i(t) \to \beta_i dt + \sigma_i dB_i(t) + \int_Y \eta_i(u)\tilde{N}(dt,du), \quad i = 1,2.$$

Here $dB_i(t) = B_i(t+dt) - B_i(t)$ is increment of standard Brownian motion. Jump sizes $\eta_i : Y \times \Omega \to \mathbb{R}, i = 1,2$ are bounded[3)], continuous and $\mathcal{B}(Y) \times \mathcal{F}_t$-measurable functions with respect to λ, where $\mathcal{B}(Y)$ is σ-algebra with respect to set Y. Hence, the potentially infectious contacts made by the infected node with another susceptible node in the population in the small time interval $[t, t+dt)$ have expected value $\beta_i dt$, while

[3)]Model assumes bounded jump sizes. If jump sizes are not bounded, then additional constrains would be necessary.

the variance is $\sigma_i^2 dt + \int_Y \eta_i^2(u)\lambda(du)dt$, $i = 1, 2$ (for more details, see [34]). Since $\beta_i > 0$, $i = 1, 2$ holds by definition, the intensities of the environmental noises are small and the variance tends to 0 as dt tends to 0, the perturbation remains positive.

New SM has the form:

$$\begin{aligned}
dS(t) &= \Big(\Lambda - \beta_1 S(t)I_1(t-\delta_1(t)) - \beta_2 S(t)I_2(t-\delta_2(t)) - (\mu+\gamma)S(t) + \delta R(t)\Big)dt \\
&\quad - \sigma_1 S(t)I_1(t-\delta_1(t))dB_1(t) - \sigma_2 S(t)I_2(t-\delta_2(t))dB_2(t) \\
&\quad - \int_Y \Big[\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big) + \eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big]S(t^-)\tilde{N}(dt,du), \\
dI_1(t) &= \Big(\beta_1 S(t)I_1(t-\delta_1(t)) - (\mu+\varepsilon_1+\gamma_1)I_1(t)\Big)dt + \sigma_1 S(t)I_1(t-\delta_1(t))dB_1(t) \\
&\quad + \int_Y \eta_1(u)S(t^-)I_1\Big((t-\delta_1(t))^-\Big)\tilde{N}(dt,du), \\
dI_2(t) &= \Big(\beta_2 S(t)I_2(t-\delta_2(t)) - (\mu+\varepsilon_2+\gamma_2)I_2(t)\Big)dt + \sigma_2 S(t)I_2(t-\delta_2(t))dB_2(t) \\
&\quad + \int_Y \eta_2(u)S(t^-)I_2\Big((t-\delta_2(t))^-\Big)\tilde{N}(dt,du), \\
dR(t) &= \Big(\gamma S(t) + \gamma_1 I_1(t) + \gamma_2 I_2(t) - (\mu+\delta)R(t)\Big)dt,
\end{aligned} \tag{3}$$

for $t \geq -\tau$ with initial value

$$S_0 = S(\xi) > 0,\ I_1^0 = I_1(\xi) \geq 0,\ I_2^0 = I_2(\xi) \geq 0,\ R_0 = R(\xi) \geq 0, \quad \xi \in [-\tau, 0], \tag{4}$$

where $(S(\xi), I_1(\xi), I_2(\xi), R(\xi)) \in \mathcal{L}^1([-\tau,0];\mathbb{R}_+^4)$. The class $\mathcal{L}^1([-\tau,0];\mathbb{R}_+^4)$ is a family of Lebesgue integrable functions from $[-\tau, 0]$ to $\mathbb{R}_+^4 = \{(x_1,\ldots,x_4) \in \mathbb{R}^4 \,|\, x_i > 0, i = 1,\ldots,6\}$. The functions $S(t^-), I_1\Big((t-\delta_1(t))^-\Big)$ and $I_2\Big((t-\delta_2(t))^-\Big)$ are left limits of $S(t), I_1(t-\delta_1(t))$ and $I_2(t-\delta_2(t))$, respectively, and the total population size at time t is

$$N(t) = S(t) + I_1(t) + I_2(t) + R(t), \quad t \geq -\tau, \tag{5}$$

which is improvement considering [38] where total population size was constant. It is naturally to assume that total population is variable because it changes during time as its compartments changes. One of the first works where varying population size had been introduced were [2], [3], [30], etc.

In recent years a lot of papers about dynamic of epidemiological models for biological viruses with Poisson or Lévy jump were proposed. Some of them are [10], [27], [34]. But, up to our knowledge, there are no results which describe evolution of two computer viruses with time dependent delays and Poisson jump. By incorporating time dependent delay and Poisson jump in our model we obtain a stochastic system for describing the spread of computer viruses with richer dynamics compared to the previous ones which could be found in the literature, allowing more accurate representation of real-world phenomena.

Remark 1.4. *Introduced systems could be more realistic if we take all parameters not to be constants, but bounded functions dependent of time. Analysis would be similar as in the case of constant parameters as it is naturally to assume that there exist lower and upper boundaries for the parameters.*

By allowing the parameters to be time dependent and bounded functions, we can capture the variability and fluctuations in these parameters. Introducing lower and upper boundaries for the parameters is a reasonable assumption, as it acknowledges that there are limits to how much these parameters can change.

This paper is structured as follows. In Section 2 preliminary results and basic assumptions are given. For the stochastic model, the existence and uniqueness of positive global solution is proven in Section 3. In addition, sufficient conditions for the extinction of viruses are derived in Theorem 4.2 (Section 4), while sufficient conditions for the strong stochastic persistence in mean of diseases are given in Theorem 5.2 (Section 5). Section 6 contains numerical simulations which illustrate theoretical results.

2. Preliminaries

In this section, we will introduce some basic notation and auxiliary results which will be used in proofs of main results (for more details see, for example, [28], [32]). At the beginning, basic terms about stochastic differential equations with time dependent delay and Poisson jump as well as Itô calculus will be stated.

As mentioned in the previous section, $(\Omega, \mathcal{F}, P)$ is a complete probability space with filtration $\{\mathcal{F}\}_{t\geq 0}$, satisfying the usual conditions (i.e. right continuous and increasing and $\mathcal{F}_0$ contains all P-null sets). Let $B(t)$ denotes m-dimensional standard Brownian motion and $\{\tilde{N}(t,\omega)\,|\,t \geq 0\}$ compensated Poisson process (λ is intensity of Poisson process). All processes are defined on same probability space and independent from each other. Now, filtration is defined as $\mathcal{F}_t = \bigcup_{i=1}^{m} \mathcal{F}_t^{B_i} \cup \mathcal{F}_t^{\tilde{N}}$, where $\mathcal{F}_t^{B_i}$, $i = 1, \ldots, m$ are σ-algebras derived from natural filtrations of Brownian motions B_i, $i = 1, \ldots, m$ while $\mathcal{F}_t^{\tilde{N}}$ is σ-algebra derived from natural filtration of compensated Poisson process $\tilde{N}$. Consider d-dimensional stochastic differential equation with time dependent delay and Poisson jump

$$dx(t) = f(x(t), x(t-\delta(t)), t)dt + g(x(t), x(t-\delta(t)), t)dB(t) + \int_Y H(x(t^-), x((t-\delta(t))^-), u)d\tilde{N}(dt, du), \quad t \geq -\tau, \tag{6}$$

with initial value $x(\theta) = \xi \in \mathcal{L}^1([-\tau, 0]; \mathbb{R}_+^4)$ and $Y \subseteq (0, +\infty)$ such that $\lambda(Y) < +\infty$. Time delay $\delta : \mathbb{R}_+ \to [0, \tau]$ is continuous and differentiable function. Function $H(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \times Y \to \mathbb{R}^d$, which represents jump diffusion coefficient, is right continuous with left limits, $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ is drift coefficient, while $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$ is Brownian diffusion coefficient.

Let $C^{2,1}(\mathbb{R}^d \times [0, +\infty); [0, +\infty))$ denotes family of nonnegative functions $V(x, t)$ defined on $\mathbb{R}^d \times [0, +\infty)$ with values in $[0, +\infty)$ which are twice continuously differentiable with respect to x and once with respect to t. If $V \in C^{2,1}(\mathbb{R}^d \times [0, +\infty); [0, +\infty))$, we define differential operator $LV : \mathbb{R}^d \times \mathbb{R}^d \times [0, +\infty) \to \mathbb{R}$ associated with equation (6) as:

$$\begin{aligned} LV(x, y, t) &= V_t'(x, t) + V_x'(x, t)f(x, y, t) + \frac{1}{2}trace[g^T(x, y, t)V_{xx}''(x, t)g(x, y, t)] \\ &\quad + \int_Y \Big(V(x + H(x^-, y^-, u), t) - V(x, t) - V_x'(x, t)H(x^-, y^-, u)\Big)\lambda(du), \end{aligned} \tag{7}$$

where

$$V_t'(x, t) = \frac{\partial V(x, t)}{\partial t}, \quad V_x'(x, t) = \Big(\frac{\partial V(x, t)}{\partial x_1}, \frac{\partial V(x, t)}{\partial x_2}, \ldots, \frac{\partial V(x, t)}{\partial x_d}\Big) \quad \text{and} \quad V_{xx}''(x, t) = \Big(\frac{\partial^2 V(x, t)}{\partial x_i x_j}\Big)_{d\times d}.$$

Using generalized Itô formula ([32], Theorem 1.16, page 9) we obtain

$$dV(x(t), y(t), t) = LV(x(t), y(t), t)dt + V_x'(x(t), t)dB(t) + \int_Y \Big(V\Big(x(t) + H\Big(x(t^-), y(t^-), u\Big), t\Big) - V(x(t), t)\Big)\tilde{N}(dt, du), \tag{8}$$

where $y(t) = x(t - \delta(t))$.

We will introduce following assumptions: Assumption 2.1 guarantees the boundedness of jump sizes, Assumption 2.2 guarantees the existence and the uniqueness of the local solution of eq. (6), while Assumption 2.3 guarantees the existence and the uniqueness of the global positive solution of eq. (6).

Assumption 2.1. *There exist positive constant* $0 < h < 1$ *such that for all* $t \geq -\tau$ *following holds:*

$$\Big|\eta_i(u)\Big| \leq h\frac{\mu}{\Lambda} \quad i = 1, 2.$$

Assumption 2.2. *(Local Lipschitz condition)*
For every $n \geq 1$ *there exist constant* $K_n > 0$ *such that for all* $t > -\tau$ *and for all* $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ *such that*

$|x_i| \vee |y_i| < n,\ i = 1, 2$ *following holds*

$$\begin{aligned} &|f(x_1, y_1, t) - f(x_2, y_2, t)|^2 \vee |g(x_1, y_1, t) - g(x_2, y_2, t)|^2 \leq K_n(|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ &\int_Y |H(x_1, y_1, u) - H(x_2, y_2, u)|^2 \lambda(du) \leq K_n(|x_1 - x_2|^2 + |y_1 - y_2|^2). \end{aligned} \tag{9}$$

Assumption 2.3. *Function* $\delta(\cdot)$ *is continuously differential and there exist constants* $k_1 \geq 0$ *and* $0 < k_2 \leq 1$ *such that*

$$k_1 \leq \delta'(t) \leq k_2.$$

Next lemmas will be very useful in our analysis (in the analysis of the extinction and the persistence of viruses).

Lemma 2.4. *(**Strong law of large numbers for continuous local martingales**, ([28], Theorem 3.4, page 12)) Let* $M = \{M(t) \,|\, t \geq 0\}$ *be a real-valued continuous local martingale vanishing at* $t = 0$. *Then*

$$\lim_{t\to\infty} [M, M](t) = \infty \quad a.s.^{4)} \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M(t)}{[M, M](t)} = 0 \quad a.s.$$

and also

$$\limsup_{t\to\infty} \frac{[M, M](t)}{t} < \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad a.s,$$

where $[M, M](t)$ *is quadratic variation process of martingale M.*

Lemma 2.5. *(**Strong law of large numbers for local martingales**, ([24], Theorem 1, page 219)) Let* $M = \{M(t),\ t \geq 0\}$ *be local martingale vanishing at* $t = 0$. *If*

$$\lim_{t\to\infty} \int_0^t \frac{d\langle M, M\rangle(s)}{(1 + s)^2} < \infty \quad a.s.$$

then

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad a.s,$$

where $\langle M, M\rangle(t)$ *is predictable quadratic variation process of martingale M.*[5)]

Remark 2.6. *Let us denote*

$$\Gamma = \left\{ (S, I_1, I_2, R) \in \mathbb{R}^4_+ \middle| \frac{\Lambda}{\mu + \varepsilon_1 + \varepsilon_2} \leq N(t) \leq \frac{\Lambda}{\mu}, \quad t \geq -\tau \right\}.$$

We will observe the dynamic of the system (3) *in the set* Γ.

3. Existence and uniqueness of a positive global solution

In this section existence and uniqueness of a positive global solution of the system of stochastic differential equations (3) will be shown.

[4)] Abbreviation a.s. stands for almost surely.

[5)] Shortly, relation $[M, M](t) = \langle M, M\rangle(t) + M_t$, where M_t is local martingale part, holds. More about this relation can be found in the standard literature, for example see [25], [33].

Theorem 3.1. *For any initial value* $(S(\xi), I_1(\xi), I_2(\xi), R(\xi)) \in \mathcal{L}^1([-\tau, 0]; \Gamma)$ *there exist unique global solution* $(S(t), I_1(t), I_2(t), R(t))$, $t \geq -\tau$ *of the system of stochastic differential equations* (3) *and the solution will remain in* Γ *a.s. Moreover,*

$$\frac{\Lambda}{\mu + \varepsilon_1 + \varepsilon_2} \leq N(t) \leq \frac{\Lambda}{\mu}, \quad t \geq -\tau. \tag{10}$$

Proof. Since the coefficients of system (3) satisfy the local Lipschitz condition (9), then for any initial value $(S(\xi), I_1(\xi), I_2(\xi), R(\xi)) \in \mathcal{L}^1([-\tau, 0]; \Gamma)$ there is unique local solution on $[-\tau, \tau_e)$ (for more details see [23]), where τ_e represents explosion time (see definition of explosion time in [28]). To prove this solution is global, we have to prove that $\tau_e = \infty$ a.s.

Lets choose $k_0 \geq 0$ sufficiently large such that for initial value holds $S(\xi), I_1(\xi), I_2(\xi), R(\xi) \in \left[\frac{1}{k_0}, k_0\right]$, for $\xi \in [-\tau, 0]$. For each $k \geq k_0$ let us define stopping time (see definition of stopping time in [28]) as

$$\tau_k = \inf_{t \in [-\tau, \tau_e)} \Big\{ \min\{S(t), I_1(t), I_2(t), R(t)\} \leq \frac{1}{k} \vee \max\{S(t), I_1(t), I_2(t), R(t)\} \geq k \Big\},$$

where we set $\inf \emptyset = \infty$. Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k \to \infty} \tau_k$. Then, $\tau_\infty \leq \tau_e$. If we show that $\tau_\infty = \infty$ a.s, then $\tau_e = \infty$ a.s. which means that $(S(t), I_1(t), I_2(t), R(t)) \in \Gamma$, a.s. for all $t \geq -\tau$, i.e. $(S(t), I_1(t), I_2(t), R(t))$ is positive global solution of system (3). Thus, we only need to prove that $\tau_\infty = \infty$ a.s. If this is not true, than there exist a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. Therefore, there is an integer $k \geq k_0$ such that

$$P\{\tau_k \leq T\} \geq \epsilon, \quad \forall k \geq k_0. \tag{11}$$

Bearing in mind (5), i.e. $N(t) = S(t) + I_1(t) + I_2(t) + R(t), \quad t \geq -\tau$ it follows that

$$dN(t) = dS(t) + dI_1(t) + dI_2(t) + dR(t).$$

Therefore, summing all four equations from (3) following ordinary differential equation is obtained

$$dN(t) = \Big(\Lambda - \mu N(t) - \varepsilon_1 I_1(t) - \varepsilon_2 I_2(t)\Big) dt.$$

For $t \in [-\tau, \tau_k)$ it follows that

$$\Big(\Lambda - \big(\mu + \varepsilon_1 + \varepsilon_2\big) N(t)\Big) dt \leq dN(t) \leq \Big(\Lambda - \mu N(t)\Big) dt.$$

Solving previous inequalities we get

$$e^{-(\mu + \varepsilon_1 + \varepsilon_2)t} \left(N(0) - \frac{\Lambda}{\mu + \varepsilon_1 + \varepsilon_2} \right) + \frac{\Lambda}{\mu + \varepsilon_1 + \varepsilon_2} \leq N(t) \leq e^{-\mu t} \left(N(0) - \frac{\Lambda}{\mu} \right) + \frac{\Lambda}{\mu}.$$

Using assumption about initial condition yields to

$$\frac{\Lambda}{\mu + \varepsilon_1 + \varepsilon_2} \leq N(t) \leq \frac{\Lambda}{\mu}, \quad t \in [-\tau, \tau_k).$$

Let us define $C^{2,1}$-function $V := V_1 + V_2 : \mathbb{R}_+^4 \to [0, \infty)$ as

$$\begin{aligned} V_1(t) &:= V_1(S(t), I_1(t), I_2(t), R(t)) \\ &= (S(t) + a_1 - a_1 \ln S(t)) + (I_1(t) + 1 - \ln I_1(t)) + (I_2(t) + 1 - \ln I_2(t)) + (R(t) + 1 - \ln R(t)), \end{aligned} \tag{12}$$

where a_1 is positive constant which has to be determined and function V_2 will be determined later. Nonnegativity of function V_1 follows from inequality $u+1-\ln u>0$, for any $u>0$.

Using (7), differential operator LV_1 associated with function V_1 has following form

$$
\begin{aligned}
LV_1(t) &= \Big(1-\frac{a_1}{S(t)}\Big)\Big(\Lambda-\beta_1S(t)I_1(t-\delta_1(t))-\beta_2S(t)I_2(t-\delta_2(t))-(\mu+\gamma)S(t)+\delta R(t)\Big)\\
&+\Big(1-\frac{1}{I_1(t)}\Big)\Big(\beta_1S(t)I_1(t-\delta_1(t))-\big(\mu+\varepsilon_1+\gamma_1\big)I_1(t)\Big)+\Big(1-\frac{1}{I_2(t)}\Big)\Big(\beta_2S(t)I_2(t-\delta_2(t))-\big(\mu+\varepsilon_2+\gamma_2\big)I_2(t)\Big)\\
&+\Big(1-\frac{1}{R(t)}\Big)\Big(\gamma S(t)+\gamma_1I_1(t)+\gamma_2I_2(t)-(\mu+\delta)R(t)\Big)+\frac{a_1\sigma_1^2I_1^2(t-\delta_1(t))}{2}+\frac{a_1\sigma_2^2I_2^2(t-\delta_2(t))}{2}\\
&+\frac{\sigma_1^2S^2(t)I_1^2(t-\delta_1(t))}{2I_1^2(t)}+\frac{\sigma_2^2S^2(t)I_2^2(t-\delta_2(t))}{2I_2^2(t)}\\
&+\int_Y\Big[S(t)-\Big(\eta_1(u)I_1\big((t-\delta_1(t))^-\big)+\eta_2(u)I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)+a_1\\
&\qquad-a_1\ln\Big[S(t)-\Big(\eta_1(u)I_1\big((t-\delta_1(t))^-\big)+\eta_2(u)I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)\Big]\\
&\qquad+I_1(t)+\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)+1-\ln\Big[I_1(t)+\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)\Big]\\
&\qquad+I_2(t)+\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)+1-\ln\Big[I_2(t)+\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)\Big]+R(t)+1-\ln R(t)\Big]\lambda(du)\\
&-\int_Y\Big[S(t)+a_1-a_1\ln S(t)+I_1(t)+1-\ln I_1(t)+I_2(t)+1-\ln I_2(t)+R(t)+1-\ln R(t)\Big]\lambda(du)\\
&-\int_Y\Big[-\Big(1-\frac{a_1}{S(t)}\Big)\Big(\eta_1(u)I_1\big((t-\delta_1(t))^-\big)+\eta_2(u)I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)\\
&\qquad+\Big(1-\frac{1}{I_1(t)}\Big)\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)+\Big(1-\frac{1}{I_2(t)}\Big)\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)\Big]\lambda(du)\\
&:=L_1+L_2,
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
L_1 &= \Big(\Lambda+(3+a_1)\mu+\varepsilon_1+\varepsilon_2+a_1\gamma+\gamma_1+\gamma_2+\delta\Big)-\mu N(t)-\varepsilon_1I_1(t)-\varepsilon_2I_2(t)-\frac{a_1}{S(t)}\Big(\Lambda+\delta R(t)\Big)\\
&-\frac{1}{R(t)}\Big(\gamma S(t)+\gamma_1I_1(t)+\gamma_2I_2(t)\Big)+a_1\Big(\beta_1I_1(t-\delta_1(t))+\beta_2I_2(t-\delta_2(t))\Big)-\frac{\beta_1S(t)I_1(t-\delta_1(t))}{I_1(t)}\\
&-\frac{\beta_2S(t)I_2(t-\delta_2(t))}{I_2(t)}+\frac{a_1}{2}\Big(\sigma_1^2I_1^2(t-\delta_1(t))+\sigma_2^2I_2^2(t-\delta_2(t))\Big)+\frac{\sigma_1^2S^2(t)I_1^2(t-\delta_1(t))}{2I_1^2(t)}+\frac{\sigma_2^2S^2(t)I_2^2(t-\delta_2(t))}{2I_2^2(t)}
\end{aligned}
$$

and

$$
\begin{aligned}
L_2 &= -a_1\int_Y\ln\Bigg[1-\frac{\Big(\eta_1(u)I_1\big((t-\delta_1(t))^-\big)+\eta_2(u)I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)}{S(t)}\Bigg]\lambda(du)-\int_Y\ln\Bigg[1+\frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\Bigg]\lambda(du)\\
&-\int_Y\ln\Bigg[1+\frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)}\Bigg]\lambda(du)-a_1\int_Y\frac{\Big(\eta_1(u)I_1\big((t-\delta_1(t))^-\big)+\eta_2(u)I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)}{S(t)}\lambda(du)\\
&+\int_Y\frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\lambda(du)+\int_Y\frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)}\lambda(du).
\end{aligned}
$$

After omitting some nonpositive terms we obtain

$$\begin{aligned} L_1 \le & \Big(\Lambda + (3+a_1)\mu + \varepsilon_1 + \varepsilon_2 + a_1\gamma + \gamma_1 + \gamma_2 + \delta\Big) - \varepsilon_1 I_1(t) - \varepsilon_2 I_2(t) + a_1\beta_1 I_1(t-\delta_1(t)) \\ & + a_1\beta_2 I_2(t-\delta_2(t)) + \frac{\sigma_1^2}{2}\Big(a_1 + k^4\Big) I_1^2(t-\delta_1(t)) + \frac{\sigma_2^2}{2}\Big(a_1 + k^4\Big) I_2^2(t-\delta_2(t)). \end{aligned} \tag{14}$$

According to Assumption 2.1 and applying Taylor formula up to second term to the functions $\ln(1-x)$, $\ln(1+y)$ and $\ln(1+z)$, where $x = \frac{(\eta_1(u)I_1((t-\delta_1(t))^-)+\eta_2(u)I_2((t-\delta_2(t))^-))S(t^-)}{S(t)}$, $y = \frac{\eta_1(u)S(t^-)I_1((t-\delta_1(t))^-)}{I_1(t)}$ and $z = \frac{\eta_2(u)S(t^-)I_2((t-\delta_2(t))^-)}{I_2(t)}$ yields to

$$\begin{aligned} L_2 \le & \, a_1 \int_Y \frac{\frac{(\eta_1(u)I_1((t-\delta_1(t))^-)+\eta_2(u)I_2((t-\delta_2(t))^-))^2 S^2(t^-)}{S^2(t)}}{2\Big(1-\theta\frac{(\eta_1(u)I_1((t-\delta_1(t))^-)+\eta_2(u)I_2((t-\delta_2(t))^-))^2 S^2(t^-)}{S^2(t)}\Big)}\lambda(du) \\ & + \int_Y \frac{\frac{\eta_1^2(u)S^2(t^-)I_1^2((t-\delta_1(t))^-)}{I_1^2(t)}}{2\Big(1-\theta\frac{\eta_1^2(u)S^2(t^-)I_1^2((t-\delta_1(t))^-)}{I_1^2(t)}\Big)}\lambda(du) + \int_Y \frac{\frac{\eta_2^2(u)S^2(t^-)I_2^2((t-\delta_2(t))^-)}{I_2^2(t)}}{2\Big(1-\theta\frac{\eta_2^2(u)S^2(t^-)I_2^2((t-\delta_2(t))^-)}{I_2^2(t)}\Big)}\lambda(du) \\ \le & \, h^2\frac{\mu^2}{\Lambda^2}k^6\lambda(Y)\Bigg(\frac{2a_1}{1-4\theta h^2\frac{\mu^2}{\Lambda^2}k^6} + \frac{1}{1-\theta h^2\frac{\mu^2}{\Lambda^2}k^6}\Bigg), \end{aligned} \tag{15}$$

where $\theta \in (0,1)$ is an arbitrary number. Substituting (15) and (14) in (13) we get

$$\begin{aligned} LV_1(t) \le & \Big(\Lambda + (3+a_1)\mu + \varepsilon_1 + \varepsilon_2 + a_1\gamma + \gamma_1 + \gamma_2 + \delta\Big) - \varepsilon_1 I_1(t) - \varepsilon_2 I_2(t) + a_1\beta_1 I_1(t-\delta_1(t)) + a_1\beta_2 I_2(t-\delta_2(t)) \\ & + \frac{\sigma_1^2}{2}\Big(a_1+k^4\Big)I_1^2(t-\delta_1(t)) + \frac{\sigma_2^2}{2}\Big(a_1+k^4\Big)I_2^2(t-\delta_2(t)) + h^2\frac{\mu^2}{\Lambda^2}k^6\lambda(Y)\Bigg(\frac{2a_1}{1-4\theta h^2\frac{\mu^2}{\Lambda^2}k^6} + \frac{1}{1-\theta h^2\frac{\mu^2}{\Lambda^2}k^6}\Bigg) \\ = & \, K_1 - \varepsilon_1 I_1(t) - \varepsilon_2 I_2(t) + a_1\beta_1 I_1(t-\delta_1(t)) + a_1\beta_2 I_2(t-\delta_2(t)) + \frac{\sigma_1^2}{2}\Big(a_1+k^4\Big)I_1^2(t-\delta_1(t)) \\ & + \frac{\sigma_2^2}{2}\Big(a_1+k^4\Big)I_2^2(t-\delta_2(t)), \end{aligned} \tag{16}$$

where

$$K_1 := \Lambda + (3+a_1)\mu + \varepsilon_1 + \varepsilon_2 + a_1\gamma + \gamma_1 + \gamma_2 + \delta + h^2\frac{\mu^2}{\Lambda^2}k^6\lambda(Y)\Bigg(\frac{2a_1}{1-4\theta h^2\frac{\mu^2}{\Lambda^2}k^6} + \frac{1}{1-\theta h^2\frac{\mu^2}{\Lambda^2}k^6}\Bigg).$$

Now we can define function V_2 as terms with delay. Let it be

$$\begin{aligned} V_2(t) := & \, V_2(S(t), I_1(t), I_2(t), R(t)) \\ = & \, a_2\int_{t-\delta_1(t)}^t I_1(s)ds + a_3\int_{t-\delta_2(t)}^t I_2(s)ds + a_4\int_{t-\delta_1(t)}^t I_1^2(s)ds + a_5\int_{t-\delta_2(t)}^t I_2^2(s)ds, \end{aligned} \tag{17}$$

where a_2, a_3, a_4 and a_5 are positive constants which have to be determined. Using (7) and Assumption 2.3 differential operator LV_2 has following form (as derivative of parametric integrals)

$$
\begin{aligned}
LV_2(t) &= a_2\Big(I_1(t) - \delta_1'(t)I_1(t-\delta_1(t))\Big) + a_3\Big(I_2(t) - \delta_2'(t)I_2(t-\delta_2(t))\Big) + a_4\Big(I_1^2(t) - \delta_1'(t)I_1^2(t-\delta_1(t))\Big)\\
&\quad + a_5\Big(I_2^2(t) - \delta_2'(t)I_2^2(t-\delta_2(t))\Big)\\
&\le a_2\Big(I_1(t) - k_1I_1(t-\delta_1(t))\Big) + a_3\Big(I_2(t) - k_1I_2(t-\delta_2(t))\Big) + a_4\Big(I_1^2(t) - k_1I_1^2(t-\delta_1(t))\Big)\\
&\quad + a_5\Big(I_2^2(t) - k_1I_2^2(t-\delta_2(t))\Big).
\end{aligned}
\tag{18}
$$

According to (12) and (17) function $V(t) = V_1(t) + V_2(t)$ has following form

$$
\begin{aligned}
V(t) &= V(S(t), I_1(t), I_2(t), R(t))\\
&= (S(t) + a_1 - a_1\ln S(t)) + (I_1(t) + 1 - \ln I_1(t)) + (I_2(t) + 1 - \ln I_2(t)) + (R(t) + 1 - \ln R(t))\\
&\quad + a_2\int_{t-\delta_1(t)}^t I_1(s)ds + a_3\int_{t-\delta_2(t)}^t I_2(s)ds + a_4\int_{t-\delta_1(t)}^t I_1^2(s)ds + a_5\int_{t-\delta_2(t)}^t I_2^2(s)ds.
\end{aligned}
\tag{19}
$$

Using (16) and (18) operator LV is

$$
\begin{aligned}
LV(t) = LV_1(t) + LV_2(t) &\le K_1 + I_1(t)\Big(a_2 - \varepsilon_1\Big) + I_2(t)\Big(a_3 - \varepsilon_2\Big) + I_1(t-\delta_1(t))\Big(a_1\beta_1 - a_2k_1\Big) + I_2(t-\delta_2(t))\Big(a_1\beta_2 - a_3k_1\Big)\\
&\quad + a_4I_1^2(t) + a_5I_2^2(t) + I_1^2(t-\delta_1(t))\Big(\frac{\sigma_1^2}{2}\Big(a_1 + k^4\Big) - a_4k_1\Big) + I_2^2(t-\delta_2(t))\Big(\frac{\sigma_2^2}{2}\Big(a_1 + k^4\Big) - a_5k_1\Big).
\end{aligned}
$$

Since we want to eliminate terms with delay, we will chose $a_1 = k_1\max\left\{\frac{\varepsilon_1}{\beta_1}, \frac{\varepsilon_2}{\beta_2}\right\}$, $a_2 = \varepsilon_1$, $a_3 = \varepsilon_2$, $a_4 = \frac{\left(k_1\max\left\{\frac{\varepsilon_1}{\beta_1}, \frac{\varepsilon_2}{\beta_2}\right\} + k^4\right)\frac{\sigma_1^2}{2}}{k_1}$ and $a_5 = \frac{\left(k_1\max\left\{\frac{\varepsilon_1}{\beta_1}, \frac{\varepsilon_2}{\beta_2}\right\} + k^4\right)\frac{\sigma_2^2}{2}}{k_1}$ which yields to

$$
LV(t) \le K_1 + a_4I_1^2(t) + a_5I_2^2(t) \le K_1 + k^2(a_4 + a_5) := C. \tag{20}
$$

Applying generalized Itô formula (8) on function $V(t)$ defined in (19), for arbitrary $k \ge k_0$ and $T > 0$ we obtain

$$
\begin{aligned}
dV(t) &= LV(t)dt + \Big(1 - \frac{a_1}{S(t)}\Big)\Big(-\sigma_1S(t)I_1(t-\delta_1(t))dB_1(t) - \sigma_2S(t)I_2(t-\delta_2(t))dB_2(t)\Big)\\
&\qquad + \Big(1 - \frac{1}{I_1(t)}\Big)\sigma_1S(t)I_1(t-\delta_1(t))dB_1(t) + \Big(1 - \frac{1}{I_2(t)}\Big)\sigma_2S(t)I_2(t-\delta_2(t))dB_2(t)\\
&\quad + \int_Y\Big[S(t) - \Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big) + \eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-) + a_1\\
&\qquad - a_1\ln\Big(S(t) - \Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big) + \eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)\Big)\\
&\qquad + I_1(t) + \eta_1(u)S(t^-)I_1\Big((t-\delta_1(t))^-\Big) + 1 - \ln\Big(I_1(t) + \eta_1(u)S(t^-)I_1\Big((t-\delta_1(t))^-\Big)\Big)\\
&\qquad + I_2(t) + \eta_2(u)S(t^-)I_2\Big((t-\delta_2(t))^-\Big) + 1 - \ln\Big(I_2(t) + \eta_2(u)S(t^-)I_2\Big((t-\delta_2(t))^-\Big)\Big)\\
&\qquad + R(t) + 1 - \ln R(t)\Big]\tilde{N}(dt, du)\\
&\quad - \int_Y\Big[S(t) + a_1 - a_1\ln S(t) + I_1(t) + 1 - \ln I_1(t) + I_2(t) + 1 - \ln I_2(t) + R(t) + 1 - \ln R(t)\Big]\tilde{N}(dt, du).
\end{aligned}
$$

$$dV(t) = LV(t)dt + \Big(a_1\sigma_1 I_1(t-\delta_1(t)) - \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)}\Big)dB_1(t) + \Big(a_1\sigma_2 I_2(t-\delta_2(t)) - \frac{\sigma_2 S(t) I_2(t-\delta_2(t))}{I_2(t)}\Big)dB_2(t)$$
$$- a_1 \int_Y \ln\Big[1 - \frac{\Big(\eta_1(u) I_1\big((t-\delta_1(t))^-\big) + \eta_2(u) I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)}{S(t)}\Big]\tilde{N}(dt,du)$$
$$- \int_Y \Big(\ln\Big[1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\Big] + \ln\Big[1 + \frac{\eta_2(u) S(t^-) I_2\big((t-\delta_2(t))^-\big)}{I_2(t)}\Big]\Big)\tilde{N}(dt,du).$$

Substituting (20) in the previous equality leads to

$$\begin{aligned} dV(t) \le{}& Cdt + \Big(a_1\sigma_1 I_1(t-\delta_1(t)) - \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)}\Big)dB_1(t) + \Big(a_1\sigma_2 I_2(t-\delta_2(t)) - \frac{\sigma_2 S(t) I_2(t-\delta_2(t))}{I_2(t)}\Big)dB_2(t) \\ & - a_1 \int_Y \ln\Big[1 - \frac{\Big(\eta_1(u) I_1\big((t-\delta_1(t))^-\big) + \eta_2(u) I_2\big((t-\delta_2(t))^-\big)\Big)S(t^-)}{S(t)}\Big]\tilde{N}(dt,du) \\ & - \int_Y \Big(\ln\Big[1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\Big] + \ln\Big[1 + \frac{\eta_2(u) S(t^-) I_2\big((t-\delta_2(t))^-\big)}{I_2(t)}\Big]\Big)\tilde{N}(dt,du). \end{aligned} \tag{21}$$

Integrating (21) from 0 to $\tau_k \wedge T$, taking expectation of both sides and using martingale property of Brownian motion and compensated process yields to

$$EV(\tau_k \wedge T) - EV(0) \le CE(\tau_k \wedge T) \quad \Rightarrow \quad EV(\tau_k \wedge T) \le V(0) + CT. \tag{22}$$

Let it be $\Omega_k = \{\tau_k \le T\}$, for $k \ge k_0$. Then, according to (11) it follows that $P(\Omega_k) \ge \epsilon$. For every $\omega \in \Omega_k$ at least one of variables $S(\tau_k,\omega), I_1(\tau_k,\omega), I_2(\tau_k,\omega)$ or $R(\tau_k,\omega)$ is equal either $\frac{1}{k}$ or k. Therefore, $V(S(\tau_k,\omega), I_1(\tau_k,\omega), I_2(\tau_k,\omega), R(\tau_k,\omega))$ is no less than either $k + 1 - \ln k$ or $\frac{1}{k} + 1 - \ln\frac{1}{k}$. So we have,

$$V(S(\tau_k,\omega), I_1(\tau_k,\omega), I_2(\tau_k,\omega), R(\tau_k,\omega)) \ge \min\Big\{k + 1 - \ln k, \frac{1}{k} + 1 - \ln\frac{1}{k}\Big\}$$

which yields to (using (22))

$$V(0) + CT \ge E\big(I_{\Omega_k} V(\tau_k \wedge T)\big) \ge \epsilon \min\Big\{k + 1 - \ln k, \frac{1}{k} + 1 - \ln\frac{1}{k}\Big\},$$

where I_{Ω_k} is indicator function of Ω_k. Letting $k \to \infty$ leads to

$$\infty > V(0) + CT = \infty,$$

which is contradiction. Therefore, $\tau_\infty = \infty$ a.s, i.e. the solution of system (3) is unique global positive. In addition, it follows that (10) holds for every $t \ge -\tau$. □

4. Extinction

In this section, we will establish sufficient conditions under which disease is extinct from the population.

Definition 4.1. *The system $\{x(t)\,|\,t \ge 0\}$ is said to be extinct almost surely if*

$$\limsup_{t\to\infty} x(t) = 0 \quad a.s.$$

Theorem 4.2. ***(Extinction of viruses)***
Let Assumption 2.1 holds and let $(S(t), I_1(t), I_2(t), R(t))$, $t \ge -\tau$ be the solution of system (3) for any initial value $(S(\xi), I_1(\xi), I_2(\xi), R(\xi)) \in \mathcal{L}^1([-\tau, 0]; \Gamma)$.

1. *If one of the following conditions is satisfied*

   - ***Case(1.1)***

$$\sigma_1^2 > 2\Big(\beta_1 - \frac{\mu^2}{\Lambda^2}(\mu + \varepsilon_1 + \gamma_1)\Big) > 0 \quad a.s. \quad i.e. \quad \mathcal{R}_0^1 := \frac{\Lambda^2}{\mu^2(\mu + \varepsilon_1 + \gamma_1)}\Big(\beta_1 - \frac{\sigma_1^2}{2}\Big) < 1, \tag{23}$$

   - ***Case(1.2)***

$$\sigma_1^2 > \frac{\beta_1^2}{2(\mu + \varepsilon_1 + \gamma_1)} \quad a.s. \quad i.e. \quad \mathcal{R}_0^{1*} := \frac{\beta_1^2}{2\sigma_1^2(\mu + \varepsilon_1 + \gamma_1)} < 1, \tag{24}$$

   then the first virus will extinct exponentially almost surely in Γ*, i.e.*

$$\limsup_{t\to+\infty} I_1(t) = 0 \quad a.s.$$

2. *If one of the following conditions is satisfied*

   - ***Case(2.1)***

$$\sigma_2^2 > 2\Big(\beta_2 - \frac{\mu^2}{\Lambda^2}(\mu + \varepsilon_2 + \gamma_2)\Big) > 0 \quad a.s. \quad i.e. \quad \mathcal{R}_0^2 := \frac{\Lambda^2}{\mu^2(\mu + \varepsilon_2 + \gamma_2)}\Big(\beta_2 - \frac{\sigma_2^2}{2}\Big) < 1, \tag{25}$$

   - ***Case(2.2)***

$$\sigma_2^2 > \frac{\beta_2^2}{2(\mu + \varepsilon_2 + \gamma_2)} \quad a.s. \quad i.e. \quad \mathcal{R}_0^{2*} := \frac{\beta_2^2}{2\sigma_2^2(\mu + \varepsilon_2 + \gamma_2)} < 1, \tag{26}$$

   then the second virus will extinct exponentially almost surely in Γ*, i.e.*

$$\limsup_{t\to+\infty} I_2(t) = 0 \quad a.s.$$

3. *If the one of the following conditions is satisfied*

   - ***Case(3.1)***

$$\begin{aligned} &\sigma_1^2 + \sigma_2^2 > 2\Big(\beta_1 + \beta_2 - \frac{\mu^2}{\Lambda^2}(2\mu + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2)\Big) > 0 \quad a.s. \quad i.e. \\ &\mathcal{R}_0^{1,2} := \frac{\Lambda^2}{\mu^2(2\mu + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2)}\Big(\beta_1 + \beta_2 - \frac{\sigma_1^2 + \sigma_2^2}{2}\Big) < 1, \end{aligned} \tag{27}$$

   - ***Case(3.2)***

$$\frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_2^2}{2\sigma_2^2} < 2\mu + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2 \quad a.s. \quad i.e. \quad \mathcal{R}_0^{1,2*} := \frac{1}{2\mu + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2}\Big(\frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_2^2}{2\sigma_2^2}\Big) < 1, \tag{28}$$

   then both viruses will extinct exponentially almost surely in Γ*, i.e.*

$$\limsup_{t\to+\infty}(I_1(t) + I_2(t)) = 0 \quad a.s.$$

Proof. *1.* We will prove the first part of theorem.

- **Case (1.1)**
  Applying generalized Itô formula (8) on function $V(t) = \ln(I_1(t) + 1)$ leads to

$$
\begin{aligned}
dV(t) = &\Big[\frac{1}{I_1(t)+1}\Big(\beta_1 S(t) I_1(t-\delta_1(t)) - \big(\mu+\varepsilon_1+\gamma_1\big) I_1(t)\Big) - \frac{\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2}\Big]dt \\
&+ \int_Y \Big[\ln\Big(I_1(t) + \eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big) + 1\Big) - \ln(I_1(t)+1) - \frac{1}{I_1(t)+1}\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)\Big]\lambda(du)dt \\
&+ \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1} dB_1(t) + \int_Y \Big(\ln\Big(I_1(t) + \eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big) + 1\Big) - \ln(I_1(t)+1)\Big)\tilde{N}(dt,du) \\
= &\Big[\frac{\beta_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1} - \frac{(\mu+\varepsilon_1+\gamma_1) I_1(t)}{I_1(t)+1} - \frac{\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2}\Big]dt \\
&+ \int_Y \Big[\ln\Big(1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big) - \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big]\lambda(du)dt \\
&+ \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1} dB_1(t) + \int_Y \ln\Big(1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\tilde{N}(dt,du).
\end{aligned}
\tag{29}
$$

We will use following inequalities in order to derive estimation for (29):

$$
S(t), I_1(t-\delta_1(t)), I_1(t) \le N(t) \le \frac{\Lambda}{\mu}; \quad \frac{1}{I_1(t)+1} \le 1; \quad -\frac{1}{I_1(t)+1} \le -\frac{\mu}{\Lambda}; \quad \ln(1+z) - z \le 0,\ z > 0.
$$

Therefore,

$$
\begin{aligned}
d\ln(I_1(t)+1) \le &\Big[\frac{\beta_1 \Lambda^2}{\mu^2} - (\mu+\varepsilon_1+\gamma_1) - \frac{\sigma_1^2 \Lambda^2}{2\mu^2}\Big]dt + \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1} dB_1(t) \\
&+ \int_Y \ln\Big(1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\tilde{N}(dt,du).
\end{aligned}
\tag{30}
$$

Integrating both sides of (30) from 0 to t and dividing with t we obtain

$$
\frac{\ln(I_1(t)+1) - \ln(I_1(0)+1)}{t} \le \frac{1}{t}\int_0^t \Big[\frac{\Lambda^2}{\mu^2}\Big(\beta_1 - \frac{\sigma_1^2}{2}\Big) - (\mu+\varepsilon_1+\gamma_1)\Big]ds + \frac{1}{t}M_1(t) + \frac{1}{t}M_2(t), \tag{31}
$$

where

$$
M_1(t) = \int_0^t \frac{\sigma_1 S(s) I_1(s-\delta_1(s))}{I_1(s)+1} dB_1(s) \tag{32}
$$

and

$$
M_2(t) = \int_0^t \int_Y \ln\Big(1 + \frac{\eta_1(u) S(s^-) I_1\big((s-\delta_1(s))^-\big)}{I_1(s)+1}\Big)\tilde{N}(ds,du) \tag{33}
$$

are continuous (local) martingale and local martingale, respectively, such that $M_1(0) = 0$ and $M_2(0) = 0$.

Also,

$$[M_1, M_1](t) = \int_0^t \frac{\sigma_1^2 S^2(s) I_1^2(s-\delta_1(s))}{(I_1(s)+1)^2} ds \le \frac{\sigma_1^2 \Lambda^4}{\mu^4} t \quad \Rightarrow \quad \limsup_{t\to\infty} \frac{[M_1, M_1](t)}{t} \le \frac{\sigma_1^2 \Lambda^4}{\mu^4} < \infty, \tag{34}$$

$$\begin{aligned} \langle M_2, M_2\rangle(t) &= \int_0^t \int_Y \ln^2\Big(1 + \frac{\eta_1(u) S(s^-) I_1\big((s-\delta_1(s))^-\big)}{I_1(s)+1}\Big)\lambda(du)ds \le \ln^2\Big(1+\frac{h\Lambda}{\mu}\Big)\lambda(Y)t \\ &\Rightarrow \lim_{t\to\infty} \int_0^t \frac{d\langle M_2, M_2\rangle(s)}{(1+s)^2} \le \ln^2\Big(1+\frac{h\Lambda}{\mu}\Big)\lambda(Y) < \infty. \end{aligned} \tag{35}$$

Hence, using Lemma 2.4 and Lemma 2.5 yields to

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{M_2(t)}{t} = 0 \quad a.s.$$

Taking limes superior of both sides of (31) and using condition (23) we can conclude that

$$\begin{aligned} \limsup_{t\to\infty} \frac{\ln(I_1(t)+1)}{t} &\le \frac{\Lambda^2}{\mu^2}\Big(\beta_1 - \frac{\sigma_1^2}{2}\Big) - (\mu + \varepsilon_1 + \gamma_1) \\ &= \big(\mu + \varepsilon_1 + \gamma_1\big)\big(\mathcal{R}_0^1 - 1\big) < 0 \quad a.s. \end{aligned} \tag{36}$$

Since $I_1(t) > 0$ holds for every $t \ge 0$ (it is proven in Theorem 3.1) and ln is increasing function, using inequality (36) there exist constant $K > 0$ such that

$$I_1(t) < I_1(t) + 1 \Rightarrow \ln I_1(t) < \ln(I_1(t)+1) < -Kt.$$

Then it follows

$$I_1(t) < I_1(t) + 1 < e^{-Kt},$$

i.e. we can conclude that,

$$\limsup_{t\to\infty} I_1(t) = 0 \quad a.s.$$

- **Case (1.2)**
Different inequalities then in the previous part of proof (Case (1.1)) will be used in order to estimate (29). Using the fact that function $f(x) = \beta_1 x - \frac{\sigma_1^2}{2}x^2$, for $x = \frac{S(t)I_1(t-\delta_1(t))}{I_1(t)+1}$, is increasing on $\Big[0, \frac{\beta_1}{\sigma_1^2}\Big]$ and therefore has maximum in $\frac{\beta_1}{\sigma_1^2}$ which is $\frac{\beta_1^2}{2\sigma_1^2}$ and $\ln(1+z) - z \le 0$, for any $z > 0$ we get

$$\begin{aligned} d\ln(I_1(t)+1) \le& \Big[\frac{\beta_1^2}{2\sigma_1^2} - (\mu + \varepsilon_1 + \gamma_1)\Big]dt + \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1} dB_1(t) \\ &+ \int_Y \ln\Big(1 + \frac{\eta_1(u) S(t^-) I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\tilde{N}(dt, du). \end{aligned} \tag{37}$$

Integrating both sides of (37) from 0 to t and dividing with t yields to

$$\frac{\ln(I_1(t)+1) - \ln(I_1(0)+1)}{t} \le \frac{1}{t}\int_0^t \Big[\frac{\beta_1^2}{2\sigma_1^2} - (\mu+\varepsilon_1+\gamma_1)\Big]ds + \frac{1}{t}M_1(t) + \frac{1}{t}M_2(t), \tag{38}$$

where $M_1(t)$ and $M_2(t)$ are defined in (32) and (33), respectively. Using the same consideration as in the previous part of proof (Case (1.1)), taking limes superior of (38) and using condition (24) we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln(I_1(t)+1)}{t} &\le \frac{\beta_1^2}{2\sigma_1^2}-(\mu+\varepsilon_1+\gamma_1)\\ &= \Big(\mu+\varepsilon_1+\gamma_1\Big)\Big(\mathcal{R}_0^{1*}-1\Big)<0 \quad a.s.\end{aligned}$$

Hence,

$$\limsup_{t\to\infty} I_1(t)=0 \quad a.s.$$

2. Extinction of second virus can be proved in the same way as we did for first virus using function $V(t):=\ln(I_2(t)+1)$ and because of that the proof will be omitted.

3. Now we will prove the final part of theorem.

- **Case (3.1)**

  Applying generalized Itô formula (8) on function $V(t):=\ln(I_1(t)+I_2(t)+1)$ it follows that

$$\begin{aligned}dV(t)=&\Big[\frac{1}{I_1(t)+I_2(t)+1}\Big(\beta_1S(t)I_1(t-\delta_1(t))-\Big(\mu+\varepsilon_1+\gamma_1\Big)I_1(t)+\beta_2S(t)I_2(t-\delta_2(t))-\Big(\mu+\varepsilon_2+\gamma_2\Big)I_2(t)\Big)\\ &-\frac{\sigma_1^2S^2(t)I_1^2(t-\delta_1(t))}{2(I_1(t)+I_2(t)+1)^2}-\frac{\sigma_2^2S^2(t)I_2^2(t-\delta_2(t))}{2(I_1(t)+I_2(t)+1)^2}\Big]dt\\ &+\int_Y\Big[\ln\Big(I_1(t)+\eta_1(u)S(t^-)I_1\Big((t-\delta_1(t))^-\Big)+I_2(t)+\eta_2(u)S(t^-)I_2\Big((t-\delta_2(t))^-\Big)+1\Big)-\ln\Big(I_1(t)+I_2(t)+1\Big)\\ &\quad-\frac{1}{I_1(t)+I_2(t)+1}\eta_1(u)S(t^-)I_1\Big((t-\delta_1(t))^-\Big)-\frac{1}{I_1(t)+I_2(t)+1}\eta_2(u)S(t^-)I_2\Big((t-\delta_2(t))^-\Big)\Big]\lambda(du)dt\\ &+\frac{\sigma_1S(t)I_1(t-\delta_1(t))}{I_1(t)+I_2(t)+1}dB_1(t)+\frac{\sigma_2S(t)I_2(t-\delta_2(t))}{I_1(t)+I_2(t)+1}dB_2(t)\\ &+\int_Y\Big[\ln\Big(I_1(t)+\eta_1(u)S(t^-)I_1\Big((t-\delta_1(t))^-\Big)+I_2(t)+\eta_2(u)S(t^-)I_2\Big((t-\delta_2(t))^-\Big)+1\Big)-\ln\Big(I_1(t)+I_2(t)+1\Big)\Big]\tilde{N}(dt,du).\end{aligned}$$

Hence,

$$\begin{aligned}dV(t)=&\Big[\frac{\beta_1S(t)I_1(t-\delta_1(t))}{I_1(t)+I_2(t)+1}+\frac{\beta_2S(t)I_2(t-\delta_2(t))}{I_1(t)+I_2(t)+1}-\frac{(\mu+\varepsilon_1+\gamma_1)I_1(t)}{I_1(t)+I_2(t)+1}-\frac{(\mu+\varepsilon_2+\gamma_2)I_2(t)}{I_1(t)+I_2(t)+1}\\ &-\frac{\sigma_1^2S^2(t)I_1^2(t-\delta_1(t))}{2(I_1(t)+I_2(t)+1)^2}-\frac{\sigma_2^2S^2(t)I_2^2(t-\delta_2(t))}{2(I_1(t)+I_2(t)+1)^2}\Big]dt\\ &+\int_Y\Big[\ln\Big(1+\frac{\Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big)+\eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)}{I_1(t)+I_2(t)+1}\Big)\\ &\quad-\frac{\Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big)+\eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)}{I_1(t)+I_2(t)+1}\Big]\lambda(du)dt\\ &+\frac{\sigma_1S(t)I_1(t-\delta_1(t))}{I_1(t)+I_2(t)+1}dB_1(t)+\frac{\sigma_2S(t)I_2(t-\delta_2(t))}{I_1(t)+I_2(t)+1}dB_2(t)\\ &+\int_Y\ln\Big(1+\frac{\Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big)+\eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)}{I_1(t)+I_2(t)+1}\Big)\tilde{N}(dt,du).\end{aligned} \tag{39}$$

We will use following inequalities in order to estimate (39):

$$S(t), I_1(t-\delta_1(t)), I_1(t), I_2(t-\delta_2(t)), I_2(t) \leq N(t) \leq \frac{\Lambda}{\mu}; \quad \frac{1}{I_1(t)+I_2(t)+1} \leq 1;$$

$$-\frac{1}{I_1(t)+I_2(t)+1} \leq -\frac{\mu}{\Lambda}; \quad \ln(1+z) - z \leq 0, \text{ for any } z > 0.$$

Therefore,

$$\begin{aligned} d\ln(I_1(t)+I_2(t)+1) \leq \Big[&\frac{\Lambda^2}{\mu^2}\big(\beta_1+\beta_2\big) - \big(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2\big) - \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\Big)\Big]dt \\ &+ \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+I_2(t)+1} dB_1(t) + \frac{\sigma_2 S(t) I_2(t-\delta_2(t))}{I_1(t)+I_2(t)+1} dB_2(t) \\ &+ \int_Y \ln\Big(1 + \frac{\Big(\eta_1(u) I_1\big((t-\delta_1(t))^-\big) + \eta_2(u) I_2\big((t-\delta_2(t))^-\big)\Big) S(t^-)}{I_1(t)+I_2(t)+1}\Big) \tilde{N}(dt,du). \end{aligned} \tag{40}$$

Integrating both sides of (40) from 0 to t and dividing with t leads to

$$\begin{aligned} \frac{\ln(I_1(t)+I_2(t)+1) - \ln(I_1(0)+I_2(0)+1)}{t} \leq \frac{1}{t}\int_0^t \Big[&\frac{\Lambda^2}{\mu^2}\big(\beta_1+\beta_2\big) - \big(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2\big) - \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}\Big)\Big]ds \\ &+ \frac{1}{t}M_3(t) + \frac{1}{t}M_4(t) + \frac{1}{t}M_5(t), \end{aligned} \tag{41}$$

where

$$M_3(t) = \int_0^t \frac{\sigma_1 S(s) I_1(s-\delta_1(s))}{I_1(s)+I_2(t)+1} dB_1(s) \tag{42}$$

and

$$M_4(t) = \int_0^t \frac{\sigma_2 S(s) I_2(s-\delta_2(s))}{I_1(s)+I_2(s)+1} dB_2(s) \tag{43}$$

are continuous (local) martingales such that $M_3(0) = 0$, $M_4(0) = 0$ and

$$M_5(t) = \int_0^t \int_Y \ln\Big(1 + \frac{\Big(\eta_1(u) I_1\big((s-\delta_1(s))^-\big) + \eta_2(u) I_2\big((s-\delta_2(s))^-\big)\Big) S(s^-)}{I_1(s)+I_2(s)+1}\Big) \tilde{N}(ds,du) \tag{44}$$

is local martingale such that $M_5(0) = 0$. Furthermore,

$$[M_3, M_3](t) = \int_0^t \frac{\sigma_1^2 S^2(s) I_1^2(s-\delta_1(s))}{(I_1(s)+I_2(s)+1)^2} ds \leq \frac{\sigma_1^2 \Lambda^4}{\mu^4} t \quad \Rightarrow \quad \limsup_{t\to\infty} \frac{[M_3, M_3](t)}{t} \leq \frac{\sigma_1^2 \Lambda^4}{\mu^4} < \infty,$$

$$[M_4, M_4](t) = \int_0^t \frac{\sigma_2^2 S^2(s) I_2^2(s-\delta_2(s))}{(I_1(s)+I_2(s)+1)^2} ds \leq \frac{\sigma_2^2 \Lambda^4}{\mu^4} t \quad \Rightarrow \quad \limsup_{t\to\infty} \frac{[M_4, M_4](t)}{t} \leq \frac{\sigma_2^2 \Lambda^4}{\mu^4} < \infty,$$

$$\langle M_5, M_5\rangle(t) = \int_0^t \int_Y \ln^2\Big(1 + \frac{\Big(\eta_1(u) I_1\big((s-\delta_1(s))^-\big) + \eta_2(u) I_2\big((s-\delta_2(s))^-\big)\Big) S(s^-)}{I_1(s)+I_2(s)+1}\Big) \lambda(du) ds \leq \ln^2\Big(1 + \frac{2h\Lambda}{\mu}\Big)\lambda(Y) t$$

$$\Rightarrow \lim_{t\to\infty}\int_0^t \frac{d\langle M_5, M_5\rangle(s)}{(1+s)^2} \le \ln^2\Big(1+\frac{2h\Lambda}{\mu}\Big)\lambda(Y) < \infty.$$

Hence, using Lemma 2.4 and Lemma 2.5 yields to

$$\lim_{t\to\infty}\frac{M_3(t)}{t}=0,\quad \lim_{t\to\infty}\frac{M_4(t)}{t}=0 \quad\text{and}\quad \lim_{t\to\infty}\frac{M_5(t)}{t}=0 \quad a.s.$$

Taking limes superior of both sides of (41) and using condition (27) we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln(I_1(t)+I_2(t)+1)}{t} &\le \frac{\Lambda^2}{\mu^2}\Big(\beta_1+\beta_2-\frac{\sigma_1^2+\sigma_2^2}{2}\Big)-(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2)\\ &= \Big(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2\Big)\Big(\mathcal{R}_0^{1,2}-1\Big)<0 \quad a.s.\end{aligned} \tag{45}$$

Since $I_1(t)>0$ and $I_2(t)>0$ holds for every $t\ge 0$ (it is proven in Theorem 3.1), and ln is increasing function, using inequality (45) there exist constant $K>0$ such that

$$I_1(t)+I_2(t) < I_1(t)+I_2(t)+1 \Rightarrow \ln(I_1(t)+I_2(t)) < \ln(I_1(t)+I_2(t)+1) < -Kt.$$

Then it follows

$$I_1(t)+I_2(t) < I_1(t)+I_2(t)+1 < e^{-Kt},$$

i.e. we can conclude that,

$$\limsup_{t\to\infty}\Big(I_1(t)+I_2(t)\Big)=0 \quad a.s.$$

- **Case (3.2)**
  Different inequalities from previous part of proof (Case (3.1)) will be used in order to estimate (39). Using the facts that function $f(x)=\beta_1 x-\frac{\sigma_1^2}{2}x^2$, for $x=\frac{S(t)I_1(t-\delta_1(t))}{I_1(t)+I_2(t)+1}$, is increasing on $\Big[0,\frac{\beta_1}{\sigma_1^2}\Big]$ and therefore has maximum in $\frac{\beta_1}{\sigma_1^2}$ which is $\frac{\beta_1^2}{2\sigma_1^2}$, the fact that function $g(y)=\beta_2 y-\frac{\sigma_2^2}{2}y^2$, for $y=\frac{S(t)I_2(t-\delta_2(t))}{I_1(t)+I_2(t)+1}$, is increasing on $\Big[0,\frac{\beta_2}{\sigma_2^2}\Big]$ and because of that has maximum in $\frac{\beta_2}{\sigma_2^2}$ which is $\frac{\beta_2^2}{2\sigma_2^2}$ and the fact that $\ln(1+z)-z\le 0$, for any $z>0$ we obtain

$$\begin{aligned} d\ln(I_1(t)+I_2(t)+1) \le &\Big[\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_2^2}{2\sigma_2^2}-\Big(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2\Big)\Big]dt\\ &+\frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+I_2(t)}dB_1(t)+\frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_1(t)+I_2(t)}dB_2(t)\\ &+\int_Y \ln\Big(1+\frac{\Big(\eta_1(u)I_1\Big((t-\delta_1(t))^-\Big)+\eta_2(u)I_2\Big((t-\delta_2(t))^-\Big)\Big)S(t^-)}{I_1(t)+I_2(t)}\Big)\tilde{N}(dt,du).\end{aligned} \tag{46}$$

Integrating both sides of (46) from 0 to t and dividing with t we get

$$\begin{aligned}\frac{\ln(I_1(t)+I_2(t)+1)-\ln(I_1(0)+I_2(0)+1)}{t} \le &\frac{1}{t}\int_0^t\Big[\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_2^2}{2\sigma_2^2}-\Big(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2\Big)\Big]ds\\ &+\frac{1}{t}M_3(t)+\frac{1}{t}M_4(t)+\frac{1}{t}M_5(t),\end{aligned} \tag{47}$$

where $M_3(t), M_4(t)$ and $M_5(t)$ are defined in (42), (43) and (44), respectively. As in the previous part of proof (Case (3.1)), taking limes superior of both sides of (47) and using condition (28) yields to

$$\limsup_{t\to\infty} \frac{\ln(I_1(t)+I_2(t)+1)}{t} \le \frac{\beta_1^2}{2\sigma_1^2} + \frac{\beta_2^2}{2\sigma_2^2} - \Big(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2\Big)$$
$$= \Big(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2\Big)\Big(\mathcal{R}_0^{1,2*}-1\Big) < 0 \quad a.s.$$

Hence,

$$\limsup_{t\to\infty} \Big(I_1(t)+I_2(t)\Big) = 0 \quad a.s.$$

□

5. Strong persistence in mean

In this section we will establish sufficient conditions for the stochastically strong persistence in mean of viruses in population.

Let us denote

$$\langle x(t)\rangle := \frac{1}{t}\int_0^t x(s)ds, \quad t \ge 0.$$

Definition 5.1. *The system $\{x(t)\,|\,t\ge 0\}$ is said to be stochastically strong persistent in mean if*

$$\liminf_{t\to\infty}\langle x(t)\rangle = \liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds > 0 \quad a.s.$$

Theorem 5.2. ***(Strong persistence in mean)***
Let Assumption 2.1 and Assumption 2.3 hold and let $(S(t), I_1(t), I_2(t), R(t))$, $t \ge -\tau$ be the solution of system (3) *for any initial value $(S(\xi), I_1(\xi), I_2(\xi), R(\xi)) \in \mathcal{L}^1([-\tau, 0]; \Gamma)$. If*

1.

$$\sigma_1^2 < \frac{2\mu^4}{\Lambda^4}\left[\frac{\beta_1\Lambda^2}{\mu^2} - \frac{\Lambda^2 h^2\lambda(Y)}{2\mu^2} - (\mu+\varepsilon_1+\gamma_1)\right] > 0 \quad a.s. \quad i.e. \quad \bar{\mathcal{R}}_0^1 := \frac{\beta_1\Lambda^2}{\mu^2\Big[\mu+\varepsilon_1+\gamma_1+\frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2}+\frac{h^2\lambda(Y)}{2}\Big)\Big]} > 1, \tag{48}$$

then the first virus will strongly persist in mean almost surely in Γ.

2.

$$\sigma_2^2 < \frac{2\mu^4}{\Lambda^4}\left[\frac{\beta_2\Lambda^2}{\mu^2} - \frac{\Lambda^2 h^2\lambda(Y)}{2\mu^2} - (\mu+\varepsilon_2+\gamma_2)\right] > 0 \quad a.s. \quad i.e. \quad \bar{\mathcal{R}}_0^2 := \frac{\beta_2\Lambda^2}{\mu^2\Big[\mu+\varepsilon_2+\gamma_2+\frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_2^2\Lambda^2}{2\mu^2}+\frac{h^2\lambda(Y)}{2}\Big)\Big]} > 1, \tag{49}$$

then the second virus will strongly persist in mean almost surely in Γ.

3.

$$\begin{aligned}&\sigma_1^2+\sigma_2^2 < \frac{2\mu^4}{\Lambda^4}\left[\Big(\beta_1+\beta_2\Big)\frac{\Lambda^2}{\mu^2} - \frac{\Lambda^2 h^2\lambda(Y)}{\mu^2} - (2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2)\right] > 0 \quad a.s. \quad i.e.\\ &\bar{\mathcal{R}}_0^{1,2} := \frac{(\beta_1+\beta_2)\Lambda^2}{\mu^2\Big[2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2+\frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2}+\frac{\sigma_2^2\Lambda^2}{2\mu^2}+h^2\lambda(Y)\Big)\Big]} > 1,\end{aligned} \tag{50}$$

then both viruses will strongly persist in mean almost surely in Γ.

Proof. 1. Let us apply generalized Itô formula (8) to the function $V := V_1 + V_2 : \mathbb{R}^4_+ \to [0, \infty)$ where

$$V_1(t) := -\ln(I_1(t)+1) \tag{51}$$

and function V_2 will be determined later. Then

$$\begin{aligned}
dV_1(t) &= \Big[-\frac{1}{I_1(t)+1}\Big(\beta_1 S(t) I_1(t-\delta_1(t)) - \big(\mu+\varepsilon_1+\gamma_1\big)I_1(t)\Big) + \frac{\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2}\Big]dt \\
&\quad + \int_Y \Big[-\ln\Big(I_1(t) + \eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)+1\Big) + \ln(I_1(t)+1) \\
&\qquad + \frac{1}{I_1(t)+1}\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)\Big]\lambda(du)dt \\
&\quad - \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) + \int_Y \Big(-\ln\Big(\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)+1\Big) + \ln(I_1(t)+1)\Big)\tilde{N}(dt,du) \\
&\le \Big[-\frac{\beta_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1} + \frac{\sigma_1^2 S^2(t) I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2} + \mu+\varepsilon_1+\gamma_1 + \frac{\beta_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1}\Big]dt \\
&\quad + \int_Y \frac{\eta_1^2(u)S^2(t^-)I_1^2\big((t-\delta_1(t))^-\big)}{2(I_1(t)+1)^2}\lambda(du)dt - \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) \\
&\quad - \int_Y \ln\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\tilde{N}(dt,du),
\end{aligned} \tag{52}$$

where we added nonnegative term $\frac{\beta_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1}$ and applied Taylor series on the function $\ln(1+a)$ up to second term, for $a = \frac{\eta_1(u)S(t^-)I_1((t-\delta_1(t))^-)}{I_1(t)+1}$. In order to estimate (52) we will use Assumption 2.1, inequalities $S(t) \le \frac{\Lambda}{\mu}$; $\frac{1}{I_1(t)+1} \le 1$ and the fact that function $f(x) = -\beta_1 x + \frac{\sigma_1^2}{2}x^2$ is increasing on $\left[\frac{\beta_1}{\sigma_1^2}, \frac{\Lambda^2}{\mu^2}\right]$ and therefore has maximum in $\frac{\Lambda^2}{\mu^2}$ $\Big($where $x = \frac{S(t)I_1(t-\delta_1(t))}{I_1(t)+1} \le \frac{\Lambda^2}{\mu^2}\Big)$. Therefore,

$$\begin{aligned}
dV_1(t) &\le \Big[-\frac{\beta_1\Lambda^2}{\mu^2} + \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{h^2\lambda(Y)}{2}\Big) + \mu+\varepsilon_1+\gamma_1 + \frac{\beta_1\Lambda}{\mu}I_1(t-\delta_1(t))\Big]dt \\
&\quad - \frac{\sigma_1 S(t) I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) - \int_Y \ln\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)}\Big)\tilde{N}(dt,du).
\end{aligned} \tag{53}$$

Now, function V_2 can be defined as

$$V_2(t) := \frac{\beta_1\Lambda}{\mu}\int_{t-\delta_1(t)}^{t} I_1(s)ds, \tag{54}$$

where using rule for derivative of parametric integral and Assumption 2.3 one can obtain

$$dV_2(t) = \frac{\beta_1\Lambda}{\mu}\Big(I_1(t) - \delta_1'(t)I_1(t-\delta_1(t))\Big) \le \frac{\beta_1\Lambda}{\mu}\Big(I_1(t) - k_1 I_1(t-\delta_1(t))\Big). \tag{55}$$

According to (51) and (54) function $V = V_1 + V_2$ is

$$V(t) = -\ln(I_1(t)+1) + \frac{\beta_1\Lambda}{\mu}\int_{t-\delta_1(t)}^{t} I_1(s)ds.$$

Therefore, using (53) and (55) it follows that

$$\begin{aligned}
dV(t) \leq& \Big[-\frac{\beta_1\Lambda^2}{\mu^2} + \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{h^2\lambda(Y)}{2}\Big) + \mu + \varepsilon_1 + \gamma_1 + \frac{\beta_1\Lambda}{\mu}I_1(t-\delta_1(t)) + \frac{\beta_1\Lambda}{\mu}\Big(I_1(t) - k_1 I_1(t-\delta_1(t))\Big)\Big]dt \\
&- \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) - \int_Y \ln\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\tilde{N}(dt,du) \\
\leq& \Big[-\frac{\beta_1\Lambda^2}{\mu^2} + \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\Lambda^2} + \frac{h^2\lambda(Y)}{2}\Big) + \mu + \varepsilon_1 + \gamma_1 + \frac{\beta_1\Lambda}{\mu}I_1(t) + \frac{\beta_1\Lambda}{\mu}C_1\Big(I_1(t-\delta_1(t)) - I_1(t-\delta_1(t))\Big)\Big]dt \\
&- \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) - \int_Y \ln\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\tilde{N}(dt,du) \\
=& \Big[-\frac{\beta_1\Lambda^2}{\mu^2} + \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{h^2\lambda(Y)}{2}\Big) + \mu + \varepsilon_1 + \gamma_1 + \frac{\beta_1\Lambda}{\mu}I_1(t)\Big]dt \\
&- \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) - \int_Y \ln\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\tilde{N}(dt,du),
\end{aligned} \tag{56}$$

where $C_1 = \max\{1, k_1\}$.

Integrating both sides of (56) from 0 to t and dividing with t yields to

$$\begin{aligned}
&\frac{-\ln(I_1(t)+1) + \frac{\beta_1\Lambda}{\mu}\int_{t-\delta_1(t)}^t I_1(s)ds + \ln(I_1(0)+1) - \frac{\beta_1\Lambda}{\mu}\int_{-\delta_1(0)}^0 I_1(s)ds}{t} \\
&\leq \frac{1}{t}\int_0^t \Big[-\frac{\beta_1\Lambda^2}{\mu^2} + \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{h^2\lambda(Y)}{2}\Big) + \mu + \varepsilon_1 + \gamma_1\Big]ds + \frac{\beta_1\Lambda}{\mu}\frac{1}{t}\int_0^t I_1(s)ds \\
&\quad - \frac{1}{t}M_1(t) - \frac{1}{t}M_2(t),
\end{aligned}$$

where $M_1(t)$ and $M_2(t)$ are defined in (32) and (33), respectively. Hence,

$$\begin{aligned}
\frac{\beta_1\Lambda}{\mu}\frac{1}{t}\int_0^t I_1(s)ds \geq& -\frac{\ln(I_1(t)+1)}{t} + \frac{\beta_1\Lambda}{\mu}\frac{1}{t}\int_{t-\delta_1(t)}^t I_1(s)ds + \frac{\ln(I_1(0)+1)}{t} \\
&- \frac{\beta_1\Lambda}{\mu}\frac{1}{t}\int_{-\delta_1(0)}^0 I_1(s)ds + \frac{\beta_1\Lambda^2}{\mu^2} - \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{h^2\lambda(Y)}{2}\Big) - (\mu + \varepsilon_1 + \gamma_1) \\
&+ \frac{1}{t}M_1(t) + \frac{1}{t}M_2(t).
\end{aligned} \tag{57}$$

Taking limes inferior of both sides of (57), using Lemma 2.4 and Lemma 2.5 (due to (34) and (35)), the fact that $-\ln(I_1(t)+1) \geq -\ln\frac{\Delta}{\mu}$, $-I_1(t) \geq -\frac{\Delta}{\mu}$ and condition (48) we obtain

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_1(s)ds \geq& \frac{\frac{\beta_1\Lambda^2}{\mu^2} - \frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{h^2\lambda(Y)}{2}\Big) - (\mu + \varepsilon_1 + \gamma_1)}{\frac{\beta_1\Lambda}{\mu}} \\
=& \frac{(\bar{\mathcal{R}}_0^1 - 1)\Big(\frac{\Lambda^2}{\mu^2}\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{h^2\lambda(Y)}{2}\Big) + \mu + \varepsilon_1 + \gamma_1\Big)}{\frac{\beta_1\Lambda}{\mu}} > 0 \quad a.s.
\end{aligned}$$

2. Strong persistence in mean of second virus can be proved in the same way as we did for first virus using function $V(t) = -\ln(I_2(t)+1) + \frac{\beta_2\Lambda}{\mu}\int_{t-\delta_2(t)}^{t} I_2(s)ds$ so the proof will be omitted.

3. We will prove strong persistence in mean for both viruses applying generalized Itô formula (8) on function $V := V_1 + V_2 : \mathbb{R}_+^4 \to [0,\infty)$ where

$$V_1(t) := -\ln(I_1(t)+1) - \ln(I_2(t)+1) \tag{58}$$

and function V_2 will be determined later. Therefore,

$$\begin{aligned}
dV_1(t) &= \Big[-\frac{1}{I_1(t)+1}\Big(\beta_1 S(t)I_1(t-\delta_1(t)) - \big(\mu+\varepsilon_1+\gamma_1\big)I_1(t)\Big) - \frac{1}{I_2(t)+1}\Big(\beta_2 S(t)I_2(t-\delta_2(t)) \\
&\quad - \big(\mu+\varepsilon_2+\gamma_2\big)I_2(t)\Big) + \frac{\sigma_1^2 S^2(t)I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2}dt + \frac{\sigma_2^2 S^2(t)I_2^2(t-\delta_2(t))}{2(I_2(t)+1)^2}\Big]dt \\
&\quad + \int_Y \Big[-\ln\Big(I_1(t) + \eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)+1\Big) + \ln(I_1(t)+1) \\
&\qquad - \ln\Big(I_2(t) + \eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)+1\Big) + \ln(I_2(t)+1) \\
&\qquad + \frac{1}{I_1(t)+1}\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big) + \frac{1}{I_2(t)+1}\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)\Big]\lambda(du)dt \\
&\quad - \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) - \frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}dB_2(t) \\
&\quad + \int_Y \Big[-\ln\Big(I_1(t) + \eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)+1\Big) + \ln(I_1(t)+1) \\
&\qquad - \ln\Big(I_2(t) + \eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)+1\Big) + \ln(I_2(t)+1)\Big]\tilde{N}(dt,du) \\
&= \Big[-\frac{\beta_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1} - \frac{\beta_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1} + (\mu+\varepsilon_1+\gamma_1)\frac{I_1(t)}{I_1(t)+1} + (\mu+\varepsilon_2+\gamma_2)\frac{I_2(t)}{I_2(t)+1} \\
&\quad + \frac{\sigma_1^2 S^2(t)I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2} + \frac{\sigma_2^2 S^2(t)I_2^2(t-\delta_2(t))}{2(I_2(t)+1)^2}\Big]dt \\
&\quad + \int_Y \Big[-\ln\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big) + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1} \\
&\qquad - \ln\Big(1 + \frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\Big) + \frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\Big]\lambda(du)dt \\
&\quad - \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) - \frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}dB_2(t) \\
&\quad - \int_Y \Big[\ln\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\Big(1 + \frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\Big)\Big]\tilde{N}(dt,du).
\end{aligned}$$

Applying Taylor formula to functions $\ln(1+a)$ and $\ln(1+b)$ $\Big($where $a = \frac{\eta_1(u)S(t^-)I_1((t-\delta_1(t))^-)}{I_1(t)+1}$ and $b = \frac{\eta_2(u)S(t^-)I_2((t-\delta_2(t))^-)}{I_2(t)+1}\Big)$ and adding nonnegative terms $\frac{\beta_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}$ and $\frac{\beta_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}$ results in the following

$$\begin{aligned}
dV_1(t) &\le \Big[-\frac{\beta_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1} + \frac{\sigma_1^2 S^2(t)I_1^2(t-\delta_1(t))}{2(I_1(t)+1)^2} + \mu + \varepsilon_1 + \gamma_1 + \frac{\beta_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1} \\
&\quad - \frac{\beta_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1} + \frac{\sigma_2^2 S^2(t)I_2^2(t-\delta_2(t))}{2(I_2(t)+1)^2} + \mu + \varepsilon_2 + \gamma_2 + \frac{\beta_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}\Big]dt
\end{aligned} \tag{59}$$

$$+\int_Y\Big[\frac{\eta_1^2(u)S^2(t^-)I_1^2\Big((t-\delta_1(t))^-\Big)}{2(I_1(t)+1)^2}+\frac{\eta_2^2(u)S^2(t^-)I_2^2\Big((t-\delta_2(t))^-\Big)}{2(I_2(t)+1)^2}\Big]\lambda(du)dt$$
$$-\frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t)-\frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}dB_2(t)$$
$$-\int_Y\Big[\ln\Big(1+\frac{\eta_1(u)S(t^-)I_1\Big((t-\delta_1(t))^-\Big)}{I_1(t)+1}\Big)\Big(1+\frac{\eta_2(u)S(t^-)I_2\Big((t-\delta_2(t))^-\Big)}{I_2(t)+1}\Big)\Big]\tilde{N}(dt,du).$$

In order to estimate (59) we will use Assumption 2.1, inequalities $S(t)\le\frac{\Lambda}{\mu}$; $\frac{1}{I_1(t)+1},\frac{1}{I_2(t)+1}\le 1$ and the fact that function $f(x)=-\beta_1 x+\frac{\sigma_1^2}{2}x^2$ is increasing on $\Big[\frac{\beta_1}{\sigma_1^2},\frac{\Lambda^2}{\mu^2}\Big]$ $\Big($where $x=\frac{S(t)I_1(t-\delta_1(t))}{I_1(t)+1}\le\frac{\Lambda^2}{\mu^2}\Big)$ and $g(y)=-\beta_2 y+\frac{\sigma_2^2}{2}y^2$ is increasing on $\Big[\frac{\beta_2}{\sigma_2^2},\frac{\Lambda^2}{\mu^2}\Big]$ $\Big($where $y=\frac{S(t)I_2(t-\delta_2(t))}{I_2(t)+1}\le\frac{\Lambda^2}{\mu^2}\Big)$. Both functions reach their maximum in $\frac{\Lambda^2}{\mu^2}$. Hence,

$$\begin{aligned}dV_1(t)\le\Big[&-(\beta_1+\beta_2)\frac{\Lambda^2}{\mu^2}+2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2+\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2}+\frac{\sigma_2^2\Lambda^2}{2\mu^2}+h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2}\\&+\beta_1\frac{\Lambda}{\mu}I_1(t-\delta_1(t))+\beta_2\frac{\Lambda}{\mu}I_2(t-\delta_2(t))\Big]dt-\frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t)-\frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}dB_2(t)\\&-\int_Y\Big[\ln\Big(1+\frac{\eta_1(u)S(t^-)I_1\Big((t-\delta_1(t))^-\Big)}{I_1(t)+1}\Big)\Big(1+\frac{\eta_2(u)S(t^-)I_2\Big((t-\delta_2(t))^-\Big)}{I_2(t)+1}\Big)\Big]\tilde{N}(dt,du).\end{aligned}\tag{60}$$

Now we can define the function V_2 as

$$V_2(t):=\beta_1\frac{\Lambda}{\mu}\int_{t-\delta_1(t)}^{t}I_1(s)ds+\beta_2\frac{\Lambda}{\mu}\int_{t-\delta_2(t)}^{t}I_2(s)ds,\tag{61}$$

where (obtained as a derivative of parametric integrals and by applying Assumption 2.3)

$$\begin{aligned}dV_2(t)&=\beta_1\frac{\Lambda}{\mu}\Big(I_1(t)-\delta_1'(t)I_1(t-\delta_1(t))\Big)+\beta_2\frac{\Lambda}{\mu}\Big(I_2(t)-\delta_2'(t)I_2(t-\delta_2(t))\Big)\\&\le\beta_1\frac{\Lambda}{\mu}\Big(I_1(t)-k_1I_1(t-\delta_1(t))\Big)+\beta_2\frac{\Lambda}{\mu}\Big(I_2(t)-k_1I_2(t-\delta_2(t))\Big).\end{aligned}\tag{62}$$

According to (58) and (61) function $V=V_1+V_2$ has following form

$$V(t)=-\ln(I_1(t)+1)-\ln(I_2(t)+1)+\beta_1\frac{\Lambda}{\mu}\int_{t-\delta_1(t)}^{t}I_1(s)ds+\beta_2\frac{\Lambda}{\mu}\int_{t-\delta_2(t)}^{t}I_2(s)ds.$$

Using (60) and (62) yields to

$$\begin{aligned}dV(t)\le\Big[&-(\beta_1+\beta_2)\frac{\Lambda^2}{\mu^2}+2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2+\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2}+\frac{\sigma_2^2\Lambda^2}{2\mu^2}+h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2}+\beta_1\frac{\Lambda}{\mu}I_1(t-\delta_1(t))\\&+\beta_1\frac{\Lambda}{\mu}\Big(I_1(t)-k_1I_1(t-\delta_1(t))\Big)+\beta_2\frac{\Lambda}{\mu}I_2(t-\delta_2(t))+\beta_2\frac{\Lambda}{\mu}\Big(I_2(t)-k_1I_2(t-\delta_2(t))\Big)\Big]dt\\&-\frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t)-\frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}dB_2(t)\end{aligned}\tag{63}$$

$$
\begin{aligned}
&- \int_Y \ln\Big[\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\Big(1 + \frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\Big)\Big]\tilde{N}(dt,du)\\
\le& \Big[-(\beta_1+\beta_2)\frac{\Lambda^2}{\mu^2} + 2\mu + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2 + \Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{\sigma_2^2\Lambda^2}{2\mu^2} + h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2} + \beta_1\frac{\Lambda}{\mu}I_1(t)\\
&+ \beta_1\frac{\Lambda}{\mu}C_1\Big(I_1(t-\delta_1(t)) - I_1(t-\delta_1(t))\Big) + \beta_2\frac{\Lambda}{\mu}I_2(t) + \beta_2\frac{\Lambda}{\mu}C_1\Big(I_2(t-\delta_2(t)) - I_2(t-\delta_2(t))\Big)\Big]dt\\
&- \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) - \frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}dB_2(t)\\
&- \int_Y \ln\Big[\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\Big(1 + \frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\Big)\Big]\tilde{N}(dt,du)\\
=& \Big[-(\beta_1+\beta_2)\frac{\Lambda^2}{\mu^2} + 2\mu + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2 + \Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{\sigma_2^2\Lambda^2}{2\mu^2} + h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2} + \beta_1\frac{\Lambda}{\mu}I_1(t) + \beta_2\frac{\Lambda}{\mu}I_2(t)\Big]dt\\
&- \frac{\sigma_1 S(t)I_1(t-\delta_1(t))}{I_1(t)+1}dB_1(t) - \frac{\sigma_2 S(t)I_2(t-\delta_2(t))}{I_2(t)+1}dB_2(t)\\
&- \int_Y \ln\Big[\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\Big(1 + \frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\Big)\Big]\tilde{N}(dt,du),
\end{aligned}
$$

where $C_1 = \max\{1, k_1\}$. Integrating both sides of (63) from 0 to t and dividing with t, we obtain following

$$
\begin{aligned}
&\frac{1}{t}\Big[-\ln(I_1(t)+1) - \ln(I_2(t)+1) + \beta_1\frac{\Lambda}{\mu}\int_{t-\delta_1(t)}^t I_1(s)ds + \beta_2\frac{\Lambda}{\mu}\int_{t-\delta_2(t)}^t I_2(s)ds\\
&\quad + \ln(I_1(0)+1) + \ln(I_2(0)+1) - \beta_1\frac{\Lambda}{\mu}\int_{-\delta_1(0)}^0 I_1(s)ds - \beta_2\frac{\Lambda}{\mu}\int_{-\delta_2(0)}^0 I_2(s)ds\Big]\\
&\le \frac{1}{t}\int_0^t\Big[-(\beta_1+\beta_2)\frac{\Lambda^2}{\mu^2} + 2\mu + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2 + \Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{\sigma_2^2\Lambda^2}{2\mu^2} + h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2}\Big]ds\\
&\quad + \beta_1\frac{\Lambda}{\mu}\frac{1}{t}\int_0^t I_1(s)ds + \beta_2\frac{\Lambda}{\mu}\frac{1}{t}\int_0^t I_2(s)ds - \frac{1}{t}M_6(t) - \frac{1}{t}M_7(t) - \frac{1}{t}M_8(t)\\
&\le -(\beta_1+\beta_2)\frac{\Lambda^2}{\mu^2} + 2\mu + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2 + \Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{\sigma_2^2\Lambda^2}{2\mu^2} + h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2}\\
&\quad + \frac{\Lambda}{\mu}C_2\frac{1}{t}\int_0^t\Big(I_1(s) + I_2(s)\Big)ds - \frac{1}{t}M_6(t) - \frac{1}{t}M_7(t) - \frac{1}{t}M_8(t),
\end{aligned}
$$

where $C_2 = \max\{\beta_1, \beta_2\}$,

$$M_6(t) = \int_0^t \frac{\sigma_1 S(s)I_1(s-\delta_1(s))}{I_1(s)+1}dB_1(s)$$

and

$$M_7(t) = \int_0^t \frac{\sigma_2 S(s)I_2(s-\delta_2(s))}{I_2(s)+1}dB_2(s)$$

are continuous (local) martingales such that $M_6(0) = 0$, $M_7(0) = 0$ and

$$M_8(t) = \int_0^t \int_Y \ln\Big[\Big(1 + \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\Big(1 + \frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\Big)\Big]\tilde{N}(dt,du)$$

is local martingale such $M_8(0) = 0$. Furthermore,

$$[M_6, M_6](t) = \int_0^t \frac{\sigma_1^2 S^2(s) I_1^2(s-\delta_1(s))}{(I_1(s)+1)^2} ds \le \frac{\sigma_1^2\Lambda^4}{\mu^4} t \quad \Rightarrow \quad \limsup_{t\to\infty} \frac{[M_6,M_6](t)}{t} \le \frac{\sigma_1^2\Lambda^4}{\mu^4} < \infty,$$

$$[M_7, M_7](t) = \int_0^t \frac{\sigma_2^2 S^2(s) I_2^2(s-\delta_2(s))}{(I_2(s)+1)^2} ds \le \frac{\sigma_2^2\Lambda^4}{\mu^4} t \quad \Rightarrow \quad \limsup_{t\to\infty} \frac{[M_7,M_7](t)}{t} \le \frac{\sigma_2^2\Lambda^4}{\mu^4} < \infty,$$

$$\begin{aligned}\langle M_8, M_8\rangle(t) &= \int_0^t \int_Y \ln^2\Big[\Big(1+ \frac{\eta_1(u)S(t^-)I_1\big((t-\delta_1(t))^-\big)}{I_1(t)+1}\Big)\Big(1+ \frac{\eta_2(u)S(t^-)I_2\big((t-\delta_2(t))^-\big)}{I_2(t)+1}\Big)\Big]\lambda(du)ds \\ &\le \ln^2\Big(1+\frac{h\Lambda}{\mu}\Big)^2 \lambda(Y)t \\ &\qquad \Rightarrow \lim_{t\to\infty}\int_0^t \frac{d\langle M_8,M_8\rangle(s)}{(1+s)^2} \le \ln^2\Big(1+\frac{h\Lambda}{\mu}\Big)^2\lambda(Y) < \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\Lambda}{\mu}C_2\frac{1}{t}\int_0^t \big(I_1(s)+I_2(s)\big)ds \ge &-\frac{\ln(I_1(t)+1)}{t} - \frac{\ln(I_2(t)+1)}{t} + \frac{\ln(I_1(0)+1)}{t} + \frac{\ln(I_2(0)+1)}{t} + \beta_1\frac{\Lambda}{\mu}\frac{1}{t}\int_{t-\delta_1(t)}^t I_1(s)ds \\ &- \beta_1\frac{\Lambda}{\mu}\frac{1}{t}\int_{-\delta_1(0)}^0 I_1(s)ds + \beta_2\frac{\Lambda}{\mu}\frac{1}{t}\int_{t-\delta_2(t)}^t I_2(s)ds - \beta_2\frac{\Lambda}{\mu}\frac{1}{t}\int_{-\delta_2(0)}^0 I_2(s)ds + (\beta_1+\beta_2)\frac{\Lambda^2}{\mu^2} \\ &- (2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2) - \Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2} + \frac{\sigma_2^2\Lambda^2}{2\mu^2} + h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2} + \frac{1}{t}M_6(t) + \frac{1}{t}M_7(t) + \frac{1}{t}M_8(t).\end{aligned} \tag{64}$$

Hence, taking limes inferior of both sides of (64), using Lemma 2.4, Lemma 2.5, the fact that $-\ln(I_1(t)+1), -\ln(I_2(t)+1) \ge -\ln\frac{\Lambda}{\mu}$; $-I_1(t), -I_2(t) \ge -\frac{\Lambda}{\mu}$ and condition (50) leads to

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \big(I_1(s)+I_2(s)\big)ds &\ge \frac{(\beta_1+\beta_2)\frac{\Lambda^2}{\mu^2} - (2\mu+\varepsilon_1+\varepsilon-2+\gamma_1+\gamma_2) - \Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2}+\frac{\sigma_2^2\Lambda^2}{2\mu^2}+h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2}}{\frac{\Lambda}{\mu}C_2} \\ &= \frac{\Big(\bar{\mathcal{R}}_0^{1,2}-1\Big)\Big(2\mu+\varepsilon_1+\varepsilon_2+\gamma_1+\gamma_2+\Big(\frac{\sigma_1^2\Lambda^2}{2\mu^2}+\frac{\sigma_2^2\Lambda^2}{2\mu^2}+h^2\lambda(Y)\Big)\frac{\Lambda^2}{\mu^2}\Big)}{\frac{\Lambda}{\mu}C_2} > 0 \quad a.s.\end{aligned}$$

Therefore, we proved theorem.
□

6. Numerical simulations

In this section, we will illustrate theoretical results from Theorems 4.2 and 5.2 using numerical simulations. We used Euler-Maruyama method for stochastic differential equations with time dependent delay and jumps (for mode details see [13]). Codes for numerical simulations were written in Wolfram Mathematica.

Parameters that describe the spread of computer viruses are explained in Table 2. Their values are taken from the annual report of Kaspersky antivirus company for year 2023 [17], while some values are assumed rationally.

Symbol	Description	Range	Units	References
Λ	number of new units of computers included in the system equipped with antivirus software	$(0, \infty)$	per year	assumed
β_1	transmission rate for first virus	$(0, 1)$	per year	[17]
β_2	transmission rate for second virus	$(0, 1)$	per year	[17]
γ	recovery rate for susceptible computers due to antivirus ability of network	$(0, 1)$	per year	[17]
μ	death rate due to computer break-down	$(0, 1)$	per year	assumed
δ	transition rate from R to S	1	per year	assumed
γ_1	recovery rate for computers infected with first virus due to antivirus ability of network	$(0, 1)$	per year	[17]
γ_2	recovery rate for computers infected with second virus due to antivirus ability of network	$(0, 1)$	per year	[17]
ϵ_1	death rate caused by first virus	$(0, 1)$	per year	[17]
ϵ_2	death rate caused by second virus	$(0, 1)$	per year	assumed
σ_1	intensity of first Brownian motion	$(0, 1)$		calculated
σ_2	intensity of second Brownian motion	$(0, 1)$		calculated
η_1	intensity of Poisson jump for first virus	$(0, 1)$		assumed
η_2	intensity of Poisson jump for second virus	$(0, 1)$		assumed

Table 2. Parameters involved in stochastic model (3).

Λ	β_1	β_2	γ	μ	δ	γ_1	γ_2	ϵ_1	ϵ_2
1	0.0639	0.0457	0.08	0.2	1	0.2633	0.2508	0.000016	0.00000001
σ_1	σ_2	η_1	η_2	S_0	I_1^0	I_2^0	R_0		
0.1708	0.2808	0.012	0.015	0.7	0.2	0.1	0		

Table 3. Parameters and initial condition taken in the extinction of disease.

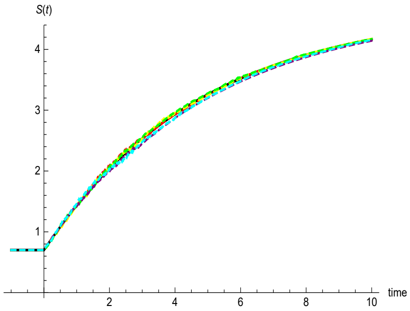
The time period is $T = 10$ years and step size is $\Delta t = \frac{1}{365}$. Delay functions are $\delta_1(t) = e^{-\lambda_1 t}$ and $\delta_2(t) = e^{-\lambda_2 t}$. For extinction we used $\lambda_1 = 7$ and $\lambda_2 = 5$. The intensity of the Poisson process is $\lambda = 2$.

In Figure 1 we illustrate the extinction results proved in Theorem 4.2 with model parameters and initial value given in Table 3. This choice of parameters satisfies the condition (28) of Theorem 4.2. Thus, we can conclude that both viruses will die out almost surely.

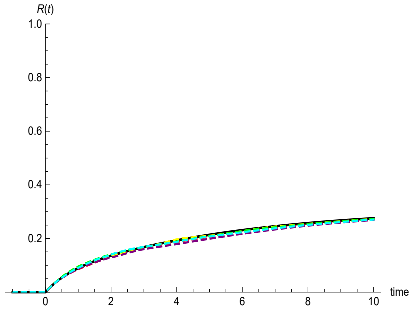
Λ	β_1	β_2	γ	μ	δ	γ_1	γ_2	ϵ_1	ϵ_2
1	0.28	0.21	0.06	0.2	1	0.28	0.26	0.000016	0.00000001
σ_1	σ_2	η_1	η_2	S_0	I_1^0	I_2^0	R_0		
0.017	0.08	0.012	0.01	0.4	0.3	0.2	0.1		

Table 4. Parameters and initial condition taken in strong persistence in mean of disease.

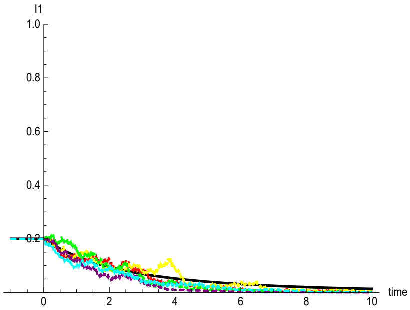
Time period is $T = 10$ years and step size is $\Delta t = \frac{1}{365}$. Delay functions are $\delta_1(t) = e^{-\lambda_1 t}$ and $\delta_2(t) = e^{-\lambda_2 t}$. For persistence in mean we used $\lambda_1 = 7$ and $\lambda_2 = 5$. Intensity of Poisson process is $\lambda = 2$.



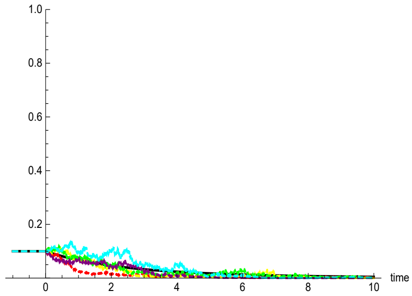
(a) S - susceptible



(b) R - recovered

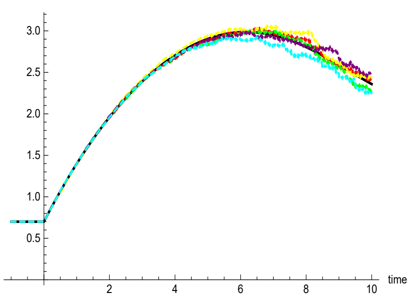


(c) I_1 - infected with first virus

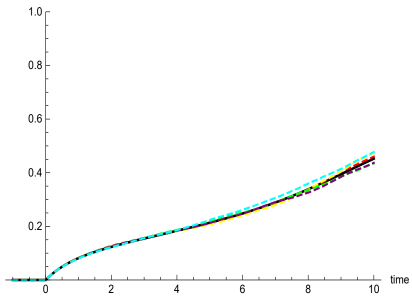


(d) I_2 - infected with second virus

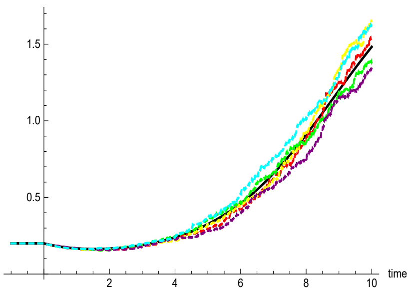
Figure 1: Extinction of disease - Deterministic (black) and five stochastic trajectories of epidemic model (3)



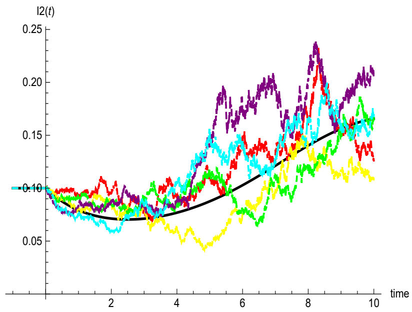
(a) S - susceptible



(b) R - recovered



(c) I_1 - infected with first virus



(d) I_2 - infected with second virus

Figure 2: Strong persistence in mean of disease - Deterministic (black) and five stochastic trajectories of epidemic model (3)

In Figure 2 we illustrate results proved in Theorem 5.2 on strong persistence in mean, with the model parameters and initial value given in Table 4. This choice of parameters satisfies the condition (50) of Theorem 5.2. Thus, we can conclude that both viruses will strongly persist in mean almost surely as stochastic trajectories fluctuate slightly around deterministic ones.

Acknowledgements

Authors express their gratitude to Editor and anonymous Referee for their constructive suggestions which led to improvement of paper.

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