Filomat 39:8 (2025), 2565–2575
https://doi.org/10.2298/FIL2508565D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on double splittings for rectangular matrices

Smrutilekha Das[a,b], Debadutta Mohanty[c], Chinmay Kumar Giri[b,*]

[a]*Department of Mathematics, Maharaja Sriram Chandra Bhanja Deo University, Odisha-757 003, India*
[b]*Department of Mathematics, Dharanidhar University, Keonjhar, Odisha-758 001, India*
[c]*Department of Mathematics, Seemanta Mahavidyalaya, Jharpokharia, Odisha-757 086, India*

Abstract. The theory of matrix splittings can be applied to derive iterative solutions for rectangular linear systems of the form $Ax = b$. Various comparison results for different subclasses of proper splittings have been proposed in the literature to enhance the convergence rate of these iterative methods. In this article, we extend the convergence theory of double proper splittings for rectangular matrices by introducing two new subclasses: double proper weak regular splitting of type II and double proper weak splitting of type II. Additionally, we present several comparison results that can be utilized to identify a more effective splitting among various options.

1. Introduction

The challenge of solving a linear system of equations

$$Ax = b,\ A \in \mathbb{R}^{m\times n},\ x \in \mathbb{R}^n,\ \text{and}\ b \in \mathbb{R}^m \tag{1}$$

is a captivating area of study within matrix theory, numerical analysis, and applied sciences. To derive numerical solutions for this system, Berman and Plemmons [2] introduced the concept of proper splitting. A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper splitting if $R(U) = R(A)$ and $N(U) = N(A)$. Here, $R(B)$ and $N(B)$ denote the range space and the null space of $B \in \mathbb{R}^{m\times n}$, respectively. Various construction techniques for proper splittings can be found in [3, 13, 20, 25]. Given $A = U - V$ as a proper splitting, Berman and Plemmons [2] proposed the iterative method

$$x^{k+1} = Tx^k + U^\dagger b,\ k = 0, 1, 2, \ldots, \tag{2}$$

where $T = U^\dagger V$ is the iteration matrix and $U^\dagger$ represents Moore-Penrose inverse of U [21]. In [2] it is established that, this iterative method converges to $A^\dagger b$ for any initial choice of x^0 if and only if the spectral radius $U^\dagger V$ is less than 1 (see Corollary 1, [2]). Therefore, the rate of convergence of the iterative method

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 65F15, 65F20.
Keywords. Spectral radius, Nonnegative matrices, Moore–Penrose inverse, Proper splitting, Double proper splitting, Convergence theorem, Comparison theorem.
Received: 07 June 2022; Accepted: 30 December 2024
Communicated by Yimin Wei
* Corresponding author: Chinmay Kumar Giri
Email addresses: smrutilekhadas.91@gmail.com (Smrutilekha Das), mohanty.deba@yahoo.com (Debadutta Mohanty), ckg2357@gmail.com (Chinmay Kumar Giri)
ORCID iDs: https://orcid.org/0009-0002-2892-0696 (Smrutilekha Das), https://orcid.org/0009-0001-1115-2945 (Debadutta Mohanty), https://orcid.org/0000-0001-8498-8076 (Chinmay Kumar Giri)

(2) depends on $\rho(U^{\dagger}V)$ and so, the spectral radius of the iteration matrix plays an important role in the comparison of the rate of convergence of different iterative methods of the same system.

In view of the above, many researchers, including Berman and Plemmons [2], Climent *et al.* [15], Jena *et al.* [18], Mishra [14], Wei *et al.* [28], Lin *et al.* [19], obtained several convergence criteria for different sub-classes of a proper splitting. Here, we mentioned a few of these classes. A proper splitting $A = U - V$ is called a

(i) proper regular splitting if $U^{\dagger} \geq 0$ and $V \geq 0$ (Definition 1.1, [18]).

(ii) proper weak regular splitting of type I if $U^{\dagger} \geq 0$ and $U^{\dagger}V \geq 0$ (Definition 1.2, [18]).

(iii) proper weak regular splitting of type II if $U^{\dagger} \geq 0$ and $VU^{\dagger} \geq 0$ (Definition 3.5, [6]).

(iv) proper weak splitting of type I if $U^{\dagger}V \geq 0$ (Definition 2, [15]).

(iv) a proper weak splitting of type II if $VU^{\dagger} \geq 0$ (Definition 2, [15]).

Numerous comparison results for these classes of splittings can be found in [4, 6–9, 14, 18, 20, 25] and the references therein, making this field rich with possibilities for exploration and advancement.

On the other hand, the method of double splitting for nonsingular matrices was first introduced by Woźnicki [30], laying the groundwork for further advancements in this area. Subsequent contributions by Shen and Huang [23], along with Miao and Zheng [24], Song and Song [16], Li and Wu [12], Li et al. [11], Wang [10], and Shekhar et al. [26, 27], have significantly enriched this theory. A notable advancement came from Jena et al. [18], who extended the concept to rectangular matrices, broadening its applicability. A splitting

$$A = P - R - S \tag{3}$$

of $A \in \mathbb{R}^{m\times n}$ is called a double proper splitting if $R(A) = R(P)$ and $N(A) = N(P)$. Building on Woźnicki's foundational work, Jena et al. [18] proposed the following iterative method:

$$x^{(k+1)} = P^{\dagger}Rx^{(k)} + P^{\dagger}Sx^{(k-1)} + P^{\dagger}b, \quad k = 1, 2, \ldots. \tag{4}$$

This can be equivalently expressed as:

$$\begin{pmatrix} x^{(k+1)} \\ x^{(k)} \end{pmatrix} = \begin{pmatrix} P^{\dagger}R & P^{\dagger}S \\ I & O \end{pmatrix} \begin{pmatrix} x^{(k)} \\ x^{(k-1)} \end{pmatrix} + \begin{pmatrix} P^{\dagger}b \\ O \end{pmatrix}, \tag{5}$$

where I is the identity matrix and O is the null matrix of appropriate order. It is well-established that the iterative method (5) converges to $A^{\dagger}b$ for any starting vectors x^0 and x^1 if and only if the spectral radius of the iteration matrix,

$$W = \begin{pmatrix} P^{\dagger}R & P^{\dagger}S \\ I & O \end{pmatrix} \tag{6}$$

is less than one, i.e., $\rho(W) < 1$. In their work, Jena et al. [18] introduced two subclasses of double proper splittings: double proper regular splitting and double proper weak regular splitting (which we refer to as double proper weak regular splitting of type I). A double proper splitting $A = P - R - S$ is called a *double proper regular splitting* if $P^{\dagger} \geq 0$, $R \geq 0$ and $S \geq 0$ and it is called a *double proper weak regular splitting of type I* if $P^{\dagger} \geq 0$, $P^{\dagger}R \geq 0$ and $P^{\dagger}S \geq 0$. The authors further demonstrated that if $A^{\dagger} \geq 0$, then the iterative method (5) associated with either the double proper regular splitting or the double proper weak regular splitting of type I converges. In 2014, Mishra [14] introduced a broader subclass known as *double proper nonnegative splitting* (which we refer to as double proper weak splitting of type I), defined by the conditions $P^{\dagger}R \geq 0$ and $P^{\dagger}S \geq 0$. Mishra [14] showed that if $A^{\dagger}P \geq 0$, then the iterative method (5) corresponding to this subclass also converges. To enhance the rate of convergence of (5), various comparison results have been established in the literature, notably in works such as [14, 17, 18]. These findings not only advance the theoretical understanding of double splitting but also provide practical insights that can significantly improve convergence rates in iterative methods.

In this article, our objective is to extend the theory of double proper slitting by introducing two new sub-classes: double proper weak regular splitting of type II and double proper weak splitting of type II. These results broaden and generalize the findings from [26].

The structure of the article is as follows: In Section 2, we introduce the notations, definitions, and some preliminary results that are essential for deriving the main outcomes. In Section 3, we prove our main results, where we investigate the convergence of the iterative method (5) related to the newly proposed sub-classes of double proper splitting. Additionally, this section includes several comparison results that assist in determining a more efficient splitting among various matrix splittings.

2. Prerequisites

Throughout the article $\mathbb{R}^{m\times n}$ represent the set of all real matrices of order $m\times n$. For any $A \in \mathbb{R}^{m\times n}$, we use A^T to denote the transpose, $R(A)$ for its range space and $N(A)$ for its null space. A matrix $A \in \mathbb{R}^{n\times n}$ is called an EP matrix, if $R(A) = R(A^T)$. Let L and M be complementary subspaces of $\mathbb{R}^n$, and let $P_{L,M}$ denote the projection onto L along M. Then, $P_{L,M}A = A$ if and only if $R(A) \subseteq L$, and $AP_{L,M} = A$ if and only if $N(A) \supseteq M$. When $L\perp M$, we write the projection as P_L. For a matrix $A \in \mathbb{R}^{n\times n}$ with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, the set of all eigenvalues is denoted by $\sigma(A)$, and the spectral radius $\rho(A)$ is defined as $\max\{| \lambda_1 |, | \lambda_2 |, \dots, | \lambda_n |\}$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be convergent if $\lim_{k\to\infty} A^k = 0$. A matrix $A \in \mathbb{R}^{n\times n}$ is convergent if and only if $\rho(A) < 1$. A matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is called non-negative if $a_{ij} \geq 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, with $a_{ij} > 0$ for at least one i, j. This is written as $A \geq 0$. Similarly, A is called positive if $a_{ij} > 0$ for every i and j. The same terms and notation apply to vectors. For matrices $A, B \in \mathbb{R}^{m\times n}$, $A \geq B$ means $A - B \geq 0$. Some fundamental results are as follows:

Lemma 2.1. (Theorem 2.1.11, [1])
Let $A \in \mathbb{R}^{n\times n}$, $A \geq 0$, $x \geq 0$, $(x \neq 0)$ and α be a positive scalar.
(i) If $\alpha x \leq Ax$, $x \geq 0$, implies $\alpha \leq \rho(A)$. Moreover, if $\alpha x < Ax$, then $\alpha < \rho(A)$.
(ii) If $Ax \leq \alpha x$, $x > 0$, implies $\rho(A) \leq \alpha$.

Theorem 2.2. (Theorem 2.20, [22])
Let $A \in \mathbb{R}^{n\times n}$ and $A \geq 0$. Then,
(i) A has a nonnegative real eigenvalue equal to its spectral radius.
(ii) To $\rho(A)$, there corresponds a nonzero eigenvector $x \geq 0$.

Lemma 2.3. (Theorem 3.15, [22])
For any $A \in \mathbb{R}^{n\times n}$, $\rho(A) < 1$ if and only if $(I - A)^{-1}$ exists and $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Theorem 2.4. (Theorem 2.21, [22])
If $A, B \in \mathbb{R}^{n\times n}$ and $A \geq B \geq 0$, then $\rho(A) \geq \rho(B)$.

Theorem 2.5. (Lemma 2.2, [23])
Let $A \in \mathbb{R}^{n\times n}$ such that $A = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix} \geq 0$ and $\rho(B + C) < 1$. Then, $\rho(A) < 1$.

For $A \in \mathbb{R}^{m\times n}$, a matrix $X \in \mathbb{R}^{n\times m}$ satisfying the four matrix equations known as Penrose equations: $AXA = A$, $XAX = X$, $(AX)^T = AX$ and $(XA)^T = XA$ is called the Moore-Penrose inverse of A. It always exists and is denoted by $A^\dagger$. (See [5, 21, 29]). Some well known properties of the Moore-Penrose inverse of $A \in \mathbb{R}^{m\times n}$ are: $R(A^\dagger) = R(A^T)$; $N(A^\dagger) = N(A^T)$; $AA^\dagger = P_{R(A)}$; $A^\dagger A = P_{R(A^T)}$. In particular, if $x \in R(A)$, then $x = A^\dagger Ax$. (For more details, see [5]). $A \in \mathbb{R}^{m\times n}$ is called semimonotone if $A^\dagger \geq 0$. Recall that the iterative method (2) is convergent if $\rho(H) < 1$. In this connection, we have the following results.

Theorem 2.6. (Theorem 3.7, [6])
Let $A = U - V$ be a proper weak regular splitting of type II. Then, $A^\dagger \geq 0$ if and only if $\rho(U^\dagger V) < 1$.

Theorem 2.7. (Lemma 3.2, [7])
Let $A = U - V$ be a proper weak splitting of type II. If $UA^\dagger \geq 0$, then $\rho(U^\dagger V) < 1$.

3. Main Results

We begin this section with the following examples, which serve as motivation for our work.

Example 3.1. *Let* $A = \begin{pmatrix} -12 & -2 \\ 6 & 5 \\ -6 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 5 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 13 & -1 \\ 3 & 0 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 10 & 0 \\ 5 & 0 \end{pmatrix} = P - R - S$. *Then $A - P - R - S$ is a double proper splitting of A. Also,* $P^\dagger = \begin{pmatrix} 0.3490 & 0.0134 & 0.3154 \\ 0.0134 & 0.1544 & 0.1275 \end{pmatrix} \geq 0$, $RP^\dagger = \begin{pmatrix} 4.5235 & 0.0201 & 3.9732 \\ 1.0470 & 0.0403 & 0.9463 \\ 0.6980 & 0.0268 & 0.6309 \end{pmatrix} \geq 0$ *and* $SP^\dagger = \begin{pmatrix} 0.3624 & 0.1678 & 0.4430 \\ 3.4899 & 0.1342 & 3.1544 \\ 1.7450 & 0.0671 & 1.5772 \end{pmatrix} \geq 0$, *but* $P^\dagger R = \begin{pmatrix} 5.2081 & -0.3490 \\ 0.8926 & -0.0134 \end{pmatrix} \ngeq 0$. *Hence, $A = P - R - S$ is not a double proper weak regular splitting of type I.*

Example 3.2. *Let* $A = \begin{pmatrix} 2 & 1 \\ -1 & 2.5 \\ 1 & 1.5 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -1 & 5 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1.5 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0.5 \end{pmatrix} = P - R - S$. *Then $A - P - R - S$ is a double proper splitting. On the other hand,* $P^\dagger = \begin{pmatrix} 0.3214 & -0.2143 & 0.1429 \\ 0.0357 & 0.1429 & 0.0714 \end{pmatrix} \ngeq 0$, $RP^\dagger = \begin{pmatrix} 0.0357 & 0.1429 & 0.0714 \\ 0.0536 & 0.2143 & 0.1071 \\ 0.0357 & 0.1429 & 0.0714 \end{pmatrix} \geq 0$ *and* $SP^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0.0357 & 0.1429 & 0.0714 \\ 0.0179 & 0.0714 & 0.0357 \end{pmatrix} \geq 0$, *but* $P^\dagger S = \begin{pmatrix} 0 & -0.1429 \\ 0 & 0.1786 \end{pmatrix} \ngeq 0$. *Hence, $A = P - R - S$ is neither double proper weak regular of type I nor a double proper weak splitting of type I.*

From the above examples one can notice that, the theories provided in [4, 14, 17, 18] are not adequate to ensure the convergence of all type splittings. To overcome with this issues, we now introduced the following class of a double proper splitting.

Definition 3.3. *A double proper splitting $A = P - R - S$ is called a double proper weak regular splitting of type II if $P^\dagger \geq 0$, $RP^\dagger \geq 0$ and $SP^\dagger \geq 0$.*

Definition 3.4. *A double proper splitting $A = P - R - S$ is called a double proper weak splitting of type II if $RP^\dagger \geq 0$ and $SP^\dagger \geq 0$.*

Remark 3.5. *A proper weak regular splitting of type II is a proper weak splitting of type II, but not conversely.*

By rewriting (3) as

$$A = P - (R + S), \tag{7}$$

we derive the following iterative method:

$$x^{k+1} = Hx^k + P^\dagger b, \ k = 0, 1, 2, \ldots, \tag{8}$$

where $H = P^\dagger(R + S)$ serves as the iteration matrix. If the splitting (3) is a double proper weak splitting of type II, then the matrices $\widetilde{H} = (R + S)P^\dagger$ and

$$\widetilde{W} = \begin{pmatrix} RP^\dagger & SP^\dagger \\ I & O \end{pmatrix} \tag{9}$$

are nonnegative. We leverage the nonnegativity and convergence properties of $\widetilde{W}$ to analyze the convergence of the iteration matrix W. To proceed, we will begin with the following Lemma.

Lemma 3.6. *The matrices W and $\widetilde{W}$ defined in* (6) *and* (9)*, respectively, have the same spectral radius.*

Proof. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector of $\widetilde{W}$ corresponding to a non-zero eigenvalue λ. Then it follows that $\widetilde{W}x = \lambda x$, which can be expressed as

$$\begin{pmatrix} RP^\dagger & SP^\dagger \\ I & O \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

From this, we derive the following equations:

$$RP^\dagger x_1 + SP^\dagger x_2 = \lambda x_1 \tag{10}$$

$$x_1 = \lambda x_2. \tag{11}$$

By pre-multiplying (10) and (11) by $P^\dagger$, we obtain

$$\begin{aligned} P^\dagger R(P^\dagger x_1) + P^\dagger S(P^\dagger x_2) &= \lambda P^\dagger x_1 \\ P^\dagger x_1 &= \lambda P^\dagger x_2. \end{aligned}$$

Letting $y_1 = P^\dagger x_1$ and $y_2 = P^\dagger x_2$, we can reformulate the system as:

$$\begin{aligned} P^\dagger R y_1 + P^\dagger S y_2 &= \lambda y_1 \\ y_1 &= \lambda y_2. \end{aligned}$$

This can be expressed in matrix form as

$$Wy = \lambda y,$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Thus, λ is an eigenvelue of W if $y \neq 0$. Suppose $y = 0$. Then we have $P^\dagger x_1 = 0$ and $P^\dagger x_2 = 0$. From equation (10), we conclude that $\lambda x_1 = 0$. Since $\lambda \neq 0$, it follows that $x_1 = 0$, which by equation (11) implies $x_2 = 0$ as well. This leads to a contradiction, as x is an eigenvector. Therefore, we conclude that $y \neq 0$. Consequently, we have $\sigma(\widetilde{W})\backslash\{0\} \subseteq \sigma(W)\backslash\{0\}$.

Conversely, let μ be a nonzero eigenvalue of W, and let $x^T = \left(x_1^T, x_2^T\right)$ be an be an associated eigenvector. This can be expressed as

$$\left(x_1^T, x_2^T\right)\begin{pmatrix} P^\dagger R & P^\dagger S \\ I & O \end{pmatrix} = \mu \left(x_1^T, x_2^T\right).$$

This yields the following equations:

$$x_1^T P^\dagger R + x_2^T = \mu x_1^T \tag{12}$$

$$x_1^T P^\dagger S = \mu x_2^T. \tag{13}$$

By post-multiplying equations (12) and (13) by $P^\dagger$, we obtain

$$\begin{aligned} x_1^T P^\dagger R P^\dagger + x_2^T P^\dagger &= \mu x_1^T P^\dagger \\ x_1^T P^\dagger S P^\dagger &= \mu x_2^T P^\dagger. \end{aligned}$$

By defining $z_1^T = x_1^T P^\dagger$ and $z_2^T = x_2^T P^\dagger$, we can reformulate the system as:

$$\begin{aligned} z_1^T R P^\dagger + z_2^T &= \mu z_1^T \\ z_1^T S P^\dagger &= \mu z_2^T, \end{aligned}$$

This leads to the equation $z^T \widetilde{W} = \mu z^T$, where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Assuming $z = 0$ implies that $x_1^T P^\dagger = 0$ and $x_2^T P^\dagger = 0$. From equation (13), we have $\mu x_2^T = 0$. Since $\mu \neq 0$, it follows that $x_2^T = 0$, which subsequently leads to $x_1^T = 0$ from (12), resulting in a contradiction.Thus, $z \neq 0$. Therefore, we conclude that W and $\widetilde{W}$ have the same set of non-zero eigenvalues, and consequently, the same spectral radius. □

The following result guarantees that the single and double iterative schemes converges simultaneously in the case of double proper weak splittings of type II.

Theorem 3.7. *Let the splitting defined by (3) be a double proper weak splitting of type II. Then the iterative method (5) is convergent if and only if the iterative method (8) is convergent.*

Proof. Assume that the iterative method (5) is convergent, implying that the spectral radius satisfies $\rho(\widetilde{W}) = \rho(W) < 1$. By Lemma 2.3, $(I - \widetilde{W})^{-1}$ exists and $(I - \widetilde{W})^{-1} \geq 0$. Through block matrix computations, we obtain the following expression:

$$(I - \widetilde{W})^{-1} = \begin{pmatrix} [I - (R+S)P^\dagger]^{-1} & [I - (R+S)P^\dagger]^{-1} S P^\dagger \\ [I - (R+S)P^\dagger]^{-1} & [I - (R+S)P^\dagger]^{-1}(I - RP^\dagger) \end{pmatrix}. \tag{14}$$

Since $(I - \widetilde{W})^{-1} \geq 0$, we deduce that $[I - (R+S)P^\dagger]^{-1} \geq 0$. By applying Lemma 2.3, it follows that $\rho((R+S)P^\dagger) = \rho(P^\dagger(R+S)) < 1$.

Conversely, assume that the iterative method (8) is convergent, meaning that $\rho(P^\dagger(R+S)) = \rho((R+S)P^\dagger) < 1$. By applying Theorem 2.5 and Lemma 3.6 together, we can conclude that $\rho(\widetilde{W}) = \rho(W) < 1$. □

If we set $U = P$ and $V = R + S$, then by applying Theorem 2.6 and Theorem 3.7, we obtain the following characterization of a semimonotone matrix via a double proper weak regular splitting of type II.

Theorem 3.8. *Let the splitting defined by (3) be double proper weak regular splitting of type II. Then, $A^\dagger \geq 0$ if and only if $\rho(W) < 1$.*

From the above result, it follows that when the splitting is either double proper regular or weak regular (of either type), the condition $A^\dagger \geq 0$ serves as an equivalent criterion for convergence. However, in the case of a double proper weak splitting of type II, this condition is not an equivalent criterion for convergence, as demonstrated by the following example.

Example 3.9. *Let* $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \\ 2 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 0.5 \\ 1 & 0.5 \\ 1 & 0.5 \end{pmatrix} - \begin{pmatrix} 0 & -1.5 \\ 0 & -1.5 \\ 0 & -1.5 \end{pmatrix} = P - R - S$ *be a proper splitting. Then,* $A^\dagger = \begin{pmatrix} 0.1667 & 0.1667 & 0.1667 \\ -0.1667 & -0.1667 & -0.1667 \end{pmatrix} \ngeq 0$, $P^\dagger = \begin{pmatrix} 0.0833 & 0.0833 & 0.0833 \\ -0.0833 & -0.0833 & -0.0833 \end{pmatrix} \ngeq 0$, $RP^\dagger = \begin{pmatrix} 0.0416 & 0.0416 & 0.0416 \\ 0.0416 & 0.0416 & 0.0416 \\ 0.0416 & 0.0416 & 0.0416 \end{pmatrix} \geq 0$ *and* $SP^\dagger = \begin{pmatrix} 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 \end{pmatrix} \geq 0$. *Hence,* $A = P - R - S$ *is a double proper weak splitting of type II. However,* $\rho(W) = 0.6781 < 1$.

In the this example, we also observe that $PA^{\dagger} = \begin{pmatrix} 0.6667 & 0.6667 & 0.6667 \\ 0.6667 & 0.6667 & 0.6667 \end{pmatrix} \geq 0$. This observation leads us to formulate another result that provides several equivalent criteria for the convergence of double proper weak splittings of type II, which follows as a consequence of Theorem 2.7 and Theorem 3.7.

Theorem 3.10. *Let the splitting defined by (3) be double proper weak splitting of type II. Then the following conditions are equivalent:*
(a) $\rho(W) < 1$.
(b) $\rho(P^{\dagger}(R+S)) = \rho((R+S)P^{\dagger}) < 1$.
(c) $PA^{\dagger} \geq 0$.
(d) $[I-(R+S)P^{\dagger}]^{-1} \geq 0$.

3.1. Comparison Theorems

Comparison theorems are essential for identifying optimal splittings that minimize the spectral radius, leading to accelerated convergence of the associated iterative schemes. Additionally, these theorems play a crucial role in evaluating the effectiveness of preconditioners. In this section, we will establish several important comparison results.

First, we present a comparison between the splittings defined by (3) and (7), assuming that both are convergent, with the former representing a double proper weak splitting of type II and the latter a proper weak splitting of type II.

Theorem 3.11. *If the splitting defined in (3) is a convergent double proper weak splitting of type II, then* $\rho(H) \leq \rho(W) < 1$.

Proof. From (14), we have

$$\begin{aligned}(I-\widetilde{W})^{-1} &= \begin{pmatrix} [I-(R+S)P^{\dagger}]^{-1} & [I-(R+S)P^{\dagger}]^{-1}SP^{\dagger} \\ [I-(R+S)P^{\dagger}]^{-1} & [I-(R+S)P^{\dagger}]^{-1}(I-RP^{\dagger}) \end{pmatrix} \\ &\geq \begin{pmatrix} [I-(R+S)P^{\dagger}]^{-1} & 0 \\ 0 & I \end{pmatrix}\end{aligned}$$

which implies $\rho((I-\widetilde{W})^{-1}) \geq \rho([I-(R+S)P^{\dagger}]^{-1})$ and hence $\rho(\widetilde{W}) = \rho(W) \geq \rho((R+S)P^{\dagger}) = \rho(H)$. □

For a double proper weak regular splitting of type II, we have the following result.

Corollary 3.12. *If the splitting defined in (3) is double proper weak regular splitting of type II and* $A^{\dagger} \geq 0$, *then* $\rho(H) \leq \rho(W) < 1$.

Let

$$A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2 \tag{15}$$

be two double proper splittings of $A \in \mathbb{R}^{m\times n}$ with the corresponding iteration matrices defined as

$$W_1 = \begin{pmatrix} P_1^{\dagger}R_1 & P_1^{\dagger}S_1 \\ I & 0 \end{pmatrix},\ W_2 = \begin{pmatrix} P_2^{\dagger}R_2 & P_2^{\dagger}S_2 \\ I & 0 \end{pmatrix}.$$

Our objective is to find such a matrix W that simplifies the computation and minimizes the spectral radius of W (for a convergent scheme), ensuring a faster convergence rate for the scheme (5). Several comparison results have been developed in [4, 14, 17] for double proper regular and double proper weak regular splittings of type I. In this work, we extend the findings of Shekhar *et al.* [26], by deriving additional comparison results for matrices that admit two double proper weak splittings of type II.

Let

$$\widetilde{W}_1 = \begin{pmatrix} R_1P_1^{\dagger} & S_1P_1^{\dagger} \\ I & 0 \end{pmatrix} \text{ and } \widetilde{W}_2 = \begin{pmatrix} R_2P_2^{\dagger} & S_2P_2^{\dagger} \\ I & 0 \end{pmatrix}.$$

We now present the following comparison result.

Theorem 3.13. *Let the splittings defined by (15) be convergent double proper weak splittings of type II. If* $AP_1^\dagger \geq AP_2^\dagger$ *and one of the following conditions holds:*
(1) $R_1P_1^\dagger \geq R_2P_2^\dagger$, *or*
(2) $S_1P_1^\dagger \leq S_2P_2^\dagger$,
then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. If $\rho(W_1) = 0$, the conclusion holds trivially. Now suppose $\rho(W_1) \neq 0$. By the definition of $\widetilde{W}_1$, we have $\widetilde{W}_1 \geq 0$. Hence by Theorem 2.2, there exists a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0$ such that $\widetilde{W}_1 x = \rho(W_1)x$, which gives the system of equations:

$$\begin{aligned} R_1P_1^\dagger x_1 + S_1P_1^\dagger x_2 &= \rho(W_1)x_1 \\ x_1 &= \rho(W_1)x_2. \end{aligned}$$

Then, it follows that

$$\begin{aligned} \widetilde{W}_2 x - \rho(W_1)x &= \begin{pmatrix} R_2P_2^\dagger x_1 + S_2P_2^\dagger x_2 - \rho(W_1)x_1 \\ x_1 - \rho(W_1)x_2 \end{pmatrix} \\ &= \begin{pmatrix} R_2P_2^\dagger x_1 + S_2P_2^\dagger x_2 - R_1P_1^\dagger x_1 - S_1P_1^\dagger x_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (R_2P_2^\dagger - R_1P_1^\dagger)x_1 + \frac{1}{\rho(W_1)}(S_2P_2^\dagger - S_1P_1^\dagger)x_1 \\ 0 \end{pmatrix} \\ &:= \begin{pmatrix} \nabla \\ 0 \end{pmatrix}, \end{aligned}$$

where $\nabla = (R_2P_2^\dagger - R_1P_1^\dagger)x_1 + \frac{1}{\rho(W_1)}(S_2P_2^\dagger - S_1P_1^\dagger)x_1$. Suppose that the first condition $R_1P_1^\dagger \geq R_2P_2^\dagger$ holds. Then

$$\nabla - \frac{1}{\rho(W_1)}(R_2P_2^\dagger - R_1P_1^\dagger)x_1 - \frac{1}{\rho(W_1)}(S_2P_2^\dagger - S_1P_1^\dagger)x_1 = (1 - \frac{1}{\rho(W_1)})(R_2P_2^\dagger - R_1P_1^\dagger)x_1 \geq 0.$$

On the other hand, utilizing the fact that $P_2P_2^\dagger = P_1P_1^\dagger$, we obtain

$$\begin{aligned} \nabla &\geq \frac{1}{\rho(W_1)}\left((R_2P_2^\dagger - R_1P_1^\dagger)x_1 + (S_2P_2^\dagger - S_1P_1^\dagger)x_1\right) \\ &= \frac{1}{\rho(W_1)}\left((R_2P_2^\dagger + S_2P_2^\dagger)x_1 - (R_1P_1^\dagger + S_1P_1^\dagger)x_1\right) \\ &= \frac{1}{\rho(W_1)}\left((R_2 + S_2)P_2^\dagger x_1 - (R_1 + S_1)P_1^\dagger)x_1\right) \\ &= \frac{1}{\rho(W_1)}\left((P_2 - A)P_2^\dagger x_1 - (P_1 - A)P_1^\dagger x_1\right) \\ &= \frac{1}{\rho(W_1)}\left(AP_1^\dagger - AP_2^\dagger\right)x_1 \geq 0. \end{aligned}$$

Therefore, we have $\widetilde{W}_2 x - \rho(W_1)x \geq 0$. Consequently, by applying Lemma 2.1 and Lemma 3.6, we arrive at the desired conclusion. Similarly, if $S_1P_1^\dagger \leq S_2P_2^\dagger$, then

$$\nabla - (R_2P_2^\dagger - R_1P_1^\dagger)x_1 - (S_2P_2^\dagger - S_1P_1^\dagger)x_1 = (\frac{1}{\rho(W_1)} - 1)(S_2P_2^\dagger - S_1P_1^\dagger)x_1 \geq 0.$$

Consequently, we have:

$$\nabla \geq (R_2P_2^\dagger - R_1P_1^\dagger)x_1 + (S_2P_2^\dagger - S_1P_1^\dagger)x_1 = (AP_1^\dagger - AP_2^\dagger)x_1 \geq 0,$$

which leads to $\widetilde{W}_2x - \rho(W_1)x \geq 0$. Thus, by applying Lemma 2.1 and Lemma 3.6, we conclude that $\rho(W_1) \leq \rho(W_2)$. □

It can be observed that the inequality $S_1P_1^\dagger - S_2P_2^\dagger \leq R_2P_2^\dagger - R_1^\dagger P_2$ implies that $(S_1 + R_1)P_1^\dagger \leq (S_2 + R_2)P_2^\dagger$, which translates to$AP_1^\dagger \geq AP_2^\dagger$. Therefore, we can derive the following result.

Theorem 3.14. *Let the splittings defined by (15) be two convergent double weak splittings of type II. If* $S_1P_1^\dagger - S_2P_2^\dagger \leq R_2P_2^\dagger - R_1P_1^\dagger$ *and one of the following conditions*
(1) $R_1P_1^\dagger \geq R_2P_2^\dagger$
(2) $S_1P_1^\dagger \leq S_2P_2^\dagger$
holds, then $\rho(W_1) \leq \rho(W_2) < 1$.

Similarly, if the inequalities $AP_1^\dagger \geq 0$ and $P_1P_2^\dagger \leq I$ hold, then pre-multiplying both sides by $AP_1^\dagger$ yields $AP_2^\dagger \leq AP_1^\dagger$.Furthermore, if $AP_2^\dagger \geq 0$ and $P_2P_1^\dagger \geq I$ are satisfied, we can conclude that $AP_1^\dagger \geq AP_2^\dagger$. Therefore, Theorem 3.13 can be reformulated with a different set of conditions, which will be presented next.

Theorem 3.15. *Let the splittings defined by (15) be two convergent double weak splittings of type II. If* $P_1P_2^\dagger \leq I$ *and* $AP_1^\dagger \geq 0$ *or* $P_2P_1^\dagger \geq I$ *and* $AP_2^\dagger \geq 0$ *, and one of the following conditions holds:*
(1) $R_1P_1^\dagger \geq R_2P_2^\dagger$, *or*
(2) $S_1P_1^\dagger \leq S_2P_2^\dagger$,
then $\rho(W_1) \leq \rho(W_2) < 1$.

By applying Theorem 2.4 in conjunction with Lemma 3.6, the following result can be proved.

Theorem 3.16. *Let the splittings defined by (15) be two convergent double proper weak splittings of type II. If* $R_1P_1^\dagger \leq R_2P_2^\dagger$ *and* $S_1P_1^\dagger \leq S_2P_2^\dagger$, *then* $\rho(W_1) \leq \rho(W_2) < 1$.

The converse of the above results is not necessarily true, as demonstrated by the following example.

Example 3.17. *Let* $A = \begin{pmatrix} 2 & 3 \\ -1 & 7.5 \\ 1 & 4.5 \end{pmatrix} = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ *be two double proper splittings, where* $P_1 = \begin{pmatrix} 2 & 4.5 \\ -1 & 11.25 \\ 1 & 6.75 \end{pmatrix}$, $R_1 = \begin{pmatrix} 0 & 0.5 \\ 0 & 2 \\ 0 & 1.25 \end{pmatrix}$, $S_1 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 2.4 & 6 \\ -1.2 & 15 \\ 1.2 & 9 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0.4 & 3 \\ -0.2 & 7.5 \\ 0.2 & 3.5 \end{pmatrix}$, $S_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Then,* $R_1P_1^\dagger = \begin{pmatrix} 0.0079 & 0.0317 & 0.0159 \\ 0.0317 & 0.1270 & 0.0635 \\ 0.0198 & 0.0794 & 0.0397 \end{pmatrix} \geq 0$, $S_1P_1^\dagger = \begin{pmatrix} 0.0159 & 0.0635 & 0.0317 \\ 0.0317 & 0.1270 & 0.0635 \\ 0.0159 & 0.0635 & 0.0317 \end{pmatrix} \geq 0$, $R_2P_2^\dagger = \begin{pmatrix} 0.1429 & 0.0714 & 0.1190 \\ 0.0357 & 0.3929 & 0.1548 \\ 0.0952 & 0.1310 & 0.1071 \end{pmatrix} \geq 0$ and $S_2P_2^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0119 & 0.0476 & 0.0238 \end{pmatrix} \geq 0$. *Thus, the splittings* $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ *are double proper weak splittings of type II. We observe that* $\rho(W_1) = 0.5142 \leq 0.5218 = \rho(W_2) < 1$, *but* $S_1P_1^\dagger \nleq S_2P_2^\dagger$.

We end this section with the following result which compares the rate of convergence of type I and type II double proper splittings of a singular matrix A.

Theorem 3.18. *Let the splittings defined by (15) be two convergent double proper weak splittings of type I and type II, respectively, of a semimonotone and EP matrix* $A \in \mathbb{R}^{n\times n}$. *If* $P_1^\dagger A \geq AP_2^\dagger$ *and one of the following conditions holds*
(1) $P_1^\dagger R_1 \geq R_2P_2^\dagger$, *or*
(2) $P_1^\dagger S_1 \leq S_2P_2^\dagger$,
then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. Since $W_1 \geq 0$, by Theorem 2.2, there exists a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0$ such that $W_1 x = \rho(W_1)x$. This leads to the following system of equations:

$$\begin{aligned} P_1^\dagger R_1 x_1 + P_1^\dagger S_1 x_2 &= \rho(W_1)x_1 \\ x_1 &= \rho(W_1)x_2. \end{aligned}$$

Then, it follows that

$$\begin{aligned} \widetilde{W}_2 x - \rho(W_1)x &= \begin{pmatrix} R_2P_2^\dagger x_1 + S_2P_2^\dagger x_2 - \rho(W_1)x_1 \\ x_1 - \rho(W_1)x_2 \end{pmatrix} \\ &= \begin{pmatrix} R_2P_2^\dagger x_1 + S_2P_2^\dagger x_2 - P_1^\dagger R_1 x_1 - P_1^\dagger S_1 x_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (R_2P_2^\dagger - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(S_2P_2^\dagger - P_1^\dagger S_1)x_1 \\ 0 \end{pmatrix} \\ &:= \begin{pmatrix} \nabla \\ 0 \end{pmatrix}, \end{aligned}$$

where $\nabla = (R_2P_2^\dagger - P_1^\dagger R_1)x_1 + \frac{1}{\rho(W_1)}(S_2P_2^\dagger - P_1^\dagger S_1)x_1$.

If $P_1^\dagger R_1 \geq R_2P_2^\dagger$ holds, then we have

$$\nabla - \frac{1}{\rho(W_1)}(R_2P_2^\dagger - P_1^\dagger R_1)x_1 - \frac{1}{\rho(W_1)}(S_2P_2^\dagger - P_1^\dagger S_1)x_1 = (1 - \frac{1}{\rho(W_1)})(R_2P_2^\dagger - P_1^\dagger R_1)x_1 \geq 0.$$

Furthermore, since P_1 and P_2 are EP matrices, we obtain the following expression:

$$\begin{aligned} \nabla &\geq \frac{1}{\rho(W_1)}\Big((R_2P_2^\dagger - P_1^\dagger R_1)x_1 + (S_2P_2^\dagger - P_1^\dagger S_1)x_1\Big) \\ &= \frac{1}{\rho(W_1)}\Big((R_2P_2^\dagger + S_2P_2^\dagger)x_1 - (P_1^\dagger R_1 + P_1^\dagger S_1)x_1\Big) \\ &= \frac{1}{\rho(W_1)}\Big((R_2 + S_2)P_2^\dagger x_1 - P_1^\dagger(R_1 + S_1)x_1\Big) \\ &= \frac{1}{\rho(W_1)}\Big((P_2 - A)P_2^\dagger x_1 - P_1^\dagger(P_1 - A)x_1\Big) \\ &= \frac{1}{\rho(W_1)}\Big((P_1^\dagger A - AP_2^\dagger)x_1\Big) \geq 0. \end{aligned}$$

Consequently, we have $\widetilde{W}_2 x - \rho(W_1)x \geq 0$. By Lemma 2.1, it follows that $\rho(W_1) \leq \rho(\widetilde{W}_2) = \rho(W_2)$. Similarly, if the inequality $P_1^\dagger S_1 \leq S_2P_2^\dagger$ holds, then we also conclude that $\rho(W_1) \leq \rho(W_2)$. □

4. Conclusion

The principal contribution of this paper is the establishment of a theoretical framework that allows researchers to focus less on the specific type of double splitting encountered during computations. We propose two new subclasses of double splittings and derive various convergence criteria for the iterative method (4). The significance of these subclasses is explored, and we present convergence results that validate their introduction, thereby broadening the applicability of the double iteration scheme (4). Additionally, we provide several comparison results that guide the selection of the most effective splitting when multiple options of the same type are available.

Acknowledgements

The authors wish to express their sincere gratitude to the anonymous reviewers for their thoughtful comments and constructive suggestions, which have substantially improved the quality of this manuscript.

References

[1] A. Berman and R.J. Plemmons, *Nonnegative matrices in the mathematical Sciences*, Classics in Applied Mathematics, SIAM, Philadelphia, 1994.

[2] A. Berman, J. Plemmons, *Cones and iterative methods for best least squares solutions of linear systems*, SIAM J. Numer. Anal. **11** (1974), 145-154.

[3] A. Berman, M. Neumann, *Proper splittings of rectangular matrices*, SIAM J. Appl. Math. **31** (1976), 307–312.

[4] A. K. Baliarsingh, D. Mishra, *Comparison results for proper nonnegative splittings of matrices*, Results. Math. **71** (2017), 93–109.

[5] A. Ben-Israel, T.N.E. Greville, *Generalized Inverses. Theory and Applications*, (2nd edition) Springer-Verlag, New York, 2003.

[6] C. K. Giri, D. Mishra, *Additional results on convergence of alternating iterations involving rectangular matrices*, Numer. Funct. Anal. Optim. **38** (2017), 160–180.

[7] C. K. Giri, D. Mishra, *Some comparison theorems for proper weak splittings of type II*, J. Anal. **25** (2017), 267–279.

[8] C. K. Giri, D. Mishra, *More on convergence theory of proper multisplittings*, Khayyam J. Math. **4** (2018), 144–154.

[9] C. K. Giri, D. Mishra, *Comparison results for proper multisplittings of rectangular matrices*, Adv. Oper. Theory **2** (2017), 334–352.

[10] C. Wang, *Comparison results for K-nonnegative double splittings of K-monotone matrices*, Calcolo, **54** (2017), 1293–1303.

[11] C. X. Li, Q. F. Cui, S. L. Wu, *Comparison theorems for single and double splittings of matrices*, J. Appl. Math. Volume 2013, Article ID 827826, 4 pages.

[12] C. X. Li, S. H. Li, *Comparison theorems of spectral radius for splittings of matrices*, J. Appl. Math. Volume 2014, Article ID 573024, 5 pages.

[13] D. Mishra, K. C. Sivakumar, *On splitting of matrices and nonnegative generalized inverses*, Oper. Matrices **6** (2012), 85–95.

[14] D. Mishra, *Nonnegative splittings for rectangular matrices*, Comput. Math. Appl. **67** (2014), 136–144.

[15] J.-J. Climent, A. Devesa, C. Carmen, *Convergence results for proper splittings*, Recent Advances in Applied and Theoretical Mathematics,1971–1976, Math. Comput. Sci. Eng., World Sci. Eng. Soc. Press, Athens, 2000.

[16] J. Song, Y. Song, *Convergence for nonnegative double splittings of matrices*, Calcolo, **48** (2011), 245–260.

[17] K. Appi Reddy, T. Kurmayya, *Comparison results for proper double Splittings of rectangular matrices*, Filomat, **32** (2018), 2273—2281.

[18] L. Jena, D. Mishra, S. Pani, *Convergence and comparison theorems for single and double decomposition of rectangular matrices*, Calcolo, **51** (2014), 141—149.

[19] L. Lin, Y. Wei, C.-W. Woo, J. Zhou, *On the convergence of splittings for semidefinite linear systems*, Linear Algebra Appl. **429** (2008) 2555–2566.

[20] N. Mishra, D. Mishra, *Two-stage iterations based on composite splittings for rectangular linear systems*, Comput. Math. Appl. **75** (2018), 2746–2756.

[21] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406–413.

[22] R. S. Varga, *Matrix Iterative Analysis* (2nd Revised and Expanded Edition), Springer-Verlag, Berlin, 2000.

[23] S.-Q. Shen, T.-Z. Huang, *Convergence and comparison theorems for double splittings of matrices*, Comput. Math. Appl. **51** (12) (2006), 1751–1760.

[24] S.-X. Miao, B. Zheng, *A note on double splittings of different monotone matrices*, Calcolo **46** (2009), 261–266.

[25] V. Shekhar, C. K. Giri, D. Mishra, *Convergence theory of iterative methods based on proper splittings and proper multisplittings for rectangular linear systems*, Filomat **34** (2020), 1835–1851.

[26] V. Shekhar, C. K. Giri and D. Mishra, *A note on double weak splittings of type II*, Linear Multilinear Algebra **70** (2022), 2297–2317.

[27] V. Shekhar, N. Mishra and D. Mishra, *On convergence theory of double K-weak splittings of type II*, Appl Math **67** (2022), 341–369.

[28] Y. Wei, X. Li, H. Wu, *Subproper and regular splittings for restricted rectangular linear system*, Appl. Math. Comput. **136** (2003), 535–547.

[29] Y. Wei, P. Stanimirović, M. Petković, it Numerical and Symbolic Computations of Generalized Inverses, Hackensack, World Scientific, 2018.

[30] Z.I. Woźnicki, *Estimation of the optimum relaxation factors in partial factorization iterative methods*, SIAM J. Matrix Anal. Appl. **14** (1993) 59–73.