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New insights on pseudoparallel submanifold in trans Sasakian manifolds

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Abstract. The aim of the present paper is to study invariant pseudo-parallel submanifolds of a trans Sasakian manifold. We have searched the necessary and sufficient conditions for an invariant pesudo-parallel submanifold to be totally geodesic under the some conditions.

1. Introduction

An almost contact manifold is odd-dimensional manifold $\widetilde{M}^{2n+1}$ which carries a field ϕ of endomorphism of the tangent space, called the structure vector field ξ, and a 1-form η-satisfying;

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where I denote the identity mapping of tangent space of each point at M. From (1), it follows

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \;\; rank(\phi) = 2n. \tag{2}$$

In this case, $\widetilde{M}^{2n+1}(\phi, \xi, \eta)$ is said to be almost contact manifold[4]. An almost contact manifold $\widetilde{M}^{2n+1}$ is called an almost contact metric manifold if a Riemannian metric tensor g satisfies

$$\begin{aligned} g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) + g(X, \phi Y) = 0 \\ \eta(X) &= g(X, \xi) \end{aligned} \tag{3}$$

for any vector fields X, Y on $\widetilde{M}$. The structure (φ, ξ, η, g) on $\widetilde{M}$ is said to be almost contact metric structure. In an almost contact metric structure (φ, ξ, η, g), the Nijenhuis tensor and the fundamental form are, respectively, defined by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y],$$

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and

$$\Phi(X,Y) = g(X,\phi Y)$$

for all $X, Y \in \Gamma(T\widetilde{M})$. An almost contact metric structure (φ, ξ, η, g) is said to be normal if

$$N_{\varphi}(X,Y) + 2d\eta(X,Y)\xi = 0.$$

If an almost contact metric structure (ϕ, ξ, η, g) is normal and satisfies

$$d\eta = \alpha\Phi \ , \ d\Phi = 2\beta\eta \wedge \Phi,$$

then it called trans-Sasakian structure, where $\alpha = \frac{1}{2n}\delta\Phi(\xi)$, $\beta = \frac{1}{2n}div(\xi)$ and δ is the codifferential of g[2].

It is well known that trans-Sasakian condition may be characterized as an almost contact metric structure satisfying

$$(\widetilde{\nabla}_X\phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \tag{4}$$

for all $X, Y \in \Gamma(T\bar{M})$[10]. It follows that

$$\widetilde{\nabla}_X\xi = -\alpha\phi X - \beta\phi^2 X. \tag{5}$$

A trans-Sasakian manifold of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are, respectively, known as coymplectic, β-Kenmotsu and α-Sasakian manifold. The different geometrical properties of trans Sasakian manifolds have studied by De and Tripathi[2–7].

Also the following relations hold in a trans Sasakian manifold

$$\begin{aligned}
\widetilde{R}(X,Y)\xi &= (\alpha^2-\beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \\
&+ Y(\alpha)\phi X - X(\alpha)\phi Y + Y(\beta)\phi^2 X - X(\beta)\phi^2 Y, &(6)\\
& \quad \eta(\widetilde{R}(X,Y)Z) = (\alpha^2-\beta^2)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}, &(7)\\
\widetilde{S}(X,\xi) &= \{2n(\alpha^2-\beta^2) - \xi(\beta)\}\eta(X) - (2n-1)X(\beta) - (\phi X)(\alpha), &(8)\\
& \quad \xi(\alpha) + 2\alpha\beta = 0, &(9)
\end{aligned}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where $\widetilde{R}$ and $\widetilde{S}$ denote the Riemannian curvature tensor and Ricci tensor of $\widetilde{M}$ with respect to Levi-Civita connection $\bar{\nabla}$, respectively[9].

Now, let M be an immersed submanifold of a semi-Riemannian manifold $(\widetilde{M}, g)$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{10}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{11}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are induced connections on M, $\Gamma(T^{\perp}M)$ and σ, A denote the second fundamental form, shape operatory of M, respectively. The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{12}$$

for all $X, Y, Z \in \Gamma(TM)$.

A submanifold M of a semi-Riemannian manifold $(\widetilde{M}, g)$ is called Chaki-pseudo parallel if its second fundamental form σ satisfies

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = 2\gamma(X)\sigma(Y,Z) + \gamma(Y)\sigma(X,Z) + \gamma(Z)\sigma(X,Y), \tag{13}$$

for all $X, Y, Z \in \Gamma(TM)$ and γ is a nowhere vanishing 1-form.

In particular, if $\gamma = 0$ then M is said to be parallel submanifold of $\widetilde{M}$.

For a submanifold M of a semi-Riemannian manifold $(\widetilde{M}, g)$, the Gauss and Weingarten equations are given by

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \\ &+ (\widetilde{\nabla}_X\sigma)(Y,Z) - (\widetilde{\nabla}_Y\sigma)(X,Z) \end{aligned} \tag{14}$$

and

$$g(\widetilde{R}(X,Y)U,V) = g(R^{\perp}(X,Y)U,V) + g([A_V, A_U]X, Y), \tag{15}$$

for all $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, where R and $R^{\perp}$ are the Riemannian curvature tensors of M and $\Gamma(T^{\perp}M)$, respectively.

As a generalization of Ricci semisymmetric manifolds, introduced the notion of Ricci pseudo symmetric manifolds[6]. We may call it as Ricci pseudo-parallel. A Riemannian manifold $(M, g), (n > 2)$ is said to be Deszcz-pseudo Ricci-symmetric if

$$(R(X,Y)\cdot S)(U,V) = L_s Q(g,S)(U,V;X,Y), \tag{16}$$

holds on $U_s = \{x \in M : (S - \frac{\tau}{n}g) \neq 0\}$, for each point $x \in M$ and $X, Y, U, V \in \Gamma(TM)$, where Q denote the Tachibana tensor, L_s is some function on U_s, R is the Riemannian curvature tensor, S is the Ricci tensor and τ is the scalar curvature of the manifold.

Definition 1.1. *A submanifold M of a semi-Riemannian manifold $(\widetilde{M}, g)$ is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel if*

$$\widetilde{R}\cdot\sigma \;,\; Q(g,\sigma) \tag{17}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma \;,\; Q(g,\widetilde{\nabla}\sigma) \tag{18}$$

$$\widetilde{R}\cdot\sigma \;,\; Q(\widetilde{S},\sigma) \tag{19}$$

and

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma \;,\; Q(S,\widetilde{\nabla}\sigma) \tag{20}$$

are linearly dependent, respectively, where Q denote the Tachibana tensor[11].

In [10], almost contact metric structures on 3-dimensional Lie algebras and the class of left invariant almost contact metric structures on the corresponding Lie groups have studied. In [7] the authors studied 3-dimensional trans-Sasakian manifolds which are locally ϕ-symmetric and have η-parallel Ricci tensor. Also [9], Chaubey searched the properties of special weakly Ricci symmetric and generalized Ricci recurrent trans Sasakian manifolds. On the other hand, the geometry of pseudo-submanifolds continues to be worked on different structures[11, 12].

Motivated by the above studies, in this paper, the geometry invariant pesudo-parallel submanifolds of a trans-Sasakian manifold is studied. The necessary and sufficient conditions are investigated under some hypotheses. The obtained results have been evaluated.

2. Invariant Pseudo-parallel Submanifolds of a Trans-Sasakian

The geometry of submanifolds of a contact metric manifold is depend on the behaviour of contact metric structure ϕ. Namely, a submanifold M of a trans-Sasakian manifold is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for any vector feld X tangent to M at every point of M. In other words, $\phi(TM) \subset (TM)$ at each point of M.

We note that, in differential geometry theory, an invariant submanifold inherits almost all properties of the ambient manifold. Therefore, invariant submanifolds are an active and fruitful research field playing a significant role in the development of modern differential geometry. In this connection, the papers related to invariant submanifolds has been studied and continues to be studied in different manifold types.

In the rest of this paper, we will assume that M an invariant submanifold of trans-Sasakian manifold $\widetilde{M}$.

Theorem 2.1. *Let M be an immersed submanifold of a trans-Sasakian manifold $\widetilde{M}(\phi, \xi, \eta, g)$. By R, we denote the Riemannian curvature tensor of submanifold M. Then the following relations hold*

$$\widetilde{R}(X, Y)\xi = R(X, Y)\xi, \tag{21}$$

$$\sigma(X, \phi Y) = \sigma(\phi X, Y) = \phi\sigma(X, Y), \tag{22}$$

$$\sigma(X, \xi) = 0. \tag{23}$$

Proof. We will not give the proof since it is a result of direct calculations. □

Theorem 2.2. *Let M be an invariant submanifold of a trans-Sasakian manifold $\widetilde{M}(\phi, \xi, \eta, g)$. Then M is a Chaki pseudo-parallel if and only if M is a totally geodesic submanifold provided $\alpha, \gamma(\xi) + \beta \neq 0$.*

Proof. Let us suppose that M is Chaki pseudo-parallel. From (13), we have

$$(\widetilde{\nabla}_X\sigma)(Y, Z) = 2\gamma(X)\sigma(Y, Z) + \gamma(Y)\sigma(X, Z) + \gamma(Z)\sigma(X, Y),$$

for all $X, Y \in (TM)$. In the last equality, taking $Y = \xi$ and by using (5) and (12), we observe

$$\gamma(\xi)\sigma(X, Z) = -\sigma(\nabla_X\xi, Z) = \sigma(\alpha\phi X + \beta\phi^2 X, Z) = \alpha\phi\sigma(X, Z) - \beta\sigma(X, Z)$$

Since σ and $\phi\sigma$ are linear independent vectors and the ambient space is trans-Sasakain manifold, we conclude that $\sigma = 0$. The converse part is trivial. □

We have following Corollary since every totally geodesic submanifold is a Chaki pseudo-parallel.

Corollary 2.3. *Let M be an invariant submanifold of a trans-Sasakian manifold $\widetilde{M}$. Then M is a parallel submanifold if and only if M is a totally geodesic.*

Theorem 2.4. *let M be an invariant pseudo-parallel submanifold of trans-Sasakian manifold $\widetilde{M}(\phi, \xi, \eta, g)$. Then M is either totally gedesic submanifold or $\lambda = \alpha^2 - \beta^2 - \xi(\beta)$*

Proof. Since M is a pseudo-parallel, there is a function λ on $\widetilde{M}$ such as

$$(\widetilde{R}(X, Y) \cdot \sigma)(U, V) = \lambda Q(g, \sigma)(U, V; X, Y),$$

for all $X, Y, U, V \in (TM)$. This leads to

$$\begin{aligned} R^{\perp}(X, Y)\sigma(U, V) &- \sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V) \\ &= -\lambda\{\sigma((X\wedge_g Y)U, V) + \sigma(U, (X\wedge_g Y)V)\}, \end{aligned}$$

which implies for $V = \xi$,

$$\sigma(R(X, Y)\xi, U) = \lambda\sigma(U, \eta(Y)X - \eta(X)Y).$$

By using (6) and (21), we have

$$\begin{aligned} & \sigma(U,(\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y]+Y(\alpha)\phi X \\ - \quad & X(\alpha)\phi Y+Y(\beta)\phi^2 X-X(\beta)\phi^2 Y)=\lambda\sigma(U,\eta(Y)X-\eta(X)Y), \end{aligned}$$

that is,

$$\begin{aligned} & \eta(Y)(\alpha^2-\beta^2-\lambda)\sigma(U,X)-\eta(X)(\alpha^2-\beta^2-\lambda)\sigma(U,Y) \\ + \quad & 2\alpha\beta\eta(Y)\phi\sigma(U,X)-2\alpha\beta\eta(X)\phi\sigma(U,Y)+Y(\alpha)\phi\sigma(U,X) \\ - \quad & X(\alpha)\phi\sigma(U,Y)-Y(\beta)\sigma(U,X)+X(\beta)\sigma(U,Y)=0. \end{aligned} \tag{24}$$

Taking $Y=\xi$ in (24) and using (23), after the necessary adjustments, we get

$$(\alpha^2-\beta^2-\xi(\beta)-\lambda)\sigma(U,X)=0, \tag{25}$$

which proves our assertions. □

Theorem 2.5. *Let M be an invariant submanifold of a trans-Sasakian manifold $\widetilde{M}$. If M is a Ricci-generalized pseudo-parallel submanifold, then at least one of the following holds*
1.) M is a totally geodesic,
2.) $\lambda=\frac{1}{2n}$,
3.) $\alpha^2-\beta^2-\xi(\beta)=0$.

Proof. Since M is an invariant Ricci-generalized pseudo-parallel submanifold, there exists a function λ on $\widetilde{M}$ such that

$$(\widetilde{R}(X,Y)\cdot\widetilde{S})(U,V)=\lambda Q(\widetilde{S},\sigma)(U,V;X,Y).$$

This yields to

$$\begin{aligned} R^{\perp}(X,Y)\sigma(U,V) \quad - \quad & \sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V) \\ = \quad & -\lambda\{\sigma((X\wedge_S Y)U,V)+\sigma(U,(X\wedge_S Y)V)\}. \end{aligned}$$

Taking $V=\xi$ and making use of (6), (21) and (23) we have

$$\sigma(U,R(X,Y)\xi)=\lambda\sigma(U,S(Y,\xi)X-S(X,\xi)Y),$$

or,

$$\begin{aligned} & \sigma(U,(\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y] \\ + \quad & Y(\alpha)\phi X-X(\alpha)\phi Y+Y(\beta)\phi^2 X-X(\beta)\phi^2 Y) \\ = \quad & \lambda\{S(\xi,Y)\sigma(U,X)-S(X,\xi)\sigma(U,Y)\}. \end{aligned} \tag{26}$$

(8) put in (26), we have

$$\begin{aligned} & (\alpha^2-\beta^2)\sigma(U,\eta(Y)X-\eta(X)Y)+2\alpha\beta\phi\sigma(U,\eta(Y)X-\eta(X)Y) \\ + \quad & \phi\sigma(Y(\alpha)X-X(\alpha)Y,U)-\sigma(Y(\beta)X-X(\beta)Y,U) \\ = \quad & \lambda\{[2n(\alpha^2-\beta^2)-\xi(\beta)]\eta(Y)-(2n-1)Y(\beta)-(\phi Y)(\alpha)\}\sigma(U,X) \\ - \quad & \lambda\{[2n(\alpha^2-\beta^2)-\xi(\beta)]\eta(X)-(2n-1)X(\beta)-(\phi X)(\alpha)\}\sigma(U,Y). \end{aligned} \tag{27}$$

Setting $Y=\xi$ in (27) and by using (23) we get

$$\begin{aligned} & [\alpha^2-\beta^2-\xi(\beta)]\sigma(U,X)+[2\alpha\beta+\xi(\alpha)]\phi\sigma(U,X) \\ = \quad & 2n\lambda[\alpha^2-\beta^2-\xi(\beta)]\sigma(U,X). \end{aligned}$$

From (9), we conclude that

$$[\alpha^2-\beta^2-\xi(\beta)][2n\lambda-1]\sigma(U,X)=0. \tag{28}$$

This proves our assertions. □

Theorem 2.6. *Let M be a invariant 2-pseudoparallel submanifold of a trans-Sasakian manifold $\widetilde{M}$. Then at least one of the following holds;*
1.) M is a totally geodesic submanifold,
2.) $\lambda = \alpha^2 - \beta^2 - \xi(\beta)$,
3.) $\widetilde{M}$ is a β-Kenmotsu manifold.

Proof. If M is a 2-pseudo parallel submanifold, then (18) implies that

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,Z) = \lambda Q(g,\sigma)(U,V,Z:X,Y).$$

That's mean

$$\begin{aligned} R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z) &- (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z) - (\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z) \\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z) = -L_{\widetilde{\nabla}\sigma}\{(\widetilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V,Z) \\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_g Y)V,Z) + (\widetilde{\nabla}_U\sigma)(V,(X\wedge_g Y)Z)\}, \end{aligned}$$

for all $X,Y,U,V,Z \in \Gamma(TM)$. Taking $X = Z = \xi$ in the last equality, we have

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) - (\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi) \\ &- (\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi) = -L_{\widetilde{\nabla}\sigma}\{(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) \\ &+ (\widetilde{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi) + (\widetilde{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi)\}. \end{aligned} \quad (29)$$

Here, the expansion of the first term

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(\xi,Y) - \sigma(\nabla_U V,\xi) - \sigma(V,\nabla_U\xi)\} \\ &= R^{\perp}(\xi,Y)\{\alpha\phi\sigma(U,V) - \beta\sigma(U,V)\}. \end{aligned} \quad (30)$$

As the second term, in view of (5) and (7)

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) &= -\sigma(V,\nabla_{R(\xi,Y)U}\xi) = \alpha\sigma(\phi R(\xi,Y)U,V) - \beta\sigma(R(\xi,Y)U,V) \\ &= \alpha\phi\sigma(R(\xi,Y)U,V) - \beta\sigma(R(\xi,Y)U,V), \end{aligned} \quad (31)$$

In the same way, for the non-zero components, we get

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi) &= -\sigma(R(\xi,Y)V,\nabla_U\xi) = \sigma(R(\xi,Y)V,\alpha\phi U + \beta\phi^2 U) \\ &= \alpha\phi\sigma(R(\xi,Y)V,U) - \beta\sigma(R(\xi,Y)V,U). \end{aligned} \quad (32)$$

For the last term of the left side of equality, making use of (5), (6) and (12), we obtain

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi) &= (\widetilde{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta))\phi^2 Y) \\ &= (\widetilde{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta))\eta(Y)\xi) \\ &- (\widetilde{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta))Y) \\ &= -\sigma(\nabla_U(\alpha^2-\beta^2-\xi(\beta))\eta(Y)\xi,V) \\ &- (\widetilde{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta))Y) \\ &= \eta(Y)(\alpha^2-\beta^2-\xi(\beta))\sigma(\alpha\phi U + \beta\phi^2 U,V) \\ &- (\widetilde{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta))Y) \\ &= \eta(Y)(\alpha\phi-\beta)(\alpha^2-\beta^2-\xi(\beta))\sigma(U,V) \\ &- (\widetilde{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta))Y). \end{aligned} \quad (33)$$

The first term of the right side of the equality give us

$$\begin{aligned}(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) &= -\sigma(\nabla_{(\xi\wedge_g Y)U}\xi, V) = \alpha\phi\sigma((\xi\wedge_g Y)U,V)\\ &- \beta\sigma((\xi\wedge_g Y)U,V)\\ &= -(\alpha\phi-\beta)\eta(U)\sigma(Y,V).\end{aligned} \tag{34}$$

Since the non-zero components of the second,

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_g Y)V) &= -\sigma(\nabla_U\xi,(\xi\wedge_g Y)V)\\ &= \sigma(\alpha\phi U+\beta\phi^2U, g(Y,V)\xi-\eta(V)Y)\\ &= -\eta(V)(\alpha\phi-\beta)\sigma(U,Y).\end{aligned} \tag{35}$$

Finally,

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi) &= (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi-Y)\\ &= (\widetilde{\nabla}_U\sigma)(\eta(Y)\xi,Y)-(\widetilde{\nabla}_U\sigma)(V,Y)\\ &= -\sigma(\nabla_U\eta(Y)\xi,V)-(\widetilde{\nabla}_U\sigma)(V,Y)\\ &= -\sigma(U[\eta(Y)]\xi+\eta(Y)\nabla_U\xi,V)-(\widetilde{\nabla}_U\sigma)(V,Y)\\ &= \eta(Y)\sigma(\alpha\phi U+\beta\phi^2U,V)-(\widetilde{\nabla}_U\sigma)(V,Y)\\ &= (\alpha\phi-\beta)\eta(Y)\sigma(U,V)-(\widetilde{\nabla}_U\sigma)(V,Y).\end{aligned} \tag{36}$$

Thus (30)-(36) put in (29) and taking $V=\xi$, we can infer

$$\begin{aligned}&- (\alpha\phi-\beta)\sigma(R(\xi,Y)\xi,U)+(\widetilde{\nabla}_U\sigma)((\alpha^2-\beta^2-\xi(\beta))Y,\xi)\\ &= \lambda\{(\alpha\phi-\beta)\sigma(U,Y)+(\widetilde{\nabla}_U\sigma)(\xi,Y)\}.\end{aligned} \tag{37}$$

From (6) and (12) we conclude

$$\begin{aligned}&(\alpha\phi-\beta)\sigma((\alpha^2-\beta^2-\xi(\beta))\phi^2Y,U)+(\alpha\phi-\beta)\sigma(\nabla_U\sigma)(\xi,Y)\\ &= \lambda\{\sigma(\nabla_U\xi,Y)-(\alpha\phi-\beta)\sigma(U,Y)\}.\end{aligned} \tag{38}$$

Consequently, we reach at

$$\alpha[\alpha^2-\beta^2-\xi(\beta)-\lambda]\sigma(U,Y)=0,$$

which proves our assertions. □

Theorem 2.7. *Let M be a invariant 2-Ricci-generalized pseudoparallel submanifold of a trans-Sasakian manifold $\widetilde{M}$. Then at least one of the following holds;*
1.) M is a totally geodesic submanifold,
2.) $\lambda=\frac{1}{2n}$,
3.) $\xi(\beta)=\alpha^2-\beta^2$.

Proof. Since M is an invariant 2-Ricci-generalized pseudoparallel submanifold of a trans-Sasakain manifold $\widetilde{M}$, there exist a function λ such that

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla})(U,V,Z)=\lambda Q(\widetilde{S},\widetilde{\nabla})(U,V,Z;X,Y).$$

This means that

$$\begin{aligned}&R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z)-(\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z)=-\lambda\{(\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z)+(\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,Z)\\ &+ (\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)Z)\},\end{aligned}$$

for all $X, Y, U, V, Z \in \Gamma(TM)$. Taking $X = V = \xi$ in the last equality, we have

$$\begin{aligned} & R^{\perp}(\xi, Y)(\widetilde{\nabla}_U\sigma)(\xi, Z) - (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi, Z) - (\widetilde{\nabla}_U\sigma)(R(\xi, Y)\xi, Z) \\ - \quad & (\widetilde{\nabla}_U\sigma)(\xi, R(\xi, Y)Z) = -\lambda\{(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi, Z) + (\widetilde{\nabla}_U\sigma)((\xi \wedge_S Y)\xi, Z) \\ + \quad & (\widetilde{\nabla}_U\sigma)(\xi, (\xi \wedge_S Y)Z)\}. \end{aligned} \tag{39}$$

Now, non-zero components of the first term

$$\begin{aligned} R^{\perp}(\xi, Y)(\widetilde{\nabla}_U\sigma)(\xi, Z) &= R^{\perp}(\xi, Y)\{\widetilde{\nabla}_U\sigma(\xi, Z) - \sigma(\nabla_U\xi, Z) - \sigma(\xi, \nabla_U Z)\} \\ &= -R^{\perp}(\xi, Y)\sigma(\nabla_U\xi, Z) = R^{\perp}(\xi, Y)\sigma(\alpha\phi U + \beta\phi^2 U, Z) \\ &= R^{\perp}(\xi, Y)(\alpha\phi - \beta)\sigma(U, Z). \end{aligned} \tag{40}$$

In the same way, after the necessary revisions, the second term give is us

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi, Z) &= -\sigma(\nabla_{R(\xi,Y)U}\xi, Z) = \sigma(\alpha\phi R(\xi, Y)U + \beta\phi^2 R(\xi, Y)U, Z) \\ &= (\alpha\phi - \beta)\sigma(R(\xi, Y)U, Z). \end{aligned} \tag{41}$$

Also, non-zero components of the third term calculations is

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi, Y)\xi, Z) &= (\widetilde{\nabla}_U\sigma)((\alpha^2 - \beta^2 - \xi(\beta))\phi^2 Y, Z) \\ &= (\widetilde{\nabla}_U\sigma)((\alpha^2 - \beta^2 - \xi(\beta))\eta(Y)\xi, Z) \\ &- (\widetilde{\nabla}_U\sigma)((\alpha^2 - \beta^2 - \xi(\beta))Y, Z) \\ &= -\sigma((\alpha^2 - \beta^2 - \xi(\beta))\eta(Y)\nabla_U\xi, Z) \\ &- (\widetilde{\nabla}_U\sigma)((\alpha^2 - \beta^2 - \xi(\beta))Y, Z) \\ &= (\alpha^2 - \beta^2 - \xi(\beta))\eta(Y)\sigma(\alpha\phi U + \beta\phi^2 U, Z) \\ &- (\widetilde{\nabla}_U\sigma)((\alpha^2 - \beta^2 - \xi(\beta))Y, Z) \\ &= (\alpha^2 - \beta^2 - \xi(\beta))\eta(Y)(\alpha\phi - \beta)\sigma(U, Z) \\ &- (\widetilde{\nabla}_U\sigma)((\alpha^2 - \beta^2 - \xi(\beta))Y, Z). \end{aligned} \tag{42}$$

After the necessary abbreviations, the fourth term is

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi, Y)Z, \xi) &= -\sigma(\nabla_U\xi, R(\xi, Y)Z) = \sigma(\alpha\phi U + \beta\phi^2 U, R(\xi, Y)Z) \\ &= (\alpha\phi - \beta)\sigma(U, R(\xi, Y)Z). \end{aligned} \tag{43}$$

For non-zero components of first term of the right side of the equality, by using (8), we have

$$\begin{aligned} (\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi, Z) &= -\sigma(\nabla_{(\xi\wedge_S Y)U}\xi, Z) = (\alpha\phi - \beta)\sigma((\xi \wedge_S Y)U, Z) \\ &= (\alpha\phi - \beta)\sigma(S(Y, U)\xi - S(\xi, U)Y, Z) \\ = -(\alpha\phi - \beta)\{[2n(\alpha^2 &- \beta^2) - \xi(\beta)]\eta(U) - (2n - 1)U(\beta) - (\phi U)(\alpha)\}\sigma(Y, Z). \end{aligned} \tag{44}$$

With similar thought, second term give us

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(S(Y, \xi)\xi - S(\xi, \xi)Y, Z) &= -\sigma(\nabla_U S(Y, \xi)\xi, Z) - (\widetilde{\nabla}_U\sigma)(S(\xi, \xi)Y, Z) \\ = S(Y, \xi)\sigma(\alpha\phi U + \beta\phi^2 U, Z) &- (\widetilde{\nabla}_U\sigma)(2n[\alpha^2 - \beta^2 - \xi(\beta)]Y, Z) \\ = (\alpha\phi - \beta)\{[2n(\alpha^2 - \beta^2 - \xi(\beta))]\eta(Y) &- (2n - 1)Y(\beta) - (\phi Y)(\alpha)\}\sigma(U, Z) \\ &- (\widetilde{\nabla}_U\sigma)(2n[\alpha^2 - \beta^2 - \xi(\beta)]Y, Z). \end{aligned} \tag{45}$$

We obtain the non-zero components of the last term

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi, S(Y,Z)\xi - S(\xi,Z)Y) &= \sigma(\nabla_U\xi, S(\xi,Z)Y) \\
&= -S(\xi,Z)\sigma(\alpha\phi U + \beta\phi^2 U, Y) \\
-(\alpha\phi - \beta)\{[2n(\alpha^2 - \beta^2 - \xi(\beta))]\eta(Z) &- (2n-1)Z(\beta) - (\phi Z)(\alpha)\}\sigma(U,Y).
\end{aligned} \tag{46}
$$

Consequently, (40)-(46) are put in (39) for $Z = \xi$ and the necessary revisions are made, we reach at

$$
\begin{aligned}
&(\alpha^2 - \beta^2 - \xi(\beta))\sigma(\nabla_U\xi, Y) + (\alpha\phi - \beta)\sigma(U, (\alpha^2 - \beta^2 - \xi(\beta))\phi^2 Z) \\
= \; &\lambda\{2n(\alpha^2 - \beta^2 - \xi(\beta))\sigma(\nabla_U\xi, Y) - (\alpha\phi - \beta)2n(\alpha^2 - \beta^2 - \xi(\beta))\sigma(U,Y)\}.
\end{aligned}
$$

for the sake of, it implies that

$$
[2n\lambda - 1][\alpha^2 - \beta^2 - \xi(\beta)]\sigma(U,Y) = 0. \tag{47}
$$

This completes the proof. □

Example 2.8. *Let $\widetilde{M}^5 = \{(x_1, x_2, x_3, x_4, t) \in \mathbf{E}^5 : t \neq 0\}$be a 5-dimensional differentiable manifold with the standart coordinate system (x_1, x_2, x_3, x_4, t). Then vector fields*

$$
E_1 = t(\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial t}), E_2 = t\frac{\partial}{\partial x_2}, E_3 = t(\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial t}), E_4 = t\frac{\partial}{\partial x_4}, E_5 = \xi = \frac{\partial}{\partial t}
$$

are the linear independent at each points of $\widetilde{M}$, that is, these vector fields a basis of tangent space of $\widetilde{M}$. Now, we define the contact structure and metric tensor ϕ and g by

$$
\phi E_1 = -E_2, \;\; \phi E_2 = E_1, \;\; \phi E_3 = -E_4, \;\; \phi E_4 = E_3, \;\; \phi E_5 = 0,
$$

and

$$
g(E_i, E_j) = \delta_{ij}, \;\; 1 \leq i, j \leq 5
$$

then we can easily verify that

$$
\phi^2 X = -X + \eta(X)\xi, \;\; g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \;\; \eta(X) = g(X,\xi).
$$

So, $\widetilde{M}^5(\phi, \xi, \eta, g)$ is a contact metric manifold. By direct calculations, we can get non-zero components of Lie-bracket as

$$
[E_i, E_5] = -\frac{1}{t}E_i, \;\; 1 \leq i \leq 4,
$$

$$
[E_1, E_2] = x_2 E_2 - t^2 E_5, \;\; [E_1, E_3] = -x_4 E_1 + x_2 E_3, \;\; [E_1, E_4] = x_2 E_4.
$$

In view of Kozsul formulae, we can find the following non-zero components of connections as

$$
\begin{aligned}
&\widetilde{\nabla}_{E_1}E_5 = -\frac{1}{t}E_1 + \frac{1}{2}t^2 E_2, \; \widetilde{\nabla}_{E_2}E_5 = -\frac{1}{2}t^2 E_1 - \frac{1}{t}E_2 \\
&\widetilde{\nabla}_{E_3}E_5 = -\frac{1}{t}E_3 + \frac{1}{2}t^2 E_4, \; \widetilde{\nabla}_{E_4}E_5 = -\frac{1}{2}t^2 E_3 - \frac{1}{t}E_4.
\end{aligned}
$$

By the straightforward calculations, by using (5),we can observe $\alpha = \frac{1}{2}t^2$ and $\beta = -\frac{1}{t}$. This tell us that $\widetilde{M}^5(\phi, \xi, \eta, g)$ is a 5-dimensional trans-Sasakain manifold with $\alpha = \frac{1}{2}t^2$ and $\beta = -\frac{1}{t}$.
Now, we consider vector fields

$$
e_1 = t(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + (x_2 + x_4)\frac{\partial}{\partial t}), \;\; e_2 = t(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}), \;\; e_3 = \frac{\partial}{\partial t}.
$$

These vector fields are the linear dependent. By $\mathbf{D}$*, let's denote the distribution spanned by these vectors. One can observe* $\mathbf{D}$ *is integrable and involutive. So their integral manifold is a submanifold of* $\widetilde{M}^5(\phi, \xi, \eta, g)$*. On can easily to see that*

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad and \quad \phi e_3 = 0. \tag{48}$$

This tell us that M is an invariant submanifold of a trans-Sasakaim manifold $\widetilde{M}^5(\phi, \xi, \eta, g)$*. On the other hand, by a direct calculation we mean that* $\sigma(e_1, e_2) = 0$*, that is, M is totally geodesic submanifold.*

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