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Jacobson's lemma and Cline's formula for minimal rank weak Drazin inverses

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Abstract. Minimal rank weak Drazin inverses of a square complex matrix M are solutions to the matrix equation $XM^{k+1} = M^k$ with the minimal rank, covering a large class of generalized inverses. In this paper, minimal rank weak Drazin inverses of $I_n - BA$ (resp. BA) are expressed in terms of that of $I_m - AB$ (resp. AB), where A and B are $m \times n$ and $n \times m$ complex matrices, respectively.

1. Introduction

The notion of minimal rank weak Drazin inverses was introduced by Campbell and Meyer [2] in 1978 for square complex matrices. Over the years, various generalized inverses have been introduced and studied intensively, such as core inverses [1], core-EP inverses [22], DMP inverses [15], weak group inverses [24], and weak core inverses [10]. They are all special classes of minimal rank weak Drazin inverses [25]. In this paper, Jacobson's lemma (resp. Cline's formula) for minimal rank weak Drazin inverses is established by expressing minimal rank weak Drazin inverses of $I_n - BA$ (resp. BA) in terms of that of $I_m - AB$ (resp. AB), where A and B are $m \times n$ and $n \times m$ complex matrices, respectively.

Throughout this paper, rings are associative with identity. Let I_n be the identity matrix of order n, and let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices.

A square matrix A over a ring R (i.e., entries of A are taken from R) is called Drazin invertible if there exists a matrix X over R such that

$$XA^{k+1} = A^k \text{ for some } k \geq 0, \quad AX^2 = X, \quad AX = XA. \tag{1}$$

Such an X is unique when it exists, and it is called the Drazin inverse of A and denoted by A^D. The smallest k satisfying the above condition is called the (Drazin) index of A and denoted by $\mathrm{Ind}(A)$.

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While every square complex matrix is Drazin invertible, this is not the case for square matrices over a general ring. Actually, the necessary and sufficient condition for a square matrix A over a ring to be Drazin invertible is that A is strongly π-regular, i.e., there exist matrices S and T over the ring such that $SA^{p+1} = A^p$ and $A^{q+1}T = A^q$ for some $p, q \geq 0$. In this case, $p, q \geq k$, and $SA^{k+1} = A^k = A^{k+1}T$, where $k = \mathrm{Ind}(A)$ [8, 9].

In [2], Campbell and Meyer introduced the notion of weak Drazin inverses for square complex matrices, which are more easier to compute than Drazin inverses and can be used in place of Drazin inverses in many cases. Recall that a matrix X is a weak Drazin inverse of A if $XA^{k+1} = A^k$ for some $k \geq 0$. In particular, weak Drazin inverses X satisfying $\mathrm{rank}(X) = \mathrm{rank}(A^D)$ (or equivalently, with the minimal rank) are called minimal rank weak Drazin inverses of A.

In the recent paper [25], some properties and characterizations of minimal rank weak Drazin inverses were given, and many generalized inverses introduced in the last decade were shown to be special classes of minimal rank weak Drazin inverses. According to [25, Theorem 2.1], $X \in \mathbb{C}^{n\times n}$ is a minimal rank weak Drazin inverse of $A \in \mathbb{C}^{n\times n}$ if and only if

$$XA^{k+1} = A^k \ \text{ for some } k \geq 0, \quad AX^2 = X. \tag{2}$$

Let A and B be $m \times n$ and $n \times m$ matrices over a ring, respectively. Jacobson's lemma (cf. [12, pp. 3–4]) states that if $I_m - AB$ is invertible, then so is $I_n - BA$, with

$$(I_n - BA)^{-1} = I_n + B(I_m - AB)^{-1}A. \tag{3}$$

Let a, b be two elements of a ring. In 2009, Patrício and Veloso da Costa [21] asked whether the Drazin invertibility of $1 - ab$ implies that of $1 - ba$, and if so then whether the indices of $1 - ab$ and $1 - ba$ are equal. Later, this question was answered affirmatively by Patrício and Hartwig [20] and by Cvetković-Ilić and Harte [7]. In [3], Castro-González, Mendes-Araújo and Patrício gave an explicit formula that expresses $(1 - ba)^D$ in terms of $(1 - ab)^D$:

$$(1 - ba)^D = 1 + b\Big((1 - ab)^D - (1 - ab)^{\pi}\Sigma_{i=0}^{k-1}(1 - ab)^i\Big)a, \tag{4}$$

where $(1 - ab)^{\pi} = 1 - (1 - ab)^D(1 - ab)$ is the spectral idempotent of $1 - ab$ and $k = \mathrm{Ind}(1 - ab)$. And in [13], Lam and Nielsen gave another formula to show that

$$(1 - ba)^D = \Big(1 - b(1 - ab)^{\pi}a\Big)^k + b(1 - ab)^D a. \tag{5}$$

In recent years, it has been shown that Jacobson's lemma has suitable analogues for various generalized inverses, and some generalizations of Jacobson's lemma have been made by studying relationships between generalized inverses of $I_m - AB$ and $I_n - CD$, when A, B, C, D satisfy some suitable conditions (cf. [4, 6, 14, 16–19, 26, 27]).

In Section 2 of this paper, we show that Rao and Mitra's χ-inverses are minimal rank weak Drazin inverses for complex matrices with indices no more than 1, and give a new characterization of minimal rank weak Drazin inverses. In Section 3, we study Jacobson's lemma for minimal rank weak Drazin inverses. Since such generalized inverses of a matrix generally do not commute with that matrix, the methods of [3] and [13] are no longer applicable. A new method is used to show that, for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$, if G is a minimal rank weak Drazin inverse of $I_m - AB$, then

$$(I_n - BA)^k(I_n + BGA)^{k+1} = I_n + B\Big(G - \Sigma_{i=0}^{k-1}(I_m - AB)^i[I_m - (I_m - AB)G]\Big)A$$

is a minimal rank weak Drazin inverse of $I_n - BA$, where $k = \mathrm{Ind}(I_m - AB)$. In a related direction, Cline's formula for minimal rank weak Drazin inverses is given by showing that if G is a minimal rank weak Drazin inverse of AB then BG^2A is a minimal rank weak Drazin inverse of BA.

2. Minimal rank weak Drazin inverses and χ-inverses

Recall that in case $\mathrm{Ind}(A) \leq 1$, the Drazin inverse A^D is called the group inverse of A and denoted by $A^{\#}$.

In 1971, Rao and Mitra considered a class of generalized inverses for square complex matrices. For $A \in \mathbb{C}^{n\times n}$, a matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$AXA = A \ \text{ and } \ \mathcal{R}(X) \subseteq \mathcal{R}(A), \tag{6}$$

is called a χ-inverse of A, where $\mathcal{R}(\cdot)$ denotes the range (column space); see [23, pp 15–16]. Clearly, (6) is equivalent to

$$AXA = A \ \text{ and } \ \mathcal{R}(X) = \mathcal{R}(A). \tag{7}$$

The necessary and sufficient condition for A to have χ-inverses is that $\mathrm{Ind}(A) \leq 1$, and both the group inverse and Baksalary and Trenkler's core inverse[1] are χ-inverses.

Every square complex matrix has minimal rank weak Drazin inverses, and every minimal rank weak Drazin inverse X of A satisfies

$$XAX = XA(A^kX^{k+1}) = (XA^{k+1})X^{k+1} = A^kX^{k+1} = X,$$

where $k = \mathrm{Ind}(A)$, but in general $AXA \neq A$.

The next result shows the relationship between minimal rank weak Drazin inverses and χ-inverses.

Proposition 2.1. *Let $A, X \in \mathbb{C}^{n\times n}$. Then A has X as a χ-inverse if and only if* $\mathrm{Ind}(A) \leq 1$ *and A has X as a minimal rank weak Drazin inverse.*

Proof. Suppose that X is a χ-inverse of A. Then by $\mathcal{R}(X) \subseteq \mathcal{R}(A)$ there exists $T \in \mathbb{C}^{n\times n}$ such that $X = AT$, and so by $AXA = A$, we can get $A = AXA = A^2TA$. It follows that $\mathrm{Ind}(A) \leq 1$ (hence $A^{\#}$ exists). Therefore, we obtain

$$XA^2 = ATA^2 = (A^{\#}A^2)TA^2 = A^{\#}(A^2TA)A = A^{\#}A^2 = A$$
$$\text{and } AX^2 = AX(AT) = (AXA)T = AT = X,$$

which imply that X is a minimal rank weak Drazin inverse of A.

Conversely, suppose that $\mathrm{Ind}(A) \leq 1$ and X is a minimal rank weak Drazin inverse of A. Then we have $XA^2 = A$ and $\mathrm{rank}(X) = \mathrm{rank}(A^{\#})$. Hence

$$AXA = AX(A^2A^{\#}) = A(XA^2)A^{\#} = A^2A^{\#} = A.$$

Also, since $\mathrm{rank}(A^{\#}) = \mathrm{rank}(A)$, we have $\mathrm{rank}(X) = \mathrm{rank}(A)$, which, together with $XA^2 = A$, imply that $\mathcal{R}(X) = \mathcal{R}(A)$. Therefore, X is a χ-inverse of A by the definition. □

The next result can be deduced from [25, Proposition 2.7]. Here, we give its proof for the convenience of the reader.

Lemma 2.2. *If $X \in \mathbb{C}^{n\times n}$ is a minimal rank weak Drazin inverse of $A \in \mathbb{C}^{n\times n}$, then*

$$XA^D = (A^D)^2, \tag{8}$$
$$XAA^D = A^D, \tag{9}$$
$$A^DAX = X. \tag{10}$$

Proof. Let $\mathrm{Ind}(A) = k$. Suppose that X is a minimal rank weak Drazin inverse of A. Then

$$XA^D = X[A^{k+1}(A^D)^{k+2}] = (XA^{k+1})(A^D)^{k+2} = A^k(A^D)^{k+2} = (A^D)^2,$$
$$XAA^D = XA^DA \overset{(8)}{=} (A^D)^2A = A^D,$$
$$A^DAX = A^DA(A^kX^{k+1}) = (A^DA^{k+1})X^{k+1} = A^kX^{k+1} = X.$$

□

The following result shows that minimal rank weak Drazin inverses not only have the same rank as the Drazin inverse, but are also similar to the Drazin inverse.

Proposition 2.3. *Let $X, A \in \mathbb{C}^{n\times n}$. Then X is a minimal rank weak Drazin inverse of A if and only if X is a weak Drazin inverse of A and is similar to A^D.*

Proof. The sufficiency is trivial. For the necessity, suppose that X is a minimal rank weak Drazin inverse of A, and let $P = I_n + AA^D - AX$, $Q = I_n - AA^D + AX$. Then

$$\begin{aligned} PQ &= (I_n + AA^D - AX)(I_n - AA^D + AX) \\ &= (I_n - AA^D + AX) + AA^D(I_n - AA^D + AX) - AX(I_n - AA^D + AX) \\ &= (I_n - AA^D + AX) + AA^D AX - (2AX - AXAA^D) \\ &\overset{(9),(10)}{=} (I_n - AA^D + AX) + AX - (2AX - AA^D) = I_n, \end{aligned}$$

and

$$\begin{aligned} QXP &= (I_n - AA^D + AX)[X(I_n + AA^D - AX)] \\ &= (I_n - AA^D + AX)XAA^D \\ &\overset{(9)}{=} (I_n - AA^D + AX)A^D = AXA^D \\ &\overset{(8)}{=} A(A^D)^2 = A^D, \end{aligned}$$

which completes the proof. □

3. Jacobson's lemma and Cline's formula for minimal rank weak Drazin inverses

In this section, we study Jacobson's lemma and Cline's formula for minimal rank weak Drazin inverses of complex matrices. The results are first given for matrices over an arbitrary ring, and then applied to complex matrices.

For two matrices A and B over an arbitrary ring, $m \times n$ and $n \times m$, respectively, and for any integer $p \geq 0$, keep in mind that

$$A(BA)^p = (AB)^p A, \tag{11}$$

$$A(I_n - BA)^p = (I_m - AB)^p A. \tag{12}$$

According to [7, 13], if $I_m - AB$ is Drazin invertible (say with index k), then

$$[I_n + B(I_m - AB)^D A](I_n - BA)^{k+1} = (I_n - BA)^k, \tag{13}$$

although $I_n + B(I_m - AB)^D A$ is in general not the Drazin inverse of $I_n - BA$.

The following is a slight modification of (13).

Lemma 3.1. *Let A, B be $m \times n$ and $n \times m$ matrices over a ring R, respectively. If G is an $m \times m$ matrix over R such that $G(I_m - AB)^{k+1} = (I_m - AB)^k$ for some $k \geq 0$, then*

$$(I_n + BGA)(I_n - BA)^{k+1} = (I_n - BA)^k. \tag{14}$$

Proof. Suppose that $G(I_m - AB)^{k+1} = (I_m - AB)^k$. Then

$$\begin{aligned} (I_n + BGA)(I_n - BA)^{k+1} &= (I_n - BA)^{k+1} + BGA(I_n - BA)^{k+1} \\ &= (I_n - BA)^{k+1} + BG(I_m - AB)^{k+1}A \\ &= (I_n - BA)^{k+1} + B(I_m - AB)^k A \\ &= (I_n - BA)^{k+1} + BA(I_n - BA)^k = (I_n - BA)^k. \end{aligned}$$

□

From Lemma 3.1 and its right-hand version, it follows that if $I_m - AB$ is Drazin invertible then so is $I_n - BA$, with $\mathrm{Ind}(I_m - AB) = \mathrm{Ind}(I_n - BA)$.

Lemma 3.2. *Let A, B be $m \times n$ and $n \times m$ matrices over a ring R, respectively. Then the following statements are equivalent:*

(1) *$I_m - AB$ is Drazin invertible;*
(2) *$I_n - BA$ is Drazin invertible;*
(3) *There exists an $m\times m$ matrix G over R such that $G(I_m-AB)^{k+1} = (I_m-AB)^k$ for some $k \geq 0$ and $(I_m-AB)G^2 = G$;*
(4) *There exists an $n\times n$ matrix H over R such that $H(I_n-BA)^{k+1} = (I_n-BA)^k$ for some $k \geq 0$ and $(I_n-BA)H^2 = H$.*

Proof. (1)⇔(2). It follows from the above discussion.

(1)⇒(3) and (2)⇒(4) are clear.

(3)⇒(1). Assume (3). It suffices to show that $(I_m - AB)^k = (I_m - AB)^{k+1}T$ for some matrix T over R. Since $(I_m - AB)G^2 = G$, we get $(I_m - AB)^{k+1}G^{k+2} = G$. Hence

$$\begin{aligned}(I_m - AB)^k &= G(I_m - AB)^{k+1}\\ &= (I_m - AB)^{k+1}G^{k+2}(I_m - AB)^{k+1}\\ &= (I_m - AB)^{k+1}[G^{k+1}(I_m - AB)^k],\end{aligned}$$

as desired.

(4)⇒(2) is similar to (3)⇒(1). □

We remark that the implication (3)⇒(1) above is due to Hartwig [11].

Theorem 3.3. *Let A, B be $m \times n$ and $n \times m$ matrices over a ring R, respectively. If G is an $m \times m$ matrix over R such that $G(I_m - AB)^{k+1} = (I_m - AB)^k$ for some $k \geq 0$ and $(I_m - AB)G^2 = G$, then*

$$H := (I_n - BA)^k(I_n + BGA)^{k+1}$$

satisfies $H(I_n - BA)^{k+1} = (I_n - BA)^k$ and $(I_n - BA)H^2 = H$.

Proof. Suppose that $G(I_m - AB)^{k+1} = (I_m - AB)^k$ and $(I_m - AB)G^2 = G$. Then by Lemma 3.2, $I_n - BA$ is Drazin invertible. Therefore,

$$\begin{aligned}H(I_n - BA)^{k+1} &= H(I_n - BA)^{2k+1}[(I_n - BA)^D]^k\\ &= (I_n - BA)^k[(I_n + BGA)^{k+1}(I_n - BA)^{2k+1}][(I_n - BA)^D]^k\\ &\overset{(14)}{=} (I_n - BA)^k(I_n - BA)^k[(I_n - BA)^D]^k = (I_n - BA)^k\end{aligned}$$

and

$$\begin{aligned}(I_n - BA)H^2 &= (I_n - BA)H(I_n - BA)^k(I_n + BGA)^{k+1}\\ &= (I_n - BA)H[(I_n - BA)^{k+1}(I_n - BA)^D](I_n + BGA)^{k+1}\\ &= (I_n - BA)[H(I_n - BA)^{k+1}](I_n - BA)^D(I_n + BGA)^{k+1}\\ &= (I_n - BA)^{k+1}(I_n - BA)^D(I_n + BGA)^{k+1}\\ &= (I_n - BA)^k(I_n + BGA)^{k+1} = H.\end{aligned}$$

□

For further calculations of H, the next two lemmas will be useful.

Lemma 3.4. *Let A, B be $m \times n$ and $n \times m$ matrices over a ring R, respectively. If an $m \times m$ matrix G over R satisfies $G(I_m - AB)G = G$, then for any integer $p \geq 1$,*

$$(I_n + BGA)^p = I_n + \Sigma_{i=1}^{p} BG^i A. \tag{15}$$

Proof. Since $G(I_m - AB)G = G$, we get $G(AB)G = G^2 - G$, which yields

$$GA(I_n + BGA) = GA + GABGA = GA + (G^2 - G)A = G^2 A.$$

It follows that

$$GA(I_n + BGA)^{p-1} = G^2 A(I_n + BGA)^{p-2} = \cdots = G^p A,$$

and so

$$\begin{aligned}(I_n + BGA)^p &= (I_n + BGA)^{p-1} + BGA(I_n + BGA)^{p-1}\\ &= (I_n + BGA)^{p-1} + BG^p A.\end{aligned}$$

Therefore, by induction on p, we can get (15). □

Lemma 3.5. [3] *Let A, B be $m \times n$ and $n \times m$ matrices over a ring, respectively. Then for any integer $p \geq 1$,*

$$(I_n - BA)^p = I_n - B\Sigma_{i=0}^{p-1}(I_n - AB)^i A.$$

Proof. Since

$$\begin{aligned}(I_n - BA)^p &= (I_n - BA)^{p-1} - BA(I_n - BA)^{p-1}\\ &= (I_n - BA)^{p-1} - B(I_n - AB)^{p-1}A,\end{aligned}$$

the result follows by induction on p. □

Theorem 3.6. *Let A, B be $m \times n$ and $n \times m$ matrices over a ring R, respectively. If G is an $m \times m$ matrix over R such that $G(I_m - AB)^{k+1} = (I_m - AB)^k$ for some $k \geq 0$ and $(I_m - AB)G^2 = G$, then*

$$H := (I_n - BA)^k (I_n + BGA)^{k+1} = I_n + B\Big(G - \Sigma_{i=0}^{k-1}(I_m - AB)^i[I_m - (I_m - AB)G]\Big)A$$

satisfies $H(I_n - BA)^{k+1} = (I_n - BA)^k$ and $(I_n - BA)H^2 = H$.

Proof. By Theorem 3.3, it suffices to show

$$H := (I_n - BA)^k (I_n + BGA)^{k+1} = I_n + B\Big(G - \Sigma_{i=0}^{k-1}(I_m - AB)^i[I_m - (I_m - AB)G]\Big)A.$$

When $k = 0$, we have $G(I_m - AB) = I_m$ and $(I_m - AB)G^2 = G$, which imply that

$$\begin{aligned}(I_m - AB)G &= (I_m - AB)G[G(I_m - AB)]\\ &= [(I_m - AB)G^2](I_m - AB) = G(I_m - AB) = I_m.\end{aligned}$$

So $I_m - (I_m - AB)G = 0$, and the result follows. We next assume $k \geq 1$. Because

$$\begin{aligned}G(I_m - AB)G &= G(I_m - AB)[(I_m - AB)^k G^{k+1}]\\ &= [G(I_m - AB)^{k+1}]G^{k+1} = (I_m - AB)^k G^{k+1} = G,\end{aligned}$$

using Lemma 3.4, we obtain

$$\begin{aligned} H &= (I_n - BA)^k[I_n + B(G + \cdots + G^k + G^{k+1})A] \\ &= (I_n - BA)^k + (I_n - BA)^k B(G + \cdots + G^k + G^{k+1})A \\ &= (I_n - BA)^k + B(I_m - AB)^k(G + G^2 + \cdots + G^{k+1})A. \end{aligned}$$

Since $(I_m - AB)G^2 = G$, it follows that

$$\begin{aligned} H &= (I_n - BA)^k + B\Big[(I_m - AB)^k G + (I_m - AB)^{k-1}G + \cdots + G\Big]A \\ &= (I_n - BA)^k + BGA + B\Sigma_{i=0}^{k-1}(I_m - AB)^i[(I_m - AB)G]A. \end{aligned}$$

Therefore by Lemma 3.5,

$$\begin{aligned} H &= I_n - B\Sigma_{i=0}^{k-1}(I_n - AB)^i A + BGA + B\Sigma_{i=0}^{k-1}(I_m - AB)^i[(I_m - AB)G]A \\ &= I_n + B\Big(G - \Sigma_{i=0}^{k-1}(I_m - AB)^i[I_m - (I_m - AB)G]\Big)A. \end{aligned}$$

□

Remark 3.7. *If the matrix G in the above theorem also commutes with $I_m - AB$, then $G = (I_m - AB)^D$. Taking $\hat{G} = G - \Sigma_{i=0}^{k-1}(I_m - AB)^i[I_m - (I_m - AB)G]$, we can get $H = I_n + B\hat{G}A$, $(I_m - AB)\hat{G} = \hat{G}(I_m - AB)$, and hence*

$$\begin{aligned} (I_n - BA)H &= I_n - BA + (I_n - BA)B\hat{G}A \\ &= I_n - BA + B(I_m - AB)\hat{G}A = I_n - BA + B\hat{G}(I_m - AB)A \\ &= I_n - BA + B\hat{G}A(I_n - BA) = (I_n + B\hat{G}A)(I_n - BA) = H(I_n - BA), \end{aligned}$$

which implies that $H = (I_n - BA)^D$.

But in general, $H \neq (I_n - BA)^D$. For example, let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = I_2$ and $G = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $I_2 - AB = I_2 - BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is of index 1, $G(I_2 - AB)^2 = I_2 - AB$ and $(I_2 - AB)G^2 = G$. But

$$H = (I_2 - BA)(I_2 + BGA)^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq (I_2 - BA)^D = I_2 - BA.$$

Applying Theorem 3.6 to complex matrices, we can get Jacobson's lemma for minimal rank weak Drazin inverses.

Theorem 3.8. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$.*

(1) *If $G \in \mathbb{C}^{m\times m}$ is a minimal rank weak Drazin inverse of $I_m - AB$, then*

$$(I_n - BA)^k(I_n + BGA)^{k+1} = I_n + B\Big(G - \Sigma_{i=0}^{k-1}(I_m - AB)^i[I_m - (I_m - AB)G]\Big)A$$

is a minimal rank weak Drazin inverse of $I_n - BA$, where $k = \mathrm{Ind}(I_m - AB)$.

(2) *If $G \in \mathbb{C}^{m\times m}$ is a χ-inverse of $I_m - AB$, then*

$$(I_n - BA)(I_n + BGA)^2 = I_n + B\Big(G - [I_m - (I_m - AB)G]\Big)A$$

is a χ-inverse of $I_n - BA$.

Proof. (1). It follows from Theorem 3.6 applied to $R = \mathbb{C}$.

(2). It follows from Proposition 2.1 and (1) applied to $k \leq 1$. □

Although $I_m - AB$ and $I_n - BA$ always have the same index when they are Drazin invertible, the situation is slightly different for AB and BA. Due to Cline [5], AB is Drazin invertible if and only if so is BA, and their indices differ at most by unit, and their Drazin inverses can be related by

$$(BA)^D = B[(AB)^D]^2 A. \tag{16}$$

This equation is known as Cline's formula.

In closing this paper we give the corresponding version of Cline's formula for minimal rank weak Drazin inverses.

Lemma 3.9. *Let A, B, G be $m \times n$, $n \times m$ and $m \times m$ matrices over a ring, respectively.*

(1) *If $G(AB)^{k+1} = (AB)^k$ for some $k \geq 0$, then $(BG^2A)(BA)^{k+2} = (BA)^{k+1}$.*
(2) *If $ABG^2 = G$, then $BA(BG^2A)^2 = BG^2A$.*

Proof. (1). If $G(AB)^{k+1} = (AB)^k$, then

$$(BG^2A)(BA)^{k+2} = BG^2[A(BA)^{k+2}] = BG^2(AB)^{k+2}A = B(AB)^kA = (BA)^{k+1}.$$

(2). If $ABG^2 = G$, then $BA(BG^2A)^2 = B[(ABG^2)(ABG^2)]A = BG^2A$. □

Proposition 3.10. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. If $G \in \mathbb{C}^{m\times m}$ is a minimal rank weak Drazin inverse of AB, then BG^2A is a minimal rank weak Drazin inverse of BA.*

Proof. Suppose that G is a minimal rank weak Drazin inverse of AB. Then $G(AB)^{k+1} = (AB)^k$ for some $k \geq 0$ and $ABG^2 = G$. By Lemma 3.9, we have $(BG^2A)(BA)^{k+2} = (BA)^{k+1}$ and $BA(BG^2A)^2 = BG^2A$, which imply that BG^2A is a minimal rank weak Drazin inverse of BA. □

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